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Existence theorems for relaxed η-α pseudomonotone and strictly η-quasimonotone generalized variational-like inequalities

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Abstract
In this paper, we prove the existence of solutions for a variational-like inequality and a generalized variational-like inequality in the relaxed η-α pseudomonotone and strictly η-quasimonotone cases in Banach spaces by using the KKM technique. The results presented in this paper improve and extend some corresponding results of several authors.

Keywords: variational-like inequality; generalized variational-like inequality; relaxed η-α pseudomonotone operator; strictly η-quasimonotone operator; solution existence

1 Introduction

The variational inequality was first introduced and studied in the finite-dimensional Euclidean space by Giannessi [1]. Variational inequality problems play a critical role in many fields of science, engineering, and economics. In the last four decades, since the time of the celebrated Hartman-Stampacchia theorem (see [2, 3]), the existence of a solution of a variational inequality, a generalized variational inequality, and other related problems has become a basic research topic which continues to attract the attention of researchers in applied mathematics (see for instance [4–13], and the references therein).

In 1995, Chang *et al.* [14] introduced and studied the problem of the existence of solutions and the perturbation problem for some kind of variational inequalities with monotone and semimonotone mappings in nonreflexive Banach spaces. Recently, Verma [15] studied a class variational inequality relaxed monotone mapping. Moreover, Fang and Huang [16] obtained the existence of solutions for variational-like inequalities with relaxed η-α monotone mappings in reflexive Banach spaces. In 2003, Facchinei and Pang [17, 18] used the degree theory to obtain a necessary and sufficient condition of variational inequality problems for continuous pseudomonotone mappings in a finite-dimensional space. In 2008, Kien *et al.* [19] proposed some extensions of the results of Facchinei and Pang [17, 18] to the case of variational inequalities and generalized variational inequalities in infinite-dimensional reflexive Banach spaces.

On the other hand, Bai *et al.* [20] introduced the new concept of relaxed η-α pseudomonotone mappings. By using the KKM technique, they obtain some existence results for variational-like inequalities with relaxed η-α pseudomonotone mappings in reflexive Banach spaces. In 2007, Wu and Huang [21] introduced the two new concepts of relaxed η-α pseudomonotonicity and relaxed η-α demipseudomonotonicity in Banach spaces. In 2009, Pourbarat and Abbasi [22] tried to replace some conditions of the work of Wu and Huang [21] with some new conditions. Moreover, they present the solvability of variational-like inequalities with relaxed η-α monotone mappings in arbitrary Banach spaces (see also in [2, 15–20] and [23–28]).

Inspired and motivated by [19], we introduce a new definition of relaxed η-α pseudomonotone mappings and prove the existence of solutions for variational-like inequality and generalized variational-like inequality with relaxed η-α pseudomonotone mappings and strictly η-quasimonotone mappings in Banach spaces by using KKM technique. The results presented in this paper improve and extend some corresponding results of several authors.

2 Preliminaries

Let X be a real reflexive Banach space with dual space X^* and $\langle \cdot, \cdot \rangle$ denoted the pairing between X^* and X. Let K be a nonempty subset of X, and 2^X denote the family of all the nonempty subset of X and $\Phi : K \to 2^{X^*}$ and $\eta : K \times K \to X$ be mappings. The *generalized variational-like inequality* defined by K and Φ, denoted by GVLI(K, Φ), is the problem of finding a point $x \in K$ such that

$$\exists x^* \in \Phi(x), \quad \langle x^*, \eta(y,x) \rangle \geq 0 \quad \forall y \in K. \tag{2.1}$$

The set of all $x \in K$ satisfying (2.1) is denoted by SOL(K, Φ). If $\Phi(x) = \{F(x)\}$ for all $x \in K$, where $F : K \to X^*$ is a single-valued mapping, then the problem GVLI(K, Φ) is called a *variational-like inequality* and the abbreviation VLI(K, F) is the problem of finding an $x \in K$ such that

$$\langle F(x), \eta(y,x) \rangle \geq 0 \quad \forall y \in K. \tag{2.2}$$

We introduce the definition of relaxed η-α pseudomonotone for α mapping which comes from a family of functions which contains all mappings α given in [20]. In fact, the new definition is an extension of Definition 2.1 in [20]. Then we recall some definitions and results which are needed in the sequel.

We introduce the family

$$\mathbf{A} = \left\{ \alpha \colon \alpha : X \to \mathbb{R}; \limsup_{t \to 0^+} \frac{\alpha(t\eta(x,y))}{t} = 0 \ \forall (x,y) \in K \times K \right\}.$$

We note that if $\alpha(tx) = k(t)\alpha(x)$, for all $x \in X$ where k is a function from $(0,\infty)$ to $(0,\infty)$ with $\lim_{t\to 0} \frac{k(t)}{t} = 0$, then $\alpha \in A$.

Definition 2.1 The mapping $F : K \to X^*$ is said to be:

(i) *Relaxed η-α pseudomonotone* if there exist $\eta : K \times K \to X$ and $\alpha : X \to \mathbb{R}$ with $\alpha \in A$, such that for every distinct points $x, y \in K$,

$$\langle F(y), \eta(x,y) \rangle \geq 0 \quad \Rightarrow \quad \langle F(x), \eta(x,y) \rangle \geq \alpha(\eta(x,y)). \tag{2.3}$$

If $\eta(x,y) = x - y$ for all distinct points x, y in K, then (2.3) becomes

$$\langle F(y), x-y\rangle \geq 0 \quad \Rightarrow \quad \langle F(x), x-y\rangle \geq \alpha(x-y),$$

and F is said to be *relaxed α pseudomonotone*.

(ii) *Strictly η-quasimonotone* if there exist $\eta : K \times K \to X$ such that for every distinct points $x, y \in K$,

$$\langle F(y), \eta(x,y)\rangle > 0 \quad \Rightarrow \quad \langle F(x), \eta(x,y)\rangle > 0. \tag{2.4}$$

If $\eta(x,y) = x - y$ for all distinct points x, y in K, then (2.4) becomes

$$\langle F(y), x-y\rangle > 0 \quad \Rightarrow \quad \langle F(x), x-y\rangle > 0,$$

and F is said to be *strictly quasimonotone*.

Definition 2.2 The mapping $\Phi : K \to 2^{X^*}$ is said to be:

(i) *Relaxed η-α pseudomonotone* if there exist $\eta : K \times K \to X$ and $\alpha : X \to \mathbb{R}$ with $\alpha \in A$,

$$\begin{aligned}&\langle y^*, \eta(x,y)\rangle \geq 0, \quad \exists y^* \in \Phi(y)\\ &\quad \Rightarrow \quad \langle x^*, \eta(x,y)\rangle \geq \alpha\bigl(\eta(x,y)\bigr), \quad \exists x^* \in \Phi(x), \forall x, y \in X.\end{aligned}$$

(ii) *Strictly η-quasimonotone* if there exist $\eta : K \times K \to X$ such that

$$\begin{aligned}&\langle F(y), \eta(x,y)\rangle > 0, \quad \exists y^* \in \Phi(y)\\ &\quad \Rightarrow \quad \langle F(x), \eta(x,y)\rangle > 0, \quad \exists x^* \in \Phi(x), \forall x, y \in X.\end{aligned}$$

Example 2.3 If $F : (-\infty, 0] \to [0, +\infty)$ define by $F(x) = x^2$ and

$$\eta(x,y) = |x-y| \quad \forall x, y \in (-\infty, 0],$$

where $c > 0$, then the mapping F is a relaxed η-α pseudomonotone mapping with

$$\alpha(z) = \begin{cases} -[|z|], & z > 0,\\ [|z|], & z \leq 0.\end{cases}$$

But it is not a relaxed α-pseudomonotone mapping. In fact, if we let $x = -1$, $y = 0$, $\langle F(y), x - y\rangle \geq 0$, but $\langle F(x), x-y\rangle < \alpha(x-y)$, which is a contradiction.

Example 2.4 If $F : (-\infty, 1) \to \mathbb{R}$ define by $F(x) = x^2 - 1$ and

$$\eta(x,y) = -c(x-y) \quad \forall x, y \in (-\infty, 1),$$

where $c > 0$. Then the mapping F is strictly η-quasimonotone but fails to be strictly quasimonotone since if $x \in (-1, 1)$ and $y < -1$, then we have $\langle F(y), x-y\rangle \geq 0$ but $\langle F(x), x-y\rangle < 0$.

Definition 2.5 ([20]) Let $F : K \to X^*$ and $\eta : K \times K \to X$ be two mappings. F is said to be *η-hemicontinuous* if, for any fixed $x, y \in K$, the mapping $f : [0,1] \to (-\infty, +\infty)$ defined by $f(t) = \langle F(x + t(y - x)), \eta(y,x)\rangle$ is continuous at 0^+.

If $\eta(x,y) = x - y$ $\forall x, y \in K$, then F is said to be *hemicontinuous*.

Definition 2.6 ([29]) A mapping $F : K \to 2^X$ is said to be a *KKM mapping* if, for any $\{x_1, \ldots, x_n\} \subset K$, $\operatorname{co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n F(x_i)$, where $\operatorname{co}\{x_1, \ldots, x_n\}$ denotes the convex hull of $x_1, \ldots, x_n$.

Lemma 2.7 ([29]) *Let K be a nonempty subset of a Hausdorff topological vector space X and let $F : K \to 2^X$ be a KKM mapping. If $F(x)$ is closed in X for every x in K and compact for some $x_0 \in K$, then*

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

Lemma 2.8 (Michael selection theorem [30]) *Let X be a paracompact space and Y be a Banach space. Then every lower semicontinuous multivalued mapping from X to the family of nonempty, closed, convex subsets of Y admits a continuous selection.*

3 Generalized variational-like inequality with relaxed η-α pseudomonotone mappings

In this section, we will discuss the existence of solutions for the following variational-like inequality and generalized variational-like inequality with relaxed η-α pseudomonotone mappings.

Theorem 3.1 *Let K be a nonempty closed convex subset of a real reflexive Banach space X. Let $F : K \to X^*$ and $\eta : K \times K \to X$ be mappings. Assume that:*

(i) *F is an η-hemicontinuous and relaxed η-α pseudomonotone;*
(ii) *$\eta(x,x) = 0$ for all $x \in K$;*
(iii) *$\eta(tx + (1-t)z, y) = t\eta(x,y) + (1-t)\eta(z,y)$ for all $x, y, z \in K$, $t \in [0,1]$.*

Then $x \in K$ is a solution of VLI(K,F) *if and only if*

$$\langle F(y), \eta(y,x)\rangle \geq \alpha(\eta(y,x)) \quad \forall y \in K. \tag{3.1}$$

Proof Suppose that $x \in K$ is a solution of VLI(K,F). Since F is relaxed η-α pseudomonotone, we have

$$\langle F(y), \eta(y,x)\rangle \geq \alpha(\eta(y,x)) \quad \forall y \in K,$$

and hence $x \in K$ is a solution of (3.1). Conversely, suppose that $x \in K$ is a solution of (3.1) and $y \in K$ be any point. Letting $x_t = ty + (1-t)x$, $t \in (0,1]$, we have $x_t \in K$. It follows from (3.1) that

$$\langle F(x_t), \eta(x_t,x)\rangle \geq \alpha(\eta(x_t,x)). \tag{3.2}$$

By the conditions of η, we have

$$\begin{aligned}\langle F(x_t), \eta(x_t, x)\rangle &= \langle F(x_t), \eta(ty + (1-t)x, x)\rangle \\ &= t\langle F(x_t), \eta(y, x)\rangle + (1-t)\langle F(x_t), \eta(x, x)\rangle \\ &= t\langle F(x_t), \eta(y, x)\rangle. \qquad (3.3)\end{aligned}$$

It follows from (3.2) and (3.3) that

$$t\langle F(x_t), \eta(y, x)\rangle = \langle F(x_t), \eta(x_t, x)\rangle \geq \alpha(\eta(x_t, x)) \quad \forall t \in (0, 1].$$

So, we have

$$\langle F(x_t), \eta(y, x)\rangle \geq \frac{\alpha(\eta(x_t, x))}{t} = \frac{\alpha(t\eta(y, x))}{t} \quad \forall t \in (0, 1].$$

Letting $t \to 0^+$, we get

$$\langle F(x), \eta(y, x)\rangle \geq 0 \quad \forall y \in K. \qquad \square$$

Theorem 3.2 *Let X be a real reflexive Banach space and $K \subset X$ be a closed convex set. Let $F : K \to X^*$ and $\eta : K \times K \to X$ be are mappings. Assume that*:

(i) *F is a relaxed η-α pseudomonotone mapping and η-hemicontinuous*;

(ii) *$\eta(x, x) = 0$ for all $x \in K$*;

(iii) *$\eta(tx + (1-t)z, y) = t\eta(x, y) + (1-t)\eta(z, y)$ for all $x, y, z \in K$, $t \in [0, 1]$ and η is lower semicontinuous*;

(iv) *$\alpha : X \to \mathbb{R}$ is lower semicontinuous.*

Then the following statements are equivalent:

(a) *There exists a reference point $x^{\text{ref}} \in K$ such that the set*

$$L_<(F, x^{\text{ref}}) := \{x \in K : \langle F(x), \eta(x, x^{\text{ref}})\rangle < \alpha(\eta(x, x^{\text{ref}}))\}$$

is bounded (*possibly empty*).

(b) *The variational-like inequality* VLI(K, F) *has a solution.*

Moreover, if there exists a vector $x^{\text{ref}} \in K$ such that the set

$$L_\leq(F, x^{\text{ref}}) := \{x \in K : \langle F(x), \eta(x, x^{\text{ref}})\rangle \leq \alpha(\eta(x, x^{\text{ref}}))\}$$

is bounded and $\eta(x, y) + \eta(y, x) = 0$ for all x, y in K, then the solution set SOL(K, F) *is nonempty and bounded.*

Proof Suppose that there exists a reference point $x^{\text{ref}} \in K$, which satisfies (a). Then there exists an open ball, denoted by Ω such that

$$L_<(F, x^{\text{ref}}) \cup \{x^{\text{ref}}\} \subset \Omega.$$

We combine this with the obvious property $\partial\Omega \cap L_<(F, x^{\text{ref}}) = \emptyset$. Thus $\langle F(x), \eta(x, x^{\text{ref}})\rangle \geq \alpha(\eta(x, x^{\text{ref}}))$ $\forall x \in K \cap \partial\Omega$. Define the set-valued mappings $T, S : K \to 2^X$, for any $x \in K$, by

$$T(x) = \{y \in K \cap \bar{\Omega} : \langle F(y), \eta(x, y)\rangle \geq 0\}$$

and

$$S(x) = \{y \in K \cap \bar{\Omega} : \langle F(x), \eta(x,y)\rangle \geq \alpha(\eta(x,y))\}.$$

We claim that T is a KKM mapping. Indeed, if T is not a KKM mapping, then there exists $\{x_1, x_2, \ldots, x_n\} \subset K$ such that $\mathrm{co}\{x_1, x_2, \ldots, x_n\} \not\subseteq \bigcup_{i=1}^n T(x_i)$. That is, there exists a $x_0 \in \mathrm{co}\{x_1, x_2, \ldots, x_n\}$, $x_0 = \sum_{i=1}^n t_i x_i$, where $t_i \geq 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^n t_i = 1$, but $x_0 \notin \bigcup_{i=1}^n T(x_i)$. By the definition of T, we have

$$\langle F(x_0), \eta(x_i, x_0)\rangle < 0, \quad i = 1, 2, \ldots, n.$$

Since $\sum_{i=1}^n t_i = 1$ for $t_i \geq 0$ $(i = 1, 2, \ldots, n)$, it follows that

$$\sum_{i=1}^n t_i \langle F(x_0), \eta(x_i, x_0)\rangle < 0.$$

On the other hand, we note that

$$\begin{aligned} \sum_{i=1}^n t_i \langle F(x_0), \eta(x_i, x_0)\rangle &= \left\langle F(x_0), \eta\left(\sum_{i=1}^n t_i x_i, x_0\right)\right\rangle \\ &= \langle F(x_0), \eta(x_0, x_0)\rangle \\ &= 0. \end{aligned}$$

It is a contradiction and this implies that T is a KKM mapping. Now we show that $T(x) \subset S(x)$ for all $x \in K$. For any given $x \in K$, let $y \in T(x)$. Thus, we have $\langle F(y), \eta(x,y)\rangle \geq 0$. Since F is a relaxed η-α pseudomonotone, we obtain $\langle F(x), \eta(x,y)\rangle \geq \alpha(\eta(x,y))$. This implies that $y \in S(x)$ and so $T(x) \subset S(x)$ for all $x \in K$. It follows that S is also a KKM mapping.

From the assumptions, we know that $S(x)$ is weakly closed. In fact, since η and α are lower semicontinuous, we see that $S(x)$ is a weakly closed subset of $K \cap \bar{\Omega}$. Since $K \cap \bar{\Omega}$ is a weakly compact and $S(x)$ is a weakly closed subset of $K \cap \bar{\Omega}$, we see that $S(x)$ is weakly compact for each $x \in K$. Thus, the conditions of Lemma 2.7 are satisfied in the weak topology. By Lemma 2.7 and Theorem 3.1, we have

$$\bigcap_{x \in K} T(x) = \bigcap_{x \in K} S(x) \neq \emptyset.$$

It follows that there exists $z \in K \cap \bar{\Omega}$ such that

$$\langle F(z), \eta(x,z)\rangle \geq 0 \quad \forall x \in K.$$

Hence $z \in \mathrm{SOL}(K,F)$.

Assume that (b) holds. We take any $x^{\mathrm{ref}} \in \mathrm{SOL}(K,F)$. That is,

$$\langle F(x^{\mathrm{ref}}), \eta(x, x^{\mathrm{ref}})\rangle \geq 0 \quad \forall x \in K.$$

By the relaxed η-α pseudomonotonicity of F, we have

$$\langle F(x), \eta(x, x^{\text{ref}})\rangle \geq \alpha(\eta(x, x^{\text{ref}})) \quad \forall x \in K.$$

Hence $L_<(F, x^{\text{ref}}) = \emptyset$ and (a) is valid.

Finally, suppose that there is some $x^{\text{ref}} \in K$ such that the set $L_\leq(F, x^{\text{ref}})$ is bounded. Then $\text{SOL}(K, F)$ is nonempty by virtue of the implication (a) $\Rightarrow$ (b). To prove that $\text{SOL}(K, F)$ is bounded, it suffices to show that $\text{SOL}(K, F) \subset L_\leq(F, x^{\text{ref}})$. Assume that $x \in \text{SOL}(K, F)$, but $x \notin L_\leq(F, x^{\text{ref}})$. Thus, we have

$$\langle F(x), \eta(y, x)\rangle \geq 0 \quad \forall y \in K \tag{3.4}$$

and

$$\langle F(x), \eta(x, x^{\text{ref}})\rangle > \alpha(\eta(x, x^{\text{ref}})). \tag{3.5}$$

Substituting $y = x^{\text{ref}}$ into the inequality in (3.4), we have

$$\langle F(x), \eta(x^{\text{ref}}, x)\rangle \geq 0. \tag{3.6}$$

This implies that $\langle F(x), \eta(x, x^{\text{ref}})\rangle \leq 0$. From (3.5) and (3.6), we have $\alpha(\eta(x, x^{\text{ref}})) < 0$. By (3.6) and F is relaxed η-α pseudomonotone, we obtain

$$\langle F(x^{\text{ref}}), \eta(x^{\text{ref}}, x)\rangle \geq \alpha(\eta(x^{\text{ref}}, x)).$$

It implies that $\langle F(x^{\text{ref}}), \eta(x, x^{\text{ref}})\rangle \leq \alpha(\eta(x, x^{\text{ref}}))$. By Theorem 3.1 and (3.5), we get $\langle F(x^{\text{ref}}), \eta(x, x^{\text{ref}})\rangle \geq 0$. Hence

$$0 \leq \langle F(x^{\text{ref}}), \eta(x, x^{\text{ref}})\rangle \leq \alpha(\eta(x, x^{\text{ref}})) < 0.$$

It is a contradiction. Therefore $x \in L_\leq(F, x^{\text{ref}})$. □

Theorem 3.3 *Let X be a real reflexive Banach space and $K \subset X$ be a closed convex set. Let $\Phi : K \to 2^{X^*}$ and $\eta : K \times K \to X$ be are mappings. Assume that:*

(i) *Φ is a lower semicontinuous multifunction with nonempty closed convex values, where X^* is endowed with the norm topology;*

(ii) *Φ is a relaxed η-α pseudomonotone mapping;*

(iii) *$\eta(x, x) = 0$ for all $x \in K$;*

(iv) *$\eta(tx + (1-t)z, y) = t\eta(x, y) + (1-t)\eta(z, y)$ for all $x, y, z \in K$, $t \in [0, 1]$ and η is lower semicontinuous;*

(v) *$\alpha : X \to \mathbb{R}$ is lower semicontinuous.*

Then the following statements are equivalent:

(a) *There exists a reference point $x^{\text{ref}} \in K$ such that the set*

$$L_<(\Phi, x^{\text{ref}}) := \left\{x \in K : \inf_{x^* \in \Phi(x)} \langle x^*, \eta(x, x^{\text{ref}})\rangle < \alpha(\eta(x, x^{\text{ref}}))\right\}$$

is bounded (possibly empty).

(b) *The generalized variational-like inequality $\text{GVLI}(K, \Phi)$ has a solution.*

Proof Since Φ is lower semicontinuous multifunction with nonempty closed convex values, by Michael's selection theorem (see for instance [30]) it admits a continuous selection; that is, there exists a continuous mapping $F : K \to X^*$ such that $F(x) \in \Phi(x)$ for every $x \in K$. If (a) holds, then there exists an open ball, denoted by Ω such that

$$L_<(\Phi, x^{\mathrm{ref}}) \cup \{x^{\mathrm{ref}}\} \subset \Omega.$$

We combine this with the obvious property $\partial\Omega \cap L_<(\Phi, x^{\mathrm{ref}}) = \emptyset$. Thus, we have

$$\langle F(x), \eta(x, x^{\mathrm{ref}})\rangle \geq \inf_{x^* \in \Phi(x)} \langle x^*, \eta(x, x^{\mathrm{ref}})\rangle \geq \alpha(\eta(x, x^{\mathrm{ref}})) \quad \forall x \in K \cap \partial\Omega.$$

Applying Theorem 3.2, we get $\mathrm{SOL}(K, F) \neq \emptyset$. For any $x \in \mathrm{SOL}(K, F)$, if we choose $x^* = F(x)$ then

$$\langle x^*, \eta(y, x)\rangle \geq 0 \quad \forall y \in K.$$

It follows that $\emptyset \neq \mathrm{SOL}(K, F) \subset \mathrm{SOL}(K, \Phi)$.

We prove that (b) $\Rightarrow$ (a). Assume that (b) holds. We take any $x^{\mathrm{ref}} \in \mathrm{SOL}(K, \Phi)$. Thus there exists $x^* \in \Phi(x^{\mathrm{ref}})$ satisfying

$$\langle x^*, \eta(y, x^{\mathrm{ref}})\rangle \geq 0 \quad \forall y \in K.$$

Because Φ is a relaxed η-α pseudomonotone, we obtain

$$\langle y^*, \eta(y, x^{\mathrm{ref}})\rangle \geq \alpha(\eta(y, x^{\mathrm{ref}})) \quad \forall y \in K, y^* \in \Phi(y).$$

It follows that

$$\inf_{y^* \in \Phi(y)} \langle y^*, \eta(y, x^{\mathrm{ref}})\rangle \geq \alpha(\eta(y, x^{\mathrm{ref}})) \quad \forall y \in K.$$

Hence $L_<(\Phi, x^{\mathrm{ref}}) = \emptyset$ and (a) is valid. □

4 Generalized variational-like inequality with strictly η-quasimonotone mappings

In this section, we will discuss the existence of solutions for the following variational-like inequality and generalized variational-like inequality with strictly η-quasimonotone mappings.

Theorem 4.1 *Let K be a nonempty closed convex subset of a real reflexive Banach space X. Let $F : K \to X^*$ and $\eta : K \times K \to X$ be mappings. Assume that*:

(i) *F is η-hemicontinuous and strictly η-quasimonotone*;
(ii) *$\eta(x, x) = 0$ for all $x \in K$*;
(iii) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$*;
(iv) *for any fixed $y, z \in K$, the mapping $x \mapsto \langle Tz, \eta(x, y)\rangle$ is convex.*

Then $x \in K$ is a solution of VLI(K, F) *if and only if*

$$\langle F(y), \eta(y, x)\rangle \geq 0 \quad \forall y \in K. \tag{4.1}$$

Proof Suppose that $x \in K$ is a solution of VLI(K,F). That is $\langle F(x), \eta(y,x)\rangle \geq 0$ $\forall y \in K$. To show that $\langle F(y), \eta(y,x)\rangle \geq 0$ $\forall y \in K$. Assume that there exists $y_0 \in K$ such that $\langle F(y_0), \eta(y_0,x)\rangle < 0$. By the property of η, we have $\langle F(y_0), \eta(x,y_0)\rangle > 0$. Since F is strictly η-quasimonotone, we have $\langle F(x), \eta(x,y_0)\rangle > 0$. By the property of η again, we get $\langle F(x), \eta(y_0,x)\rangle < 0$. It is a contradiction. Hence $\langle F(y), \eta(y,x)\rangle \geq 0$ $\forall y \in K$.

Conversely, suppose that $x \in K$ is a solution of (4.1) and $y \in K$ is arbitrary. Letting $x_t = ty + (1-t)x$, $t \in (0,1]$, we have $x_t \in K$. It follows from (4.1) that

$$\bigl\langle F(x_t), \eta(x_t,x)\bigr\rangle \geq 0. \tag{4.2}$$

By assumption, we have

$$\begin{aligned}\bigl\langle F(x_t), \eta(x_t,x)\bigr\rangle &= \bigl\langle F(x_t), \eta\bigl(ty+(1-t)x,x\bigr)\bigr\rangle \\ &\leq t\bigl\langle F(x_t), \eta(y,x)\bigr\rangle + (1-t)\bigl\langle F(x_t), \eta(x,x)\bigr\rangle \\ &= t\bigl\langle F(x_t), \eta(y,x)\bigr\rangle.\end{aligned} \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\bigl\langle F(x_t), \eta(y,x)\bigr\rangle \geq 0 \quad \forall t \in (0,1].$$

Since F is η-hemicontinuous and letting $t \to 0^+$, we get

$$\bigl\langle F(x), \eta(y,x)\bigr\rangle \geq 0 \quad \forall y \in K.$$

□

Theorem 4.2 *Let X be a real reflexive Banach space and $K \subset X$ be a closed convex set. Let $F : K \to X^*$ and $\eta : K \times K \to X$ be are mappings. Assume that:*

(i) *F is a strictly η-quasimonotone mapping and η-hemicontinuous;*
(ii) *$\eta(x,x) = 0$ for all $x \in K$;*
(iii) *$\eta(x,y) + \eta(y,x) = 0$ for all $x,y \in K$;*
(iv) *for any fixed $y,z \in K$, the mapping $x \mapsto \langle Tz, \eta(x,y)\rangle$ is convex and η is lower semicontinuous.*

Then the following statements are equivalent:

(a) *There exists a reference point $x^{\mathrm{ref}} \in K$ such that the set*

$$L_<\bigl(F,x^{\mathrm{ref}}\bigr) := \bigl\{x \in K : \bigl\langle F(x), \eta\bigl(x,x^{\mathrm{ref}}\bigr)\bigr\rangle < 0\bigr\}$$

is bounded (possibly empty).

(b) *The variational-like inequality* VLI(K,F) *has a solution.*

Moreover, if there exists a vector $x^{\mathrm{ref}} \in K$ such that the set

$$L_\leq\bigl(F,x^{\mathrm{ref}}\bigr) := \bigl\{x \in K : \bigl\langle F(x), \eta\bigl(x,x^{\mathrm{ref}}\bigr)\bigr\rangle \leq 0\bigr\}$$

is bounded, then the solution set SOL(K,F) *is nonempty and bounded.*

Proof Suppose that (a) holds. Then there exists a reference point $x^{\mathrm{ref}} \in K$ and an open ball, denoted by Ω such that

$$L_<\bigl(F,x^{\mathrm{ref}}\bigr) \cup \bigl\{x^{\mathrm{ref}}\bigr\} \subset \Omega.$$

We combine this with the obvious property $\partial\Omega \cap L_{<}(F, x^{\text{ref}}) = \emptyset$. Thus $\langle F(x), \eta(x, x^{\text{ref}})\rangle \geq \alpha(\eta(x, x^{\text{ref}}))$ $\forall x \in K \cap \partial\Omega$. Defined the set-valued mappings $T, S : K \to 2^X$, for any $x \in K$, by

$$T(x) = \{y \in K \cap \bar{\Omega} : \langle F(y), \eta(x, y)\rangle \geq 0\}$$

and

$$S(x) = \{y \in K \cap \bar{\Omega} : \langle F(x), \eta(x, y)\rangle \geq 0\}.$$

Since η is lower semicontinuous, we find that $T(x)$ and $S(x)$ are weakly closed subsets of $K \cap \bar{\Omega}$. We claim that T is a KKM mapping. Similar to the proof of Theorem 3.2 we show that T is a KKM mapping. Now we show that $T(x) \subset S(x)$ for all $x \in K$. For any given $x \in K$, we let $y \in T(x)$. That is, $\langle F(y), \eta(x, y)\rangle \geq 0$. Since F is strictly η-quasimonotone, we have $\langle F(x), \eta(x, y)\rangle \geq 0$. This implies that $y \in S(x)$ and so $T(x) \subset S(x)$ for all $x \in K$. It follows that S is also a KKM mapping. Since $K \cap \bar{\Omega}$ is weakly compact and $S(x)$ is a weakly closed subset of $K \cap \bar{\Omega}$, we find that $S(x)$ is weakly compact for each $x \in K$. Thus, the condition of Lemma 2.7 is satisfied in the weak topology. By Lemma 2.7 and Theorem 4.1, we have

$$\bigcap_{x \in K} T(x) = \bigcap_{x \in K} S(x) \neq \emptyset.$$

It follows that there exists $z \in K \cap \bar{\Omega}$ such that

$$\langle F(z), \eta(x, z)\rangle \geq 0 \quad \forall x \in K.$$

Hence $z \in \text{SOL}(K, F)$.

Assume that (b) holds. We take any $x^{\text{ref}} \in \text{SOL}(K, F)$, that is,

$$\langle F(x^{\text{ref}}), \eta(x, x^{\text{ref}})\rangle \geq 0 \quad \forall x \in K.$$

By the strict η-quasimonotonicity of F, we have

$$\langle F(x), \eta(x, x^{\text{ref}})\rangle \geq 0 \quad \forall x \in K.$$

Hence $L_{<}(F, x^{\text{ref}}) = \emptyset$ and (a) is valid.

Finally, suppose that there is some $x^{\text{ref}} \in K$ such that the set $L_{\leq}(F, x^{\text{ref}})$ is bounded. Then $\text{SOL}(K, F)$ is nonempty by virtue of the implication (a) $\Rightarrow$ (b). To prove that $\text{SOL}(K, F)$ is bounded, it suffices to show that $\text{SOL}(K, F) \subset L_{\leq}(F, x^{\text{ref}})$. Assume that $x \in \text{SOL}(K, F)$. Thus, we have

$$\langle F(x), \eta(y, x)\rangle \geq 0 \quad \forall y \in K. \tag{4.4}$$

Substituting $y = x^{\text{ref}}$ into the inequality in (4.4), we have

$$\langle F(x), \eta(x^{\text{ref}}, x)\rangle \geq 0. \tag{4.5}$$

This implies that $\langle F(x), \eta(x, x^{\text{ref}})\rangle \leq 0$. Therefore $x \in L_{\leq}(F, x^{\text{ref}})$. □

Theorem 4.3 *Let X be a real reflexive Banach space and $K \subset X$ be a closed convex set. Let $\Phi : K \to 2^{X^*}$ and $\eta : K \times K \to X$ be are mappings. Assume that*:

(i) *Φ is a lower semicontinuous multifunction with nonempty closed convex values, where X^* is endowed with the norm topology*;
(ii) *Φ is a strictly η-quasimonotone mapping*;
(iii) *$\eta(x,x) = 0$ for all $x \in K$*;
(iv) *$\eta(x,y) + \eta(y,x) = 0$ for all $x,y \in K$*;
(v) *for any fixed $y,z \in K$, the mapping $x \mapsto \langle Tz, \eta(x,y)\rangle$ is convex and η is lower semicontinuous.*

Then the following statements are equivalent:

(a) *There exists a reference point $x^{\mathrm{ref}} \in K$ such that the set*

$$L_<(\Phi, x^{\mathrm{ref}}) := \Big\{x \in K : \inf_{x^* \in \Phi(x)} \langle x^*, \eta(x, x^{\mathrm{ref}})\rangle < 0\Big\}$$

is bounded (possibly empty).

(b) *The generalized variational-like inequality* $\mathrm{GVLI}(K,\Phi)$ *has a solution.*

Proof Since Φ is a lower semicontinuous multifunction with nonempty closed convex values, by Michael's selection theorem (see for instance [30]) it admits a continuous selection; that is, there exists a continuous mapping $F : K \to X^*$ such that $F(x) \in \Phi(x)$ for every $x \in K$. If (a) holds, then there exists an open ball, denoted by Ω, such that

$$L_<(\Phi, x^{\mathrm{ref}}) \cup \{x^{\mathrm{ref}}\} \subset \Omega.$$

We combine this with the obvious property $\partial\Omega \cap L_<(\Phi, x^{\mathrm{ref}}) = \emptyset$. Then we have

$$\langle F(x), \eta(x, x^{\mathrm{ref}})\rangle \geq \inf_{x^* \in \Phi(x)} \langle x^*, \eta(x, x^{\mathrm{ref}})\rangle \geq 0 \quad \forall x \in K \cap \partial\Omega.$$

Applying Theorem 4.2, we get $\mathrm{SOL}(K,F) \neq \emptyset$. For any $x \in \mathrm{SOL}(K,F)$, if we choose $x^* = F(x)$ then

$$\langle x^*, \eta(y,x)\rangle \geq 0 \quad \forall y \in K.$$

It follows that $\emptyset \neq \mathrm{SOL}(K,F) \subset \mathrm{SOL}(K,\Phi)$.

We prove that (b) $\Rightarrow$ (a). Assume that (b) holds. We take any $x^{\mathrm{ref}} \in \mathrm{SOL}(K,\Phi)$. Thus there exists $x^* \in \Phi(x^{\mathrm{ref}})$ satisfying

$$\langle x^*, \eta(y, x^{\mathrm{ref}})\rangle \geq 0 \quad \forall y \in K.$$

Because Φ is strictly η-quasimonotone and Theorem 4.1, we obtain

$$\langle y^*, \eta(y, x^{\mathrm{ref}})\rangle \geq 0 \quad \forall y \in K, y^* \in \Phi(y).$$

It follows that

$$\inf_{y^* \in \Phi(y)} \langle y^*, \eta(y, x^{\mathrm{ref}})\rangle \geq 0 \quad \forall y \in K.$$

Hence $L_<(\Phi, x^{\mathrm{ref}}) = \emptyset$ and (a) is valid. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The work presented here was carried out in collaboration between all authors. SP, C-FW and AA defined the research theme. SP and C-FW designed theorems and methods of proof and interpreted the results. AA proved the theorems, interpreted the results and wrote the paper. All authors read and approved the final manuscript.

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