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Preconditioning methods for solving a general split feasibility problem

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Dedicated to Professor Shih-sen Chang on the occasion of his 80th birthday.

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Abstract

We introduce and study a new general split feasibility problem (GSFP) in a Hilbert space. This problem generalizes the split feasibility problem (SFP). The GSFP extends the SFP with a nonlinear continuous operator. We apply the preconditioning methods to increase the efficiency of the CQ algorithm, two general preconditioning CQ algorithms for solving the GSFP are presented. We also propose a new inexact method to approximate the preconditioner. The convergence theorems are established under the projections with respect to special norms. Some numerical results illustrate the efficiency of the proposed methods.

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1 Introduction

As preconditioning methods can improve the condition number of the ill-posed system matrix, the convergence rate of the iterative algorithm can also be improved [1]. In [2, 3], a preconditioning method is applied to modify the projected Landweber algorithm for solving a linear feasibility problem (LFP). The modified algorithm is

$$x_{n+1} = P_C\big[x_n - \tau DA^*(Ax_n - b)\big], \quad n \geq 0,$$

where $\tau \in (0, 2/\|DA^*A\|)$, $A : X \to Y$ is a linear and continuous operator, $\|\cdot\|$ means 2-norm, X and Y are Hilbert spaces and $b \in Y$ is the datum of the problem, corrupted by noise or experimental errors.

While under the nonlinear conditions, Auslender and Dafermos [4, 5] proposed an algorithm to solve variational inequalities (VI),

$$x_{n+1} = P_S\big[x_n - \tau_n G^{-1}F(x_n)\big], \quad n \geq 0, \tag{1.1}$$

where P_S is the projection operator onto S with respect to the norm $\|\cdot\|_G$. Bertsekas and Gafni [6] and Marcotte and Wu [7] improved it with variable symmetric positive defined matrices G_n, Fukushima [8] modified it by a relaxed projection method with half-space;

then in [9], Yang established the convergence of Auslender's algorithm under the weak co-coercivity of F.

Further, the general variational inequality problem (GVIP) has been investigated by many authors (see [10–13]). It is to find $u^* \in \mathbb{R}^n$ such that $g(u^*) \in K$ and

$$\langle F(u^*), g(u) - g(u^*)\rangle \geq 0, \quad g(u) \in K,$$

where K is a nonempty closed convex set in $\mathbb{R}^n$, $F, g : R^n \to R^n$ are nonlinear operators. In [12], Santos and Scheimberg extended and applied (1.1) to solve the GVI.

However, a general split feasibility problem (GSFP) equals to a GVI, and preconditioning methods for solving the GSFP have not been studied. By introducing a convex minimization problem, the split feasibility problem (SFP) is equivalent to a variational inequality problem (VIP), which involves a Lipschitz continuous and inversely strong monotone (ism) operator, see [14–22]. Similarly, by the same way, in this paper we introduce that a new GSFP equals to a GVI involving a Lipschitz continuous and co-coercive operator [9, 17–19].

Otherwise, Mohammad and Abdul [23] considered a general split feasibility in infinite-dimensional real Hilbert spaces. It is to find x^* such that

$$x^* \in \bigcap_{i=1}^{\infty} C_i, \qquad Ax^* \in \bigcap_{j=1}^{\infty} Q_j,$$

where $A : H_1 \to H_2$ is a bounded linear operator, $\{C_i\}_{i=1}^{\infty}$ and $\{Q_j\}_{j=1}^{\infty}$ are the families of nonempty closed convex subsets of H_1 and H_2, respectively.

Let C and Q be nonempty closed convex subsets in real Hilbert spaces H_1 and H_2, respectively. We consider a general split feasibility problem which is different from the one in [23]. Our GSFP is to find

$$x^* \in H_1, g(x^*) \in C \text{ such that } Ag(x^*) \in Q, \tag{1.2}$$

where $A : H_1 \to H_2$ is a bounded linear operator and $g : H_1 \to C$ is a continuous operator. We see that the SFP in [24] and the GSFP in [23] are particular cases of GSFP (1.2). It has applications in many special fields such as signal decryption, demodulating the digital signal and noise processing, *etc.* In order to solve GSFP (1.2), two preconditioning algorithms are developed in this paper following the iterative scheme

$$g(x_{n+1}) = P_C\big[g(x_n) - \gamma DA^*(I - P_Q)Ag(x_n)\big], \quad n \geq 0,$$

where the two general constraints C and Q deal with projections with respect to the norms corresponding to some symmetric positive definite matrices. Define the solution set of (1.2) $\Gamma = \{x^* \in H_1 \mid Ag(x^*) \in Q\}$, as Γ is nonempty, and by virtue of the related G-co-coercive operator, we can establish the convergence of the proposed algorithms.

The paper is organized as follows. Section 2 presents two useful propositions. In Section 3, we define the algorithms with fixed preconditioner, variable preconditioner, relaxed projection and preconditioner approximation and analyze the convergence. Numerical results are reported in Section 4. Finally, Section 5 gives some concluding remarks.

2 Preliminaries

In what follows, we state some concepts and propositions.

The SFP is to find a point $x^* \in C$ such that $Ax^* \in Q$, where $A : H_1 \to H_2$ is a bounded linear operator [24].

Let G be a symmetric positive definite matrix, and set $D = G^{-1}$. Then the norm $\|\cdot\|_G$ is defined by $\|x\|_G^2 = \langle x, Gx\rangle$ for $\forall x \in H$. We denote by P_C the projection operator onto C with respect to the norm $\|\cdot\|_G$ [6], *i.e.*,

$$P_C(x) = \arg\min_{y\in C}\bigl\{\|x-y\|_G\bigr\}.$$

Let $\lambda_{\min}$ and $\lambda_{\max}$ be the minimum and the largest eigenvalues of G, respectively. Then for the 2-norm $\|\cdot\|$ [18, 19], we have

$$\lambda_{\min}\|x\|^2 \le \|x\|_G^2 \le \lambda_{\max}\|x\|^2, \quad \forall x \in H. \tag{2.1}$$

Proposition 2.1 [7, 10] *Let C be a nonempty closed convex subset in H, for $\forall x, y \in H$ and $\forall z \in C$, the G-projection operator onto C has the following properties*:

(i) $\langle G(x - P_C x), z - P_C(x)\rangle \le 0$;
(ii) $\|x \pm y\|_G^2 = \|x\|_G^2 \pm 2\langle x, Gy\rangle + \|y\|_G^2$;
(iii) $\|P_C(x) - P_C(y)\|_G^2 \le \|x - y\|_G^2 - \|(P_C(x) - x) - (P_C(y) - y)\|_G^2$.

Let $\tilde{D}$ be a symmetric positive definite matrix, and $A^T\tilde{D} = DA^T$. Then the norm $\|\cdot\|_{\tilde{D}}$ is defined by $\|y\|_{\tilde{D}}^2 = \langle y, \tilde{D}y\rangle$ for $\forall y \in H$. We denote by P_Q the projection operator onto Q with respect to the norm $\|\cdot\|_{\tilde{D}}$. According to the SFP, when GSFP (1.2) has no solution (refer to [2, 14, 15]), we can define

$$f_g(x) = \frac{1}{2}\bigl\|(I - P_Q)Ag(x)\bigr\|^2$$

and

$$f_{\tilde{D}}^g(x) = \frac{1}{2}\bigl\|Ag(x) - P_QAg(x)\bigr\|_{\tilde{D}}^2 = \bigl\langle\tilde{D}\bigl(Ag(x) - P_QAg(x)\bigr), Ag(x) - P_QAg(x)\bigr\rangle,$$

$f_{\tilde{D}}^g(x)$ is also convex and continuously differentiable in H. Its gradient operator is

$$\nabla f_{\tilde{D}}^g(x) = DA^T(I - P_Q)Ag(x).$$

As $D = I$, we define

$$\nabla f_g(x) = A^T(I - P_Q)Ag(x),$$

$\nabla f_g(x)$ is also Lipschitz continuous.

Proposition 2.2 *If we consider the constrained minimization problem*

$$\min\bigl\{f_{\tilde{D}}^g(x) \mid x \in H_1 \text{ s.t. } g(x) \in C\bigr\},$$

its stationary point $x^* \in H_1$ *satisfies*

$$\begin{cases} x^* \in H, g(x^*) \in C \text{ such that} \\ \langle \nabla f_D^g(x^*), g(x) - g(x^*) \rangle \geq 0, \quad \forall g(x) \in C, \end{cases}$$

which is a general variational inequality involving a Lipschitz continuous and G-co-coercive operator.

Proof For $\forall x, y \in H$, from (2.1) and Lemma 8.1 in [15], we have

$$\begin{aligned} \left\| \nabla f_D^g(x) - \nabla f_D^g(y) \right\|_G^2 &\leq \lambda_{\max}(G) \left\| \nabla f_D^g(x) - \nabla f_D^g(y) \right\|^2 \\ &\leq \frac{\lambda_{\max}^2(D)}{\lambda_{\min}(D)} \left\| \nabla f_g(x) - \nabla f_g(y) \right\|^2 \\ &\leq \frac{\lambda_{\max}^2(D)}{\lambda_{\min}(D)} L^2 \left\| g(x) - g(y) \right\|^2 \\ &\leq \frac{\lambda_{\max}^3(D)}{\lambda_{\min}(D)} L^2 \left\| g(x) - g(y) \right\|_G^2, \end{aligned}$$

where L is the largest eigenvalue of $A^T A$; therefore, the operator ∇f_D^g is Lipschitz continuous,

$$\begin{aligned} &\left\langle \nabla f_D^g(x) - \nabla f_D^g(y), g(x) - g(y) \right\rangle \\ &\quad \geq \lambda_{\min}(D) \left\langle \nabla f_g(x) - \nabla f_g(y), g(x) - g(y) \right\rangle \\ &\quad \geq \frac{\lambda_{\min}(D)}{L} \left\| \nabla f_g(x) - \nabla f_g(y) \right\|^2 \\ &\quad \geq \frac{\lambda_{\min}(D)}{L \cdot \lambda_{\max}(D)} \left\| \nabla f_g(x) - \nabla f_g(y) \right\|_D^2 \\ &\quad = \frac{\lambda_{\min}(D)}{L \cdot \lambda_{\max}(D)} \left\langle D\big(\nabla f_g(x) - \nabla f_g(y)\big), GD\big(\nabla f_g(x) - \nabla f_g(y)\big) \right\rangle \\ &\quad = \frac{\lambda_{\min}(D)}{L \cdot \lambda_{\max}(D)} \left\| \nabla f_D^g(x) - \nabla f_D^g(y) \right\|_G^2. \end{aligned}$$

Thus, the operator ∇f_D^g is co-coercive. □

3 Main results

In this section, we propose several modified CQ algorithms with preconditioning techniques and prove the convergence.

3.1 General preconditioning CQ algorithm

In this part, we have our first algorithm with fixed stepsize and preconditioner to solve GSFP (1.2). The algorithm is as follows.

Algorithm 3.1 Choose $\forall x_0 \in H_1$ such that $g(x_0) \in C$, and let $x_n \in H_1$ such that $g(x_n) \in C$, then we calculate x_{n+1} such that

$$g(x_{n+1}) = P_C\big[g(x_n) - \gamma \nabla f_D^g(x_n)\big], \quad n \geq 0, \tag{3.1}$$

where $\gamma \in (0, \frac{2}{L \cdot L_D})$, L and L_D are the largest eigenvalues of $A^T A$ and D, respectively.

Now we establish the weak convergence of Algorithm 3.1.

Theorem 3.1 *Suppose that the operators* $g : H_1 \to C$ *and* $g^{-1} : C \to H_1$ *are continuous. If* $\Gamma \neq \emptyset$, *then the sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges to the solution of GSFP* (1.2).

Proof Firstly, for $\forall x^* \in \Gamma$, we have

$$g(x^*) = P_C\bigl[g(x^*) - \gamma \nabla f_D^g(x^*)\bigr].$$

From (3.1), (iii), (ii) and the definition of ism, we have

$$\begin{aligned}
&\bigl\|g(x_{n+1}) - g(x^*)\bigr\|_G^2 \\
&\quad = \bigl\|P_C\bigl[g(x_n) - \gamma \nabla f_D^g(x_n)\bigr] - P_C\bigl[g(x^*) - \gamma \nabla f_D^g(x^*)\bigr]\bigr\|_G^2 \\
&\quad \leq \bigl\|g(x_n) - g(x^*) - \gamma\bigl[\nabla f_D^g(x_n) - \nabla f_D^g(x^*)\bigr]\bigr\|_G^2 \\
&\quad \leq \bigl\|g(x_n) - g(x^*)\bigr\|_G^2 - 2\gamma\bigl\langle g(x_n) - g(x^*), \nabla f_g(x_n) - \nabla f_g(x^*)\bigr\rangle \\
&\qquad + \gamma^2\bigl\langle \nabla f_D^g(x_n) - \nabla f_D^g(x^*), \nabla f_g(x_n) - \nabla f_g(x^*)\bigr\rangle \\
&\quad \leq \bigl\|g(x_n) - g(x^*)\bigr\|_G^2 - \left(\frac{2\gamma}{L} - \gamma^2 L_D\right)\bigl\|\nabla f_g(x_n) - \nabla f_g(x^*)\bigr\|^2,
\end{aligned} \tag{3.2}$$

as $\frac{2\gamma}{L} - \gamma^2 L_D > 0$, which implies that the sequence $\{\|g(x_n) - g(x^*)\|_G\}_{n\in\mathbb{N}}$ is monotonically decreasing, then we can obtain that the sequence $\{\|g(x_n) - g(x^*)\|_G\}_{n\in\mathbb{N}}$ is also convergent, especially, the sequence $\{g(x_n)\}_{n\in\mathbb{N}}$ is bounded. Consequently, we get from (3.2)

$$\lim_{n\to\infty}\bigl\|\nabla f_g(x_n) - \nabla f_g(x^*)\bigr\| = \lim_{n\to\infty}\bigl\|\nabla f_g(x_n)\bigr\| = 0. \tag{3.3}$$

Moreover, for each $g(x_n) \in C$, from (iii) and (2.1), we have

$$\begin{aligned}
\bigl\|g(x_{n+1}) - g(x_n)\bigr\|_G &\leq \gamma\bigl\|\nabla f_D(x_n)\bigr\|_G \\
&\leq \gamma L_D\bigl\|\nabla f_g(x_n)\bigr\|_G \leq \gamma L_D\sqrt{\lambda_{\max}(G)}\bigl\|\nabla f_g(x_n)\bigr\|.
\end{aligned}$$

Then by virtue of (3.3) we have

$$\lim_{n\to\infty}\bigl\|g(x_{n+1}) - g(x_n)\bigr\|_G = 0. \tag{3.4}$$

Hence, there exists a subsequence $\{g(x_j)\}_{j\in\overline{N}}$ of $\{g(x_n)\}_{n\in\mathbb{N}}$ such that

$$\lim_{j\to\infty}\bigl\|g(x_{j+1}) - g(x_j)\bigr\|_G = 0.$$

Thus, $\{g(x_j)\}_{j\in\overline{N}}$ is also bounded.

Let $\bar{x}$ be an accumulation point of $\{x_n\}$, then the subsequence of $\{x_n\}$, $\{x_j\}_{j\in\overline{N}} \to \bar{x}$ as $j \to \infty$. Because of the continuity of g, there exists an accumulation point $g(\bar{x}) \in C$ of the sequence $\{g(x_n)\}_{n\in\mathbb{N}}$; for the subsequence $\{g(x_j)\}_{j\in\overline{N}}$, we have $\{g(x_j)\}_{j\in\overline{N}} \to g(\bar{x})$ as $j \to \infty$. After that, from (3.3) we obtain

$$\lim_{j\to\infty} \|\nabla f_g(x_{n_j})\| = \|\nabla f_g(\bar{x})\| = \|A^T Ag(\bar{x}) - A^T P_Q Ag(\bar{x})\| = 0,$$

that is, $Ag(\bar{x}) \in Q$.

We use $\bar{x}$ in place of x^* in (3.2) and obtain that $\{\|g(x_n) - g(\bar{x})\|_G\}$ is convergent. Because its subsequence $\{\|g(x_{n_j}) - g(\bar{x})\|_G\} \to 0$, then we get that $\{g(x_n)\}_{n\in\mathbb{N}}$ converges to $g(\bar{x})$ as $j \to \infty$. As well as g^{-1} is continuous, we finally have

$$\lim_{n\to\infty} x_n = \bar{x} \in \Gamma.$$

Therefore, $\bar{x}$ is a solution of GSFP (1.2). □

From Algorithm 3.1 and Theorem 3.1 we can deduce the following results easily.

Corollary 3.1 *If $g = I$, then GSFP* (1.2) *reduces to SFP, Algorithm* 3.1 *also reduces to a preconditioning CQ* (*PCQ*) *algorithm for* $\forall x_0 \in H_1$:

$$x_{n+1} = P_C\bigl[x_n - \gamma \nabla f_D(x_n)\bigr], \quad n \ge 0, \tag{3.5}$$

where $\gamma \in (0, \frac{2}{L\cdot L_D})$, L and L_D are the largest eigenvalues of $A^T A$ and D, respectively. P_C and P_Q are still the projection operators onto C and Q with respect to the norm $\|\cdot\|_D$ and $\|\cdot\|_{\tilde{D}}$, respectively.

Corollary 3.2 *If $g = I$, $D = I$, $\tilde{D} = I$, then GSFP reduces to SFP, then Algorithm* 3.1 *reduces to the CQ algorithm proposed in* [25].

Corollary 3.3 *If $g = I$, $\tilde{D} = I$, P_C and P_Q are the projections onto C and Q with respect to the norm $\|\cdot\|$, set $F(x_n) = DA^T(I - P_Q)ADx_n$,* (3.5) *transforms into the algorithm in* [26]

$$x_{n+1} = P_C\bigl[x_n - \gamma F(x_n)\bigr], \quad n \ge 0,$$

where $\gamma \in (0, \frac{2}{L})$, $L = \|DA^T\|^2$. Then the GSFP reduces to the extended split feasibility problem (*ESFP*) *in* [26].

3.2 An algorithm with variable projection metric

The algorithms above can speed the convergence of CQ algorithm, but the stepsize and preconditioner are fixed. In this subsection, we extend the results in [6] and construct an iterative scheme with variable stepsize and preconditioner D_n from one iteration to the next. As a key role, D_n will change arbitrarily or following some rules to achieve the convergence progress and better results.

Let D_n and $\tilde{D}_n$ be two symmetric positive definite matrices for $n = 0, 1, 2, \ldots$. Denote by P_C and P_Q the projections onto C and Q with respect to the norm $\|\cdot\|_{D_n}$ and $\|\cdot\|_{\tilde{D}_n}$, respectively. Let χ be a set of symmetric positive definite matrices, we have the following algorithm.

Algorithm 3.2 Choose $\forall x_0 \in H_1$ such that $g(x_0) \in C$, and let $x_n \in H_1$ such that $g(x_n) \in C$; for $\forall D_n \in \chi$, we compute x_{n+1} such that

$$g(x_{n+1}) = P_C\big[g(x_n) - \gamma_n D_n \nabla f_g(x_n)\big], \quad n \geq 0, \tag{3.6}$$

where $\gamma_n \in (0, \frac{2}{L \cdot M_D})$, L is the largest eigenvalue of $A^T A$, M_D is the minimum value of all the largest eigenvalue values L_{D_n} to matrices D_n.

Remark 3.1 Define $d_n = \|g(x_{n+1}) - g(x_n)\|_{G_n}$, then for the next iteration, D_{n+1} is either chosen arbitrarily from χ or equivalent to D_n. It is conditional on whether d_n has decreased or not. More particularly, we define a scalar $\bar{d}_n$ with initial value $\bar{d}_0 = \infty$. Having chosen a scalar $\alpha \in (0, 1)$ at the nth iteration, then $\bar{d}_{n+1}$ is calculated by

$$\bar{d}_{n+1} = \begin{cases} \alpha \bar{d}_n, & \text{if } d_n \leq \bar{d}_n; \\ \bar{d}_n, & \text{if } d_n > \bar{d}_n, \end{cases}$$

then we select

$$D_{n+1} = \begin{cases} \forall D \in \chi, & \text{if } \bar{d}_{n+1} < \bar{d}_n; \\ D_n, & \text{if } \bar{d}_{n+1} = \bar{d}_n. \end{cases}$$

Theorem 3.2 *If* $\Gamma \neq \emptyset$, *then the sequence* $\{x_n\}$ *generated by Algorithm* 3.2 *converges to the solution of GSFP* (1.2).

Proof To obtain variable D_n at each iteration and keep the convergence of Algorithm 3.2, d_n must be a descending behavior for $n = 0, 1, 2, \ldots$. We first show that

$$\underline{\lim}_{n \to \infty} d_n = 0. \tag{3.7}$$

Indeed if (3.7) is not true, we have $\underline{\lim}_{n \to \infty} d_n > 0$, then D_n must have changed a finite number of times, we set that this number is $\kappa \in N$. Therefore, let $x^* \in \Gamma$ be a solution of GSFP, refer to (3.2) and for $n > \kappa$, we have

$$\begin{aligned} \big\|g(x_{n+1}) - g\big(x^*\big)\big\|_{G_\kappa}^2 \leq{}& \big\|g(x_n) - g\big(x^*\big)\big\|_{G_\kappa}^2 \\ &- \left(\frac{2\gamma_\kappa}{L} - \gamma_\kappa^2 L_{D_\kappa}\right)\big\|\nabla f_g(x_n) - \nabla f_g\big(x^*\big)\big\|^2, \end{aligned}$$

as $M_D = \min\{L_{D_n} \mid n = 0, 1, \ldots \kappa\}$, $\frac{2\gamma_\kappa}{L} - \gamma_\kappa^2 L_{D_\kappa} > 0$. Then, following the proof of Theorem 3.1, we also get

$$\lim_{n \to \infty} \big\|\nabla f_g(x_n)\big\| = 0 \tag{3.8}$$

and

$$\lim_{n \to \infty} d_n = \lim_{n \to \infty} \big\|g(x_{n+1}) - g(x_n)\big\|_{G_\kappa} = 0. \tag{3.9}$$

Equation (3.9) contradicts the above hypothesis, so (3.7) is true.

By using (3.6) and (iii), for the nth iteration, we have

$$\begin{aligned} d_n &= \|g(x_{n+1}) - g(x_n)\|_{G_n}^2 \\ &\geq \frac{1}{2}\|g(x_n) - P_C[g(x_n)]\|_{G_n}^2 - \|g(x_{n+1}) - P_C[g(x_n)]\|_{G_n}^2 \\ &= \frac{1}{2}\|g(x_n) - P_C[g(x_n)]\|_{G_n}^2 - \|P_C[g(x_n) - \gamma_n D_n \nabla f_g(x_n)] - P_C[g(x_n)]\|_{G_n}^2 \\ &\geq \frac{1}{2}\|g(x_n) - P_C[g(x_n)]\|_{G_n}^2 - \gamma_n L_{D_n}\|\nabla f_g(x_n)\|_{G_n}^2, \end{aligned}$$

where $\gamma_n L_{D_n} > 0$. Then from (2.1) and (3.8) we know that

$$\lim_{n\to\infty}\|\nabla f_g(x_n)\|_{G_n} = \lim_{n\to\infty}\|A^T(I - P_Q)Ag(x_n)\|_{G_n} = 0,$$

so $Ag(x_n) \in Q$. By virtue of (3.7), we also have

$$\underline{\lim}_n \|g(x_n) - P_C[g(x_n)]\| = 0.$$

This means that at least a subsequence of $\{g(x_n)\}_{n\in\mathbb{N}}$ converges to a solution of $g(\bar{x}) \in \Gamma$. Similar to the argumentation of accumulation in the proof of Theorem 3.1, we know that $\{x_n\}$ converges to a solution of GSFP. □

3.3 Some methods for execution

In Algorithms 3.1 and 3.2, there still exists difficulty to implement the projections P_C and P_Q with respect to the defined norms, especially when C and Q are general closed convex sets. According to the relaxed method in [8, 27, 28], we consider the above algorithm in which the closed convex subsets C and Q are the following particular formula:

$$C = \{g(x) \in H_1 \mid c(g(x)) \leq 0\} \quad \text{and} \quad Q = \{Ag(x) \in H_2 \mid q(Ag(x)) \leq 0\},$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex functions. C_n and Q_n are given as

$$\begin{aligned} C_n &= \{g(x) \in H_1 \mid c(g(x_n)) + \langle \xi_n, g(x) - g(x_n)\rangle \leq 0\}, \\ Q_n &= \{Ag(x) \in H_1 \mid q(Ag(x_n)) + \langle \eta_n, Ag(x) - Ag(x_n)\rangle \leq 0\}, \end{aligned}$$

where $\xi_n \in \partial c(g(x_n))$, $\eta_n \in \partial q(Ag(x_n))$.

Here, we also replace P_C and P_Q by P_{C_n} and P_{Q_n}. However, in this paper, take Algorithm 3.2 for example, the projections are with respect to the norms corresponding to G_n and $\tilde{D}_n$, we should use the following methods to calculate them. For $\forall z \in H_1$ and $\forall y \in H_2$,

$$P_{C_n}(z) = \begin{cases} z - D_n \frac{c[g(x_n)] + \langle D_n \xi_n, z - g(x_n)\rangle}{\|\xi_n\|_{D_n}^2}\xi_n, & \text{if } c[g(x_n)] + \langle D_n \xi_n, z - g(x_n)\rangle > 0; \\ z, & \text{otherwise} \end{cases}$$

and

$$P_{Q_n}(y) = \begin{cases} y - \tilde{D}_n \frac{q[Ag(x_n)] + \langle \tilde{D}_n \eta_n, y - Ag(x_n)\rangle}{\|\eta_n\|_{\tilde{D}_n}^2}\eta_n, & \text{if } q[Ag(x_n)] + \langle \tilde{D}_n \eta_n, y - Ag(x_n)\rangle > 0; \\ y, & \text{otherwise.} \end{cases}$$

Set $z = g(x_n) - \gamma_n D_n \nabla f_g(x_n)$, $y = Ag(x_n)$, let $\bar{x} \in H_1$ be an accumulation of $\{x_n\}_{n\in\mathbb{N}}$. From the proof above, it is easy to deduce that

$$\lim_{n\to\infty} c\big[g(x_n)\big] + \big\langle D_n\xi_n, z - g(x_n)\big\rangle = c\big[g(\bar{x})\big] \le 0,$$
$$\lim_{n\to\infty} q\big[Ag(x_n)\big] + \big\langle \tilde{D}_n\eta_n, y - Ag(x_n)\big\rangle = q\big[Ag(\bar{x})\big] \le 0.$$

Therefore, $g(\bar{x}) \in C \subseteq C_n$, $Ag(\bar{x}) \in Q \subseteq Q_n$, with the projections P_{C_n} and P_{Q_n}, $\bar{x}$ is a solution of GSFP.

Next, we present a new approximation method to estimate γ_n and D_n in Algorithm 3.2. If $\Gamma \neq \emptyset$, for $\forall x^* \in \Gamma$ and $n \ge 0$, such that

$$D_n \nabla f_g\big(x^*\big) = D_n A^T Ag\big(x^*\big) - D_n A^T P_{Q_n} Ag\big(x^*\big) = 0,$$

under the ideal condition, if $D_n A^T A \approx I$, the solution is done, but unfortunately, $(A^T A)^{-1}$ cannot be calculated directly when A is a large matrix in practice. As

$$A^T Ag\big(x^*\big) = A^T P_{Q_n} Ag\big(x^*\big) = \lambda g\big(x^*\big),$$

where λ is an eigenvalue of $A^T A$. Let $D_0 = I$, for the nth iteration, we have the next $j \times j$ approximation of D_{n+1}

$$D_{n+1}^{jj} = \begin{cases} \frac{[g(x_n)]_j}{[A^T P_{Q_n} Ag(x_n)]_j}, & \text{if } [g(x_n)]_j \neq 0 \text{ and } [A^T P_{Q_n} Ag(x_n)]_j \neq 0; \\ D_n^{jj}, & \text{otherwise,} \end{cases}$$

where $j = 1, 2, \ldots$. So, at the nth iteration, let l_{D_n} be the minimum eigenvalue of D_n, take $M_{D_n} \approx \min\{L_{D_k} \mid k = 0, 1, \ldots, n\}$, $L_n \approx \max\{(l_{D_k})^{-1} \mid k = 0, 1, \ldots, n\}$, the variable stepsize is approximated by

$$\gamma_n = \frac{\rho_n}{L_n \cdot M_{D_n}}, \quad \rho_n \in (0, 2), n = 1, 2, \ldots.$$

4 Numerical results

We consider the following problem from [29] in a finite dimensional Hilbert space:

Let $C = \{x \in H_1 \mid c(x) \le 0\}$, where $c(x) = -x_1 + x_2^2 + \cdots + x_N^2$, and $Q = \{y \in H_2 \mid q(y) \le 0\}$, where $q(y) = y_1 + y_2^2 + \cdots + y_M^2 - 1$. $A_{M\times N}$ is a random matrix where every element of A is in $(0, 1)$ satisfying $\Gamma \neq \emptyset$. Let x_0 be a random vector in H_1 where every element of x_0 is in $(0, 1)$.

We set $\|x_{n+1} - x_n\| \le \varepsilon$ as the stop rule, and let $N = 10$, $M = 20$, $g = I$, $\tilde{D}_n = I$, for $n \ge 0$. Using the methods in Section 3.3, we compare Algorithm 3.2 with the relaxed CQ algorithm (RCQ) in [30], with different ε and initial values. The results can be seen in Table 1. We see that the proposed methods in this paper behave better.

5 Concluding remarks

In this paper, we have discussed a new general split feasibility problem, which is related to the general variational inequalities involving a co-coercive operator. By using the G-norm

Table 1 The comparison between the results of preconditioning and relaxed CQ algorithms

ε	Algorithms	n	CPU (s)	$c(x)$	$q(y)$
0.01	RCQ	16	0.0112	–0.1665	57.1092
	3.2	22	0.0049	–0.1333	1.2636
0.001	RCQ	200	0.0216	–0.2684	1.2708
	3.2	13	0.0039	–0.0770	3.5300E–02
0.0001	RCQ	360	0.0295	–0.1039	1.0710E–01
	3.2	60	0.0081	–0.0479	1.4900E–02
0.00001	RCQ	464	0.0363	–0.2561	7.5000E–03
	3.2	21	0.0043	–0.0566	3.3809E–04

method, variable modulus method and relaxed method, two modified projection algorithms for solving the GSFP and some approximate methods for algorithm executing have been presented. The numerical results show that by preconditioning method, the convergence speed of CQ algorithm can be improved, but the way to obtain variable stepsize in the paper is inexact. To continue to improve it or combine it with the methods in [14] and [28] is another interesting subject.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors take equal roles in deriving results and writing of this paper. All authors read and approved the final manuscript.

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