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Weighted boundedness for Toeplitz type operators related to strongly singular integral operators

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Abstract

In this paper, we show the sharp maximal function estimates for the Toeplitz type operators related to the strongly singular integral operators. As an application, we obtain the boundedness of the operators on weighted Lebesgue and Triebel-Lizorkin spaces.

MSC: 42B20; 42B25

Keywords: Toeplitz operator; strongly singular integral operator; sharp maximal function; Triebel-Lizorkin space; weighted Lipschitz function

1 Introduction and Preliminaries

As a development of singular integral operators [1, 2], their commutators have been well studied. In [3–5], the authors prove that the commutators generated by the singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo [6] proves a similar result when singular integral operators are replaced by the fractional integral operators. In [7–9], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained. In [10, 11], the boundedness for the commutators generated by the singular integral operators and the weighted *BMO* and Lipschitz functions on $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained. In [12, 13], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by *BMO* and Lipschitz functions is obtained. In this paper, we will study the Toeplitz type operators related to the strongly singular integral operator and the weighted Lipschitz functions.

First, let us introduce some notation. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q \left|f(y) - f_Q\right| dy,$$

where we write $f_Q = |Q|^{-1} \int_Q f(x)\,dx$. It is well known that [1, 2]

$$M^{\#}(f)(x) \approx \sup_{Q\ni x} \inf_{c\in C} \frac{1}{|Q|} \int_Q \left|f(y) - c\right| dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

For $\eta > 0$, set $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < 1$ and $1 \le r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^r\, dy \right)^{1/r}.$$

The A_p weight is defined by [1]

$$A_p = \left\{ w \in L^1_{\mathrm{loc}}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}\, dx \right)^{p-1} < \infty \right\},$$
$$1 < p < \infty,$$

and

$$A_1 = \left\{ w \in L^p_{\mathrm{loc}}(R^n) : M(w)(x) \le Cw(x), \text{ a.e.} \right\}.$$

The $A(p,q)$ weight is defined by [14], for $1 < p, q < \infty$,

$$A(p,q) = \left\{ w > 0 : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)^q\, dx \right)^{1/q} \left(\frac{1}{|Q|} \int_Q w(x)^{-p/(p-1)}\, dx \right)^{(p-1)/p} < \infty \right\}.$$

Given a non-negative weight function w, for $1 \le p < \infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x)\, dx \right)^{1/p} < \infty.$$

For $\beta > 0$, $p > 1$ and the non-negative weight function w, let $\dot{F}_p^{\beta,\infty}(w)$ be the weighted homogeneous Triebel-Lizorkin space [9].

For $0 < \beta < 1$ and the non-negative weight function w, the weighted Lipschitz space $\mathrm{Lip}_\beta(w)$ is the space of functions b such that

$$\|b\|_{\mathrm{Lip}_\beta(w)} = \sup_Q \frac{1}{w(Q)^{1+\beta/n}} \int_Q |b(y) - b_Q|\, dy < \infty.$$

Remark (1) For $b \in \mathrm{Lip}_\beta(w)$, $w \in A_1$ and $x \in Q$, it is well known that

$$|b_Q - b_{2^kQ}| \le Ck\|b\|_{\mathrm{Lip}_\beta(w)} w(x) w(2^kQ)^{\beta/n}.$$

(2) Let $b \in \mathrm{Lip}_\beta(w)$ and $w \in A_1$. By [15], we know that the spaces $\mathrm{Lip}_\beta(w)$ coincide and the norms $\|b\|_{\mathrm{Lip}_\beta(w)}$ are equivalent with respect to different values of $1 \le p \le \infty$.

Definition Let $T : S \to S'$ be a bounded linear operator. T is called a strongly singular integral operator if it satisfies the following conditions:

(i) T extends to a bounded operator on $L^2(R^n)$;

(ii) there exists a function $K(x,y)$ continuous away from the diagonal on $R^n \times R^n$ such that

$$\left|K(x,y) - K(x,z)\right| + \left|K(y,x) - K(z,x)\right| \leq C|y-z|^{\delta}|x-z|^{n+\delta/\varepsilon}$$

if $2|y-z|^{\varepsilon} \leq |x-z|$ for some $0 < \delta \leq 1$, $0 < \varepsilon < 1$, and
$(Tf,g) = \int_{R^n}\int_{R^n} K(x,y)f(y)g(x)\,dy\,dx$ for $f,g \in S$ with disjoint support;

(iii) for some $(1-\varepsilon)n/2 \leq \beta < n/2$, T and T^* extend to a bounded operator from $L^q(R^n)$ into $L^2(R^n)$, where $1/q = 1/2 + \beta/n$.

Let b be a locally integrable function on R^n. The Toeplitz type operator related to T is defined by

$$T_b = \sum_{k=1}^{m} T^{k,1} M_b T^{k,2},$$

where $T^{k,1}$ are strongly singular integral operators or $\pm I$ (the identity operator), $T^{k,2}$ are bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, \ldots, m$, $M_b(f) = bf$.

Note that the commutator $[b,T](f) = bT(f) - T(bf)$ is a particular case of the Toeplitz type operators T_b. The Toeplitz type operators T_b are non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors [4, 5]. In [16–19], the boundedness of the strongly singular integral operator is obtained. In [20], a sharp function estimate of the strongly singular integral operator is obtained. In [21], the boundedness of the strongly singular integral operators and their commutators is obtained. In [13], the Toeplitz type operators related to the strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions is obtained. Our works are motivated by these papers. The main purpose of this paper is to prove sharp maximal inequalities for the Toeplitz type operators T_b. As applications, we obtain the weighted L^p-norm inequality and the Triebel-Lizorkin space boundedness for the Toeplitz type operators T_b.

We need the following preliminary lemmas.

Lemma 1 ([16]) *Let T be a strongly singular integral operator. Then T is bounded on $L^p(w)$ for $w \in A_p$ with $1 < p < \infty$, and when $((1-\varepsilon)n + 2\beta)/2\beta < u \leq 2$, $0 < u/v \leq \delta$, T is bounded from $L^u(R^n)$ into $L^v(R^n)$.*

Lemma 2 ([15]) *For any cube Q, $b \in \mathrm{Lip}_\beta(w)$, $0 < \beta < 1$, and $w \in A_1$, we have*

$$\sup_{x\in Q}\left|b(x) - b_Q\right| \leq C\|b\|_{\mathrm{Lip}_\beta(w)} w(Q)^{1+\beta/n}|Q|^{-1}.$$

Lemma 3 ([9]) *For $0 < \beta < 1$, $1 < p < \infty$, and $w \in A_\infty$, we have*

$$\|f\|_{\dot{F}_p^{\beta,\infty}(w)} \approx \left\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|f(x) - f_Q\right|dx\right\|_{L^p(w)}$$
$$\approx \left\|\sup_{Q\ni\cdot}\inf_c\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|f(x) - c\right|dx\right\|_{L^p(w)}.$$

Lemma 4 ([1]) *Let* $0 < p < \infty$ *and* $w \in \bigcup_{1 \le r < \infty} A_r$. *Then, for any smooth function* f *for which the left-hand side is finite,*

$$\int_{R^n} M(f)(x)^p w(x)\,dx \le C \int_{R^n} M^{\#}(f)(x)^p w(x)\,dx.$$

Lemma 5 ([14]) *Suppose that* $0 < \eta < n$, $1 < s < p < n/\eta$, $1/q = 1/p - \eta/n$, *and* $w \in A(p,q)$. *Then*

$$\left\| M_{\eta,s}(f) \right\|_{L^q(w^q)} \le C \|f\|_{L^p(w^p)}.$$

2 Theorems and proofs

We shall prove the following theorems.

Theorem 1 *Let* $w \in A_1$, $0 < \beta < 1$, $b \in \mathrm{Lip}_\beta(w)$, *and* $((1-\varepsilon)n + 2\beta)/2\beta < s < n/\beta$. *If* $g \in L^p(R^n)$ $(1 < p < \infty)$ *and* $T_1(g) = 0$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M^{\#}\big(T_b(f)\big)(\tilde{x}) \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^m M_{\beta,s}\big(T^{k,2}(f)\big)(\tilde{x}).$$

Theorem 2 *Let* $w \in A_1$, $0 < \beta < \min(1, \delta/\varepsilon)$, $((1-\varepsilon)n + 2\beta)/2\beta < s < n/\beta$, *and* $b \in \mathrm{Lip}_\beta(w)$. *If* $g \in L^p(R^n)$ $(1 < p < \infty)$ *and* $T_1(g) = 0$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$\sup_{Q \ni \tilde{x}} \inf_{c \in R^n} \frac{1}{|Q|^{1+\beta/n}} \int_Q \big|T_b(f)(x) - c\big|\,dx \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^m M_s\big(T^{k,2}(f)\big)(\tilde{x}).$$

Theorem 3 *Let* $w \in A_1$, $0 < \beta < 1$, $1/q = 1/p - \beta/n$, *and* $b \in \mathrm{Lip}_\beta(w)$. *If* $g \in L^p(R^n)$ $(1 < p < \infty)$ *and* $T_1(g) = 0$, *then* T_b *is bounded from* $L^p(w)$ *to* $L^q(w^{q/p - q(1+\beta/n)})$.

Theorem 4 *Let* $w \in A_1$, $0 < \beta < \min(1, \delta/\varepsilon)$, $1 < p < n/m\beta$, $1/q = 1/p - \beta/n$, *and* $b \in \mathrm{Lip}_\beta(w)$. *If* $g \in L^p(R^n)$ $(1 < p < \infty)$ *and* $T_1(g) = 0$, *then* T_b *is bounded from* $L^p(w)$ *to* $\dot{F}_q^{\beta,\infty}(w^{q/p - q(1+\beta/n)})$.

Proof of Theorem 1 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 that the following inequality holds:

$$\frac{1}{|Q|} \int_Q \big|T_b(f)(x) - C_0\big|\,dx \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^m M_{\beta,s}\big(T^{k,2}(f)\big)(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are T $(k = 1, \dots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We have the following two cases.

Case 1. $d > 1$. Write

$$T_b(f)(x) = T_{b-b_Q}(f)(x) = T_{(b-b_Q)\chi_{2Q}}(f)(x) + T_{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x).$$

Then

$$\frac{1}{|Q|} \int_Q \big|T_b(f)(x) - f_2(x_0)\big|\,dx \le \frac{1}{|Q|} \int_Q \big|f_1(x)\big|\,dx + \frac{1}{|Q|} \int_Q \big|f_2(x) - f_2(x_0)\big|\,dx = I_1 + I_2.$$

For I_1, by Hölder's inequality, boundedness of T, and Lemma 2, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \left|T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)\right|dx \\
&\leq \left(\frac{1}{|Q|}\int_{R^n}\left|T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)\right|^s dx\right)^{1/s} \\
&\leq C|Q|^{-1/s}\left(\int_{2Q}\left|M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)\right|^s dx\right)^{1/s} \\
&\leq C|Q|^{-1/s}\left(\int_{2Q}\left(\left|b(x)-b_Q\right|\left|T^{k,2}(f)(x)\right|\right)^s dx\right)^{1/s} \\
&\leq C|Q|^{-1/s}\sup_{x\in 2Q}\left|b(x)-b_{2Q}\right|\left(\int_Q\left|T^{k,2}(f)(x)\right|^s dx\right)^{1/s} \\
&\leq C|Q|^{-1/s}\|b\|_{\mathrm{Lip}_\beta(w)}\frac{w(2Q)^{1+\beta/n}}{|2Q|}|Q|^{1/s-\beta/n}\left(\frac{1}{|Q|^{1-s\beta/n}}\int_Q\left|T^{k,2}(f)(x)\right|^s dx\right)^{1/s} \\
&\leq C\|b\|_{\mathrm{Lip}_\beta(w)}\left(\frac{w(Q)}{|Q|}\right)^{1+\beta/n}M_{\beta,s}\left(T^{k,2}(f)\right)(\tilde{x}) \\
&\leq C\|b\|_{\mathrm{Lip}_\beta(w)}w(\tilde{x})^{1+\beta/n}M_{\beta,s}\left(T^{k,2}(f)\right)(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_1 &\leq \sum_{k=1}^m \frac{1}{|Q|}\int_Q\left|T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)\right|dx \\
&\leq C\|b\|_{\mathrm{Lip}_\beta}w(\tilde{x})^{1+\beta/n}\sum_{k=1}^m M_{\beta,s}\left(T^{k,2}(f)\right)(\tilde{x}).
\end{aligned}$$

For I_2, by $d>1$ and $2|x-x_0|^\varepsilon \leq |y-x_0|$ for $x\in Q$ and $y\in(2Q)^c$, we obtain, for $x\in Q$,

$$\begin{aligned}
&\left|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x)-T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x_0)\right| \\
&\leq \int_{(2Q)^c}\left|b(y)-b_{2Q}\right|\left|K(x,y)-K(x_0,y)\right|\left|T^{k,2}(f)(y)\right|dy \\
&\leq C\int_{(2Q)^c}\left|b(y)-b_{2Q}\right|\frac{|x_0-x|^\delta}{|x_0-y|^{n+\delta/\varepsilon}}\left|T^{k,2}(f)(y)\right|dy \\
&\leq Cd^\delta\sum_{j=1}^\infty\int_{2^jd\leq|y-x_0|<2^{j+1}d}\left(2^jd\right)^{-n-\delta/\varepsilon}\left|b(y)-b_{2^{j+1}Q}\right|\left|T^{k,2}(f)(y)\right|dy \\
&\quad + Cd^\delta\sum_{j=1}^\infty\left(2^jd\right)^{-n-\delta/\varepsilon}|b_{2^{j+1}Q}-b_{2Q}|\int_{2^jd\leq|y-x_0|<2^{j+1}d}\left|T^{k,2}(f)(y)\right|dy \\
&\leq C\|b\|_{\mathrm{Lip}_\beta(w)}d^{\delta-\delta/\varepsilon} \\
&\quad \times\sum_{j=1}^\infty 2^{-j\delta/\varepsilon}\left(\frac{w(2^{j+1}Q)}{|2^{j+1}Q|}\right)^{1+\beta/n}\left(\frac{1}{|2^{j+1}Q|^{1-s\beta/n}}\int_{2^{j+1}Q}\left|T^{k,2}(f)(y)\right|^s dy\right)^{1/s} \\
&\quad + C\|b\|_{\mathrm{Lip}_\beta(w)}d^{\delta-\delta/\varepsilon}
\end{aligned}$$

$$\times \sum_{j=1}^{\infty} j2^{-j\delta/\varepsilon} w(\tilde{x}) \left(\frac{w(2^{j+1}Q)}{|2^{j+1}Q|}\right)^{\beta/n} \left(\frac{1}{|2^{j+1}Q|^{1-s\beta/n}} \int_{2^{j+1}Q} \left|T^{k,2}(f)(y)\right|^s dy\right)^{1/s}$$
$$\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,s}\left(T^{k,2}(f)\right)(\tilde{x}),$$

thus

$$I_2 \le \frac{1}{|Q|}\int_Q \sum_{k=1}^m \left|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x) - T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x_0)\right| dx$$
$$\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^m M_{\beta,s}\left(T^{k,2}(f)\right)(\tilde{x}).$$

Case 2. $d \le 1$. Set $\tilde{Q} = Q(x_0, d^\varepsilon)$ and write

$$T_b(f)(x) = T_{b-b_Q}(f)(x) = T_{(b-b_Q)\chi_{2\tilde{Q}}}(f)(x) + T_{(b-b_Q)\chi_{(2\tilde{Q})^c}}(f)(x) = f_1(x) + f_2(x).$$

Then

$$\frac{1}{|Q|}\int_Q \left|T_b(f)(x) - f_2(x_0)\right| dx \le \frac{1}{|Q|}\int_Q \left|f_1(x)\right| dx + \frac{1}{|Q|}\int_Q \left|f_2(x) - f_2(x_0)\right| dx = I_3 + I_4.$$

For I_3, since $((1-\varepsilon)n + 2\beta)/2\beta \le s < \infty$, there exists q such that $r < s$, $0 < r/q \le \varepsilon$, and T is bounded from $L^r(R^n)$ into $L^q(R^n)$. By using the same argument as in the proof of I_1, we get

$$\frac{1}{|Q|}\int_Q \left|T^{k,1}M_{(b-b_Q)\chi_{2\tilde{Q}}}T^{k,2}(f)(x)\right| dx$$
$$\le \left(\frac{1}{|Q|}\int_{R^n} \left|T^{k,1}M_{(b-b_Q)\chi_{2\tilde{Q}}}T^{k,2}(f)(x)\right|^q dx\right)^{1/q}$$
$$\le C|Q|^{-1/q}\left(\int_{R^n} \left|\left(b(x) - b_{2Q}\right)f_1(x)\right|^r dx\right)^{1/r}$$
$$\le C|Q|^{-1/q}\left(\int_{2\tilde{Q}} \left(\left|b(x) - b_{2\tilde{Q}}\right|^r + |b_{2\tilde{Q}} - b_{2Q}|^r\right)\left|T^{k,2}(f)(x)\right|^r dx\right)^{1/r}$$
$$\le C\|b\|_{\mathrm{Lip}_\beta(w)}|Q|^{-1/q}\left(w(\tilde{Q})^{1+\beta/n}|\tilde{Q}|^{-1} + w(\tilde{x})w(\tilde{Q})^{\beta/n}\right)\left(\int_{2\tilde{Q}} \left|T^{k,2}(f)(x)\right|^r dx\right)^{1/r}$$
$$\le C\|b\|_{\mathrm{Lip}_\beta(w)} d^{n(\varepsilon/r-1/q)} w(\tilde{x})\left(\frac{w(\tilde{Q})}{|\tilde{Q}|}\right)^{\beta/n}\left(\frac{1}{|\tilde{Q}|^{1-s\beta/n}}\int_{\tilde{Q}} \left|T^{k,2}(f)(x)\right|^s dx\right)^{1/s}$$
$$\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,s}\left(T^{k,2}(f)\right)(\tilde{x}),$$

thus

$$I_3 \le \sum_{k=1}^m \frac{1}{|Q|}\int_Q \left|T^{k,1}M_{(b-b_Q)\chi_{2\tilde{Q}}}T^{k,2}(f)(x)\right| dx$$
$$\le C\|b\|_{\mathrm{Lip}_\beta} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^m M_{\beta,s}\left(T^{k,2}(f)\right)(\tilde{x}).$$

For I_4, by using the same argument as in the proof of I_2, we get, for $x \in Q$,

$$\begin{aligned}
&\left|T^{k,1}M_{(b-b_Q)\chi_{(2\tilde{Q})^c}}T^{k,2}(f)(x) - T^{k,1}M_{(b-b_Q)\chi_{(2\tilde{Q})^c}}T^{k,2}(f)(x_0)\right| \\
&\quad \le \int_{(2\tilde{Q})^c} \left|b(y) - b_{2Q}\right|\left|K(x,y) - K(x_0,y)\right|\left|T^{k,2}(f)(y)\right|\,dy \\
&\quad \le C\int_{(2\tilde{Q})^c} \left|b(y) - b_{2Q}\right|\left|f(y)\right|\frac{|x_0 - x|^\delta}{|x_0 - y|^{n+\delta/\varepsilon}}\,dy \\
&\quad \le Cd^\delta \sum_{j=1}^\infty \left(2^j d^\varepsilon\right)^{-n-\delta/\varepsilon}\int_{2^{j+1}\tilde{Q}} \left|b(y) - b_{2^{j+1}\tilde{Q}}\right|\left|T^{k,2}(f)(y)\right|\,dy \\
&\qquad + Cd^\delta \sum_{j=1}^\infty \left(2^j d^\varepsilon\right)^{-n-\delta/\varepsilon}|b_{2^{j+1}\tilde{Q}} - b_{2\tilde{Q}}|\int_{2^{j+1}\tilde{Q}} \left|T^{k,2}(f)(y)\right|\,dy \\
&\qquad + Cd^\delta \sum_{j=1}^\infty \left(2^j d^\varepsilon\right)^{-n-\delta/\varepsilon}|b_{2\tilde{Q}} - b_{2Q}|\int_{2^{j+1}\tilde{Q}} \left|T^{k,2}(f)(y)\right|dy \\
&\quad \le C\|b\|_{\mathrm{Lip}_\beta(w)} \sum_{j=1}^\infty 2^{-j\delta/\varepsilon}\left(\frac{w(2^{j+1}\tilde{Q})}{|2^{j+1}\tilde{Q}|}\right)^{1+\beta/n}\left(\frac{1}{|2^{j+1}\tilde{Q}|^{1-s\beta/n}}\int_{2^{j+1}Q}\left|T^{k,2}(f)(y)\right|^s dy\right)^{1/s} \\
&\qquad + C\|b\|_{\mathrm{Lip}_\beta(w)} \\
&\qquad \times \sum_{j=1}^\infty j2^{-j\delta/\varepsilon}w(\tilde{x})\left(\frac{w(2^{j+1}\tilde{Q})}{|2^{j+1}\tilde{Q}|}\right)^{\beta/n}\left(\frac{1}{|2^{j+1}\tilde{Q}|^{1-s\beta/n}}\int_{2^{j+1}\tilde{Q}}\left|T^{k,2}(f)(y)\right|^s dy\right)^{1/s} \\
&\qquad + C\|b\|_{\mathrm{Lip}_\beta(w)} \sum_{j=1}^\infty 2^{-j\delta/\varepsilon}w(\tilde{x})\left(\frac{w(\tilde{Q})}{|\tilde{Q}|}\right)^{\beta/n}\left(\frac{1}{|2^{j+1}\tilde{Q}|^{1-s\beta/n}}\int_{2^{j+1}\tilde{Q}}\left|T^{k,2}(f)(y)\right|^s dy\right)^{1/s} \\
&\quad \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,s}\left(T^{k,2}(f)\right)(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_4 &\le \frac{1}{|Q|}\int_Q \sum_{k=1}^m \left|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x) - T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x_0)\right|\,dx \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n}\sum_{k=1}^m M_{\beta,s}\left(T^{k,2}(f)\right)(\tilde{x}).
\end{aligned}$$

These complete the proof of Theorem 1. □

Proof of Theorem 2 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 that the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q \left|T_b(f)(x) - C_0\right|\,dx \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n}\sum_{k=1}^m M_s\left(T^{k,2}(f)\right)(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are T ($k = 1,\dots,m$). Fix a cube $Q = Q(x_0,d)$ and $\tilde{x} \in Q$. We have the following two cases.

Case 1. $d > 1$. Similar to the proof of Theorem 1, we have

$$T_b(f)(x) = T_{b-b_Q}(f)(x) = T_{(b-b_Q)\chi_{2Q}}(f)(x) + T_{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x)$$

and

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q \left|T_b(f)(x) - f_2(x_0)\right| dx \\
&\quad\le \frac{1}{|Q|^{1+\beta/n}}\int_Q \left|f_1(x)\right| dx + \frac{1}{|Q|^{1+\beta/n}}\int_Q \left|f_2(x) - f_2(x_0)\right| dx = II_1 + II_2.
\end{aligned}$$

By using the same argument as in the proof of Theorem 1, we get

$$\begin{aligned}
II_1 &\le \sum_{k=1}^m \frac{C}{|Q|^{\beta/n}} \sup_{x\in 2Q}\left|b(x) - b_{2Q}\right| |Q|^{-1/s}\left(\int_{2Q}\left|T^{k,2}(f)(x)\right|^s dx\right)^{1/s} \\
&\le \sum_{k=1}^m \|b\|_{\mathrm{Lip}_\beta(w)}\left(\frac{w(Q)}{|Q|}\right)^{1+\beta/n}\left(\frac{1}{|2Q|}\int_{2Q}\left|T^{k,2}(f)(x)\right|^s dx\right)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n}\sum_{k=1}^m M_s\left(T^{k,2}(f)\right)(\tilde{x}), \\
II_2 &\le \frac{1}{|Q|^{1+\beta/n}}\int_Q\int_{(2Q)^c}\left|b(y) - b_{2Q}\right|\left|K(x,y) - K(x_0,y)\right|\left|T^{k,2}(f)(y)\right| dy\,dx \\
&\le \frac{C}{|Q|^{1+\beta/n}}\int_Q\int_{(2Q)^c}\left|b(y) - b_{2Q}\right|\frac{|x_0 - x|^\delta}{|x_0 - y|^{n+\delta/\varepsilon}}\left|T^{k,2}(f)(y)\right| dy\,dx \\
&\le Cd^\delta\sum_{j=1}^\infty \left(2^j d\right)^{-n-\delta/\varepsilon}\int_{2^{j+1}Q}\left|b(y) - b_{2^{j+1}Q}\right|\left|T^{k,2}(f)(y)\right| dy \\
&\quad + Cd^\delta\sum_{j=1}^\infty \left(2^j d\right)^{-n-\delta/\varepsilon}|b_{2^{j+1}Q} - b_{2Q}|\int_{2^{j+1}Q}\left|T^{k,2}(f)(y)\right| dy \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} d^{\delta-\delta/\varepsilon} \\
&\quad\times\sum_{j=1}^\infty 2^{j(\beta-\delta/\varepsilon)}\left(\frac{w(2^{j+1}Q)}{|2^{j+1}Q|}\right)^{1+\beta/n}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\left|T^{k,2}(f)(y)\right|^s dy\right)^{1/s} \\
&\quad + C\|b\|_{\mathrm{Lip}_\beta(w)} d^{\delta-\delta/\varepsilon} \\
&\quad\times\sum_{j=1}^\infty j2^{j(\beta-\delta/\varepsilon)} w(\tilde{x})\left(\frac{w(2^{j+1}Q)}{|2^{j+1}Q|}\right)^{\beta/n}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\left|T^{k,2}(f)(y)\right|^s dy\right)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n}\sum_{k=1}^m M_s\left(T^{k,2}(f)\right)(\tilde{x}).
\end{aligned}$$

Case 2. $d \le 1$. Set $\tilde{Q} = Q(x_0, d^\rho)$, where $\rho = (\delta - \beta)/(\delta/\varepsilon - \beta) < \varepsilon$, and write

$$T_b(f)(x) = T_{b-b_Q}(f)(x) = T_{(b-b_Q)\chi_{2\tilde{Q}}}(f)(x) + T_{(b-b_Q)\chi_{(2\tilde{Q})^c}}(f)(x) = f_1(x) + f_2(x)$$

and

$$\begin{aligned}&\frac{1}{|Q|^{1+\beta/n}}\int_Q\big|T_b(f)(x)-f_2(x_0)\big|\,dx\\&\quad\le\frac{1}{|Q|^{1+\beta/n}}\int_Q\big|f_1(x)\big|\,dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q\big|f_2(x)-f_2(x_0)\big|\,dx=II_3+II_4.\end{aligned}$$

By using the same argument as in the proof of Theorem 1, for $((1-\varepsilon)n+2\beta)/2\beta\le s<\infty$, there exists q such that $r<s$, $0<r/q\le\varepsilon$, and T is bounded from $L^r(R^n)$ into $L^q(R^n)$, and we get

$$\begin{aligned}II_3&\le\sum_{k=1}^m\frac{1}{|Q|}\int_Q\big|T^{k,1}M_{(b-b_Q)\chi_{2\tilde Q}}T^{k,2}(f)(x)\big|\,dx\\&\le\sum_{k=1}^m\frac{1}{|Q|^{\beta/n}}\biggl(\frac{1}{|Q|}\int_{R^n}\big|T^{k,1}M_{(b-b_Q)\chi_{2\tilde Q}}T^{k,2}(f)(x)\big|^q\,dx\biggr)^{1/q}\\&\le C\sum_{k=1}^m d^{-\beta-n/q}\biggl(\int_{R^n}\big|\big(b(x)-b_{2Q}\big)f_1(x)\big|^r\,dx\biggr)^{1/r}\\&\le C\sum_{k=1}^m d^{-\beta-n/q}\biggl(\int_{2\tilde Q}\big(\big|b(x)-b_{2\tilde Q}\big|^r+|b_{2\tilde Q}-b_{2Q}|^r\big)\big|T^{k,2}(f)(x)\big|^r\,dx\biggr)^{1/r}\\&\le C\sum_{k=1}^m\|b\|_{\mathrm{Lip}_\beta(w)}d^{-\beta-n/q}\big(w(\tilde Q)^{1+\beta/n}|\tilde Q|^{-1}+w(\tilde x)w(\tilde Q)^{\beta/n}\big)\biggl(\int_{2\tilde Q}\big|T^{k,2}(f)(x)\big|^r\,dx\biggr)^{1/r}\\&\le C\sum_{k=1}^m\|b\|_{\mathrm{Lip}_\beta(w)}d^{\rho(n/s+\beta)-\beta-n/q}w(\tilde x)\biggl(\frac{w(\tilde Q)}{|\tilde Q|}\biggr)^{\beta/n}\biggl(\frac{1}{|\tilde Q|}\int_{\tilde Q}\big|T^{k,2}(f)(x)\big|^s\,dx\biggr)^{1/s}\\&\le C\|b\|_{\mathrm{Lip}_\beta(w)}w(\tilde x)^{1+\beta/n}\sum_{k=1}^m M_s\big(T^{k,2}(f)\big)(\tilde x),\end{aligned}$$

$$\begin{aligned}II_4&\le\sum_{k=1}^m\frac{1}{|Q|^{1+\beta/n}}\int_Q\int_{(2\tilde Q)^c}\big|b(y)-b_{2Q}\big|\big|K(x,y)-K(x_0,y)\big|\big|T^{k,2}(f)(y)\big|\,dy\,dx\\&\le\sum_{k=1}^m\frac{C}{|Q|^{1+\beta/n}}\int_Q\int_{(2\tilde Q)^c}\big|b(y)-b_{2Q}\big|\frac{|x_0-x|^\delta}{|x_0-y|^{n+\delta/\varepsilon}}\big|T^{k,2}(f)(y)\big|\,dy\,dx\\&\le C\sum_{k=1}^m d^{\delta-\beta}\sum_{j=1}^\infty\big(2^jd^\rho\big)^{-n-\delta/\varepsilon}\int_{2^{j+1}\tilde Q}\big|b(y)-b_{2^{j+1}\tilde Q}\big|\big|T^{k,2}(f)(y)\big|\,dy\\&\quad+C\sum_{k=1}^m d^{\delta-\beta}\sum_{j=1}^\infty\big(2^jd^\rho\big)^{-n-\delta/\varepsilon}|b_{2^{j+1}\tilde Q}-b_{2\tilde Q}|\int_{2^{j+1}\tilde Q}\big|T^{k,2}(f)(y)\big|\,dy\\&\quad+C\sum_{k=1}^m d^{\delta-\delta}\sum_{j=1}^\infty\big(2^jd^\rho\big)^{-n-\delta/\varepsilon}|b_{2\tilde Q}-b_{2Q}|\int_{2^{j+1}\tilde Q}\big|T^{k,2}(f)(y)\big|\,dy\\&\le C\sum_{k=1}^m\|b\|_{\mathrm{Lip}_\beta(w)}\sum_{j=1}^\infty 2^{j(\beta-\delta/\varepsilon)}\biggl(\frac{w(2^{j+1}\tilde Q)}{|2^{j+1}\tilde Q|}\biggr)^{1+\beta/n}\biggl(\frac{1}{|2^{j+1}\tilde Q|}\int_{2^{s+1}Q}\big|T^{k,2}(f)(y)\big|^s\,dy\biggr)^{1/s}\\&\quad+C\sum_{k=1}^m\|b\|_{\mathrm{Lip}_\beta(w)}\end{aligned}$$

$$\times \sum_{j=1}^{\infty} j2^{j(\beta-\delta/\varepsilon)} w(\tilde{x}) \left(\frac{w(2^{j+1}\tilde{Q})}{|2^{j+1}\tilde{Q}|} \right)^{\beta/n} \left(\frac{1}{|2^{j+1}\tilde{Q}|} \int_{2^{j+1}\tilde{Q}} \left| T^{k,2}(f)(y) \right|^s dy \right)^{1/s}$$
$$+ C \sum_{k=1}^{m} \|b\|_{\mathrm{Lip}_\beta(w)} \sum_{j=1}^{\infty} j2^{j(\beta-\delta/\varepsilon)} w(\tilde{x}) \left(\frac{w(\tilde{Q})}{|\tilde{Q}|} \right)^{\beta/n} \left(\frac{1}{|2^{j+1}\tilde{Q}|} \int_{2^{j+1}\tilde{Q}} \left| T^{k,2}(f)(y) \right|^s dy \right)^{1/s}$$
$$\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^{m} M_s\big(T^{k,2}(f)\big)(\tilde{x}).$$

This completes the proof of Theorem 2. □

Proof of Theorem 3 Choose $1 < s < p$ in Theorem 1, notice that $w^{q/p-q(1+\beta/n)} \in A_\infty$ and $w^{1/p} \in A(p,q)$, and we have, by Lemmas 1, 4, and 5,

$$\begin{aligned}
\|T_b(f)\|_{L^q(w^{q/p-q(1+\beta/n)})} &\le \|M(T_b(f))\|_{L^q(w^{q/p-q(1+\beta/n)})} \\
&\le C\|M^\#(T_b(f))\|_{L^q(w^{q/p-q(1+\beta/n)})} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} \sum_{k=1}^{m} \|M_{\beta,s}(T^{k,2}(f)) w^{1+\beta/n}\|_{L^q(w^{q/p-q(1+\beta/n)})} \\
&= C\|b\|_{\mathrm{Lip}_\beta(w)} \sum_{k=1}^{m} \|M_{\beta,s}(T^{k,2}(f))\|_{L^q(w^{q/p})} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} \sum_{k=1}^{m} \|T^{k,2}(f)\|_{L^p(w)} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} \|f\|_{L^p(w)}.
\end{aligned}$$

This completes the proof of Theorem 3. □

Proof of Theorem 4 Choose $1 < s < p$ in Theorem 2. By using Lemma 3, we obtain

$$\begin{aligned}
&\|T_b(f)\|_{\dot{F}_q^{\beta,\infty}(w^{q/p-q(1+\beta/n)})} \\
&\quad\le C\|b\|_{\mathrm{Lip}_\beta(w)} \sum_{k=1}^{m} \|M_s(T^{k,2}(f)) w^{1+\beta/n}\|_{L^q(w^{q/p-q(1+\beta/n)})} \\
&\quad= C\|b\|_{\mathrm{Lip}_\beta(w)} \sum_{k=1}^{m} \|M_s(T^{k,2}(f))\|_{L^q(w^{q/p})} \\
&\quad\le C\|b\|_{\mathrm{Lip}_\beta(w)} \sum_{k=1}^{m} \|T^{k,2}(f)\|_{L^p(w)} \\
&\quad\le C\|b\|_{\mathrm{Lip}_\beta(w)} \|f\|_{L^p(w)}.
\end{aligned}$$

This completes the proof of the theorem. □

Remark A typical example of strongly singular integral operators is a class of multiplier operators whose symbol is given by $\exp(i|\xi|^\varepsilon)/|\xi|^\delta$ for $0 < \varepsilon < 1$ and $\delta > 0$ [18–20, 22].

Competing interests

The author declares that they have no competing interests.

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