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On mean curvature integrals of the outer parallel body of the projection of a convex body

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Abstract
In this paper, we obtain expressions of the mean curvature integrals of two outer parallel bodies, where the outer parallel bodies are in the distance ρ of a projection body in different space ($\mathbb{R}^n$ and $L_{r[O]}$). These mean curvature integrals are the generalizations of Santaló's results. As corollaries, we establish mean values of the mean curvature integrals and Minkowski quermassintegrals of two outer parallel bodies, respectively.

MSC: 52A20; 53C65

Keywords: mean curvature integral; the Minkowski quermassintegral; outer parallel body; convex body

1 Introduction

The mean curvature integral is a basic concept in integral geometry. It connects many geometric invariants, such as area, the Euler-Poincaré characteristic, the degree of the spherical Gauss map, the Gauss-Kronecker curvature and so on. Also it has close relation to the Minkowski quermassintegral of convex body. Meanwhile, the mean curvature integral plays an important role in Chern fundamental kinematic formula. It is well known that kinematic formulas are very important and classical in integral geometry.

Under the assumptions that $\mathbb{R}^n$ is the n-dimensional Euclidean space and $L_{r[O]}$ is an r-dimensional linear subspace through a fixed point O, Santaló [1] investigated the ith mean curvature integral $M_i^{(n)}$ of a flattened convex body K in $\mathbb{R}^n$ and established the expression of $M_i^{(n)}$ in terms of $M_j^{(r)}$, where $M_j^{(r)}$ is the jth mean curvature integral of K in $L_{r[O]}$. On the basis of [1], Chen and Yang [2] investigated $M_i^{(n)}$ of a flattened convex body K in space forms and gave the expression of it in terms of $M_j^{(r)}$, where $M_j^{(r)}$ is the jth mean curvature integral of K in r-dimensional geodesic submanifold, their work extends the result of Santaló in [1]. In [3], Zhou and Jiang investigated $M_i^{(n)}$ of the projection body $K_\rho^{(r)}$ as a flattened convex body of $\mathbb{R}^n$.

In this paper, we investigate the ith mean curvature integral $M_i^{(n)}$ of $\partial(K_r')_\rho^{(n)}$ and $\partial(K_r')_\rho^{(r)}$, naturally, where $(K_r')_\rho^{(n)}$ and $(K_r')_\rho^{(r)}$ are the outer parallel bodies of K_r' in $\mathbb{R}^n$ and $L_{r[O]}$, respectively. We give the expressions of $M_i^{(n)}$ in terms of $M_j^{(r)}$. Besides, we obtain the mean value of $M_i^{(n)}$. Our main results are the following theorems.

Theorem 1 *Let K be a convex body with C^2 boundary ∂K in $\mathbb{R}^n$. Let $(K_r')_\rho^{(n)}$ be the outer parallel body of K_r' in the distance ρ in $\mathbb{R}^n$, where K_r' is the orthogonal projection of K on the r-dimensional linear subspace $L_{r[O]} \subseteq \mathbb{R}^n$. Denote by $M_i^{(n)}(\partial(K_r')_\rho^{(n)})$ ($i = 0, 1, \dots, n-1$) the mean curvature integrals of $(K_r')_\rho^{(n)}$ and by $M_i^{(r)}(\partial K_r')$ ($i = 0, 1, \dots, r-1$) the mean curvature integrals of K_r' in $L_{r[O]}$. Then*:

(1) *If $i \geq n-r$, then*

$$M_i^{(n)}\big(\partial\big(K_r'\big)_\rho^{(n)}\big) = \sum_{q=i}^{n-1} \binom{n-i-1}{q-i} \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} \frac{O_q}{O_{q-n+r}} M_{q-n+r}^{(r)}\big(\partial K_r'\big)\rho^{q-i}. \tag{1.1}$$

(2) *If $i \leq n-r-1$, then*

$$\begin{aligned} M_i^{(n)}\big(\partial\big(K_r'\big)_\rho^{(n)}\big) &= \binom{n-i-1}{n-r-i-1}\binom{n-1}{n-r-1}^{-1} O_{n-r-1} V_r\big(K_r'\big)\rho^{n-r-i-1} \\ &\quad + \sum_{q=n-r}^{n-1} \binom{n-i-1}{q-i} \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} \frac{O_q}{O_{q-n+r}} M_{q-n+r}^{(r)}\big(\partial K_r'\big)\rho^{q-i}, \end{aligned} \tag{1.2}$$

where $V_r(K_r')$ denotes the r-dimensional volume of K_r'.

Theorem 2 *Let K be a convex body with C^2 boundary ∂K in $\mathbb{R}^n$. Let $(K_r')_\rho^{(r)}$ be the outer parallel body of K_r' in the distance ρ in $L_{r[O]}$, where K_r' is the orthogonal projection of K on the r-dimensional linear subspace $L_{r[O]} \subseteq \mathbb{R}^n$. Denote by $M_i^{(n)}(\partial(K_r')_\rho^{(r)})$ ($i = 0, 1, \dots, n-1$) the mean curvature integrals of $(K_r')_\rho^{(r)}$ as a flattened convex body of $\mathbb{R}^n$ and by $M_i^{(r)}(\partial K_r')$ ($i = 0, 1, \dots, r-1$) the mean curvature integrals of K_r' as a convex body of $L_{r[O]}$. Then*:

(1) *If $i \geq n-r$, then*

$$M_i^{(n)}\big(\partial\big(K_r'\big)_\rho^{(r)}\big) = \frac{\binom{r-1}{i-n+r}}{\binom{n-1}{i}} \frac{O_i}{O_{i-n+r}} \sum_{q=0}^{2n-i-r-1} \binom{2n-i-r-1}{q} M_{i-n+r+q}^{(r)}\big(\partial\big(K_r'\big)\big)\rho^q. \tag{1.3}$$

(2) *If $i = n-r-1$, then*

$$\begin{aligned} &M_{n-r-1}^{(n)}\big(\partial\big(K_r'\big)_\rho^{(r)}\big) \\ &\quad = \binom{n-1}{n-r-1}^{-1} O_{n-r-1}\left[V_r\big(K_r'\big) + \sum_{q=0}^{r-1} \frac{\rho^{q+1}}{q+1}\binom{r-1}{q} M_q^{(r)}\big(\partial\big(K_r'\big)\big)\right]. \end{aligned} \tag{1.4}$$

(3) *If $i < n-r$, then*

$$M_i^{(n)}\big(\partial\big(K_r'\big)_\rho^{(r)}\big) = 0, \tag{1.5}$$

where $V_r(K_r')$ denotes the r-dimensional volume of K_r'.

Especially, letting $\rho \to 0$, Theorem 2 reduces to Lemma 1 (in Section 2) proved by Santaló in 1957 (see [1, 4, 5]). In fact, the main result of [3] and Theorem 2 are similar in nature, but the coefficient in [3] is a little inappropriate. Note that the results of [1, 4, 5] play an important role in integral geometry and differential geometry and are widely used (see [3, 5–7]).

2 Preliminaries

A set in the Euclidean space $\mathbb{R}^n$ is called convex if and only if it contains, with each pair of its points, the entire line segment joining them. A convex set with nonempty interior is called a convex body. The boundary ∂K of a convex body K is a convex hypersurface.

Let K be a convex body in $\mathbb{R}^n$, then ∂K is an $(n-1)$-dimensional convex hypersurface. Assuming that ∂K is of class C^2 and P is a point of ∂K, we choose $e_1, \dots, e_{n-1}$ to be the principal curvature directions at the point P. Further, we suppose that $k_1, \dots, k_{n-1}$ are the principal curvatures at the point P, which correspond to the principal curvature directions.

Consider the Gauss map $G : p \to N(p)$, whose differential

$$dG_P : x'(t) \to N'(t) \quad \big(x(0) = P\big) \tag{2.1}$$

satisfies Rodrigues' equations,

$$dG_p(e_i) = -k_i e_i, \quad i = 1, \dots, n-1. \tag{2.2}$$

Then we have the mean curvature

$$H = \frac{1}{n-1}(k_1 + \cdots + k_{n-1}) = -\frac{1}{n-1}\operatorname{trace}(dG_P), \tag{2.3}$$

along with the Gauss-Kronecker curvature,

$$K = k_1 \cdots k_{n-1} = (-1)^{n-1}\det(dG_P). \tag{2.4}$$

The ith order mean curvature is the ith order elementary symmetric function of the principal curvatures. We denote by H_i the ith order mean curvature normalized such that

$$\prod_{i=1}^{n-1}(1 + tk_i) = \sum_{i=0}^{n-1} H_i t^i. \tag{2.5}$$

Thus, $H_1 = H$ is the mean curvature and H_{n-1} is the Gauss-Kronecker curvature.

The ith order mean curvature integral $M_i^{(n)}$ of ∂K at P is defined by

$$M_i^{(n)}(\partial K) = \int_{\partial K} H_i \, d\sigma = \binom{n-1}{i}^{-1} \int_{\partial K} \{k_{j_1}, \dots, k_{j_i}\} \, d\sigma, \quad i = 1, \dots, n-1, \tag{2.6}$$

where $\{k_{j_1}, \dots, k_{j_i}\}$ denotes the ith elementary symmetric function of the principal curvatures and $d\sigma$ is the area element of ∂K. As a particular case, let $M_0^{(n)}(\partial K) = F$ be the area of ∂K, for completeness. Moreover, we have $M_{n-1}^{(n)} = O_{n-1}$, where O_{n-1} denotes the area of the $(n-1)$-dimensional unit sphere and its value is given by the formula

$$O_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{2.7}$$

For instance, if $n = 2$, and K is a plane convex figure in $\mathbb{R}^2$, then $M_0^{(2)} = F(K)$ and $M_1^{(2)} = 2\pi$. If $n = 3$, and K is a convex body in $\mathbb{R}^3$, then $M_0^{(3)} = F(K)$, $M_2^{(3)} = 4\pi$ and $M_1^{(3)}$ is the integral of mean curvature of ∂K. See [5, 7] for a detailed description.

On the other hand, we consider all the $(n-r)$-dimensional linear subspaces $L_{n-r[O]}$ through a fixed point O. Let K'_{n-r} be the orthogonal projection of K onto $L_{n-r[O]}$, denote by $V(K'_{n-r})$ the volume of K'_{n-r} and by $dL_{n-r[O]}$ the densities of the Grassmann manifold $G_{n-r,r}$. Then the mean value of the projected volumes $E(V(K'_{n-r}))$ is

$$E\bigl(V\bigl(K'_{n-r}\bigr)\bigr) = \frac{I_r(K)}{m(G_{n-r,r})} = \frac{O_{r-1}\cdots O_1 O_0}{O_{n-1}\cdots O_{n-r}} I_r(K), \quad r = 1, 2, \ldots, n-1, \tag{2.8}$$

where Grassmann manifold $G_{n-r,r}$ is the set of unoriented r-planes of $\mathbb{R}^n$ through a fixed point, $m(G_{n-r,r})$ is the volume of $G_{n-r,r}$ given by

$$m(G_{n-r,r}) = m(G_{r,n-r}) = \int_{G_{r,n-r}} dL_{r[O]} = \frac{O_{n-1}\ldots O_{n-r}}{O_{r-1}\cdots O_1 O_0} \tag{2.9}$$

and

$$I_r(K) = \int_{G_{n-r,r}} V\bigl(K'_{n-r}\bigr)\, dL_{n-r[O]} = \int_{G_{r,n-r}} V\bigl(K'_{n-r}\bigr)\, dL_{r[O]}. \tag{2.10}$$

For completeness, we define

$$I_0(K) = V(K) \quad \text{(the } n\text{-dimensional volume of } K). \tag{2.11}$$

The Minkowski quermassintegral is introduced by Minkowski and is defined by

$$\begin{aligned} W_r^{(n)}(K) &= \frac{(n-i)O_{n-1}}{nO_{n-r-1}} E\bigl(V\bigl(K'_{n-r}\bigr)\bigr) \\ &= \frac{(n-r)O_{r-1}\cdots O_0}{nO_{n-2}\cdots O_{n-r-1}} I_r(K), \quad r = 1, 2, \ldots, n-1. \end{aligned} \tag{2.12}$$

In particular, we put $W_0^{(n)}(K) = I_0(K) = V(K)$, $W_n^{(n)}(K) = \frac{O_{n-1}}{n}$.

The outer parallel body K_ρ in the distance ρ of a convex figure K is the union of all solid spheres of radius ρ the centers of which are points of K. Then we have the following Steiner formula for the outer parallel body K_ρ ($\rho \geq 0$):

$$V(K_\rho) = \sum_{i=0}^{n} \binom{n}{i} W_i^{(n)}(K)\rho^i. \tag{2.13}$$

As a consequence of the Steiner formula we have

$$W_i^{(n)}(K_\rho) = \sum_{j=0}^{n-i} \binom{n-i}{j} W_{i+j}^{(n)}(K)\rho^j, \quad i = 0, 1, \ldots, n. \tag{2.14}$$

Moreover, we have the relation between the mean curvature integrals of ∂K and the Minkowski quermassintegrals of K (see [4, 5, 7]), that is, the Cauchy formula

$$M_i^{(n)}(\partial K) = nW_{i+1}^{(n)}(K), \quad i = 0, 1, \ldots, n-1. \tag{2.15}$$

Note that the Minkowski quermassintegrals $W_i^{(n)}$ are well defined for any convex figure, whereas $M_j^{(n)}$ makes sense only if ∂K is of class C^2.

Let K be a convex body in the r-dimensional linear subspace $L_{r[O]} \subseteq \mathbb{R}^n$, and $M_q^{(r)}(\partial K)$ the mean curvature integrals of K as a convex surface of $L_{r[O]}$. Consider K as a flattened convex body of $\mathbb{R}^n$, Santaló obtained the following lemma with respect to the mean curvature integral in 1957 (see [1, 4, 5]).

Lemma 1 *Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $L_{r[O]}$ be the r-dimensional linear subspace through a fixed point O in $\mathbb{R}^n$. Let K be a convex body of the dimension r in $L_{r[O]}$. Then K can be considered both as a convex body in $L_{r[O]}$ and as a flattened convex body in $\mathbb{R}^n$. Then the qth mean curvature integral $M_q^{(n)}(\partial K)$ satisfies the conditions*:

(1) *If $q \geq n-r$, then*

$$M_q^{(n)}(\partial K) = \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} \frac{O_q}{O_{q-n+r}} M_{q-n+r}^{(r)}(\partial K). \tag{2.16}$$

(2) *If $q = n-r-1$, then*

$$M_{n-r-1}^{(n)}(\partial K) = \binom{n-1}{n-r-1}^{-1} O_{n-r-1} V_r(K), \tag{2.17}$$

where $V_r(K)$ denotes the r-dimensional volume of K.

(3) *If $q < n-r-1$, then*

$$M_q^{(n)}(\partial K) = 0. \tag{2.18}$$

Later, Jiang and Zeng [8] investigated the integral of $M_i^{(n)}$ of $\partial(K_r')_\rho^{(r)}$ on the Grassmann manifold $G_{r,n-r}$ and obtained the mean value of these mean curvature integrals.

Lemma 2 *Let K be a convex body with C^2 boundary ∂K in $\mathbb{R}^n$ and let K_r' be the orthogonal projection of K on the r-dimensional subspace $L_{r[O]} \subseteq \mathbb{R}^n$. Denote by $M_i^{(r)}(\partial K_r')$ $(i = 0,1,\ldots,r-1)$ the mean curvature integrals of K_r' as a convex body of $L_{r[O]}$ and by $M_i^{(n)}(\partial K)$ $(i = 0,1,\ldots,n-1)$ the mean curvature integrals of K in $\mathbb{R}^n$. Then*

$$\int_{G_{r,n-r}} M_i^{(r)}\bigl(\partial K_r'\bigr)\, dL_{r[O]} = \frac{O_{n-2}\cdots O_{n-r}}{O_{r-2}\cdots O_0} M_{n-r+i}^{(n)}(\partial K). \tag{2.19}$$

3 Proofs of the main theorems and some corollaries

Proof of Theorem 1 We apply the Cauchy formula (2.15) to the convex body $(K_r')_\rho^{(n)}$, then

$$M_i^{(n)}\bigl(\partial\bigl(K_r'\bigr)_\rho^{(n)}\bigr) = nW_{i+1}^{(n)}\bigl(\bigl(K_r'\bigr)_\rho^{(n)}\bigr), \quad i = 0,1,\ldots,n-1. \tag{3.1}$$

Applying (2.14) to the convex body K_r', we have

$$W_i^{(n)}\bigl(\bigl(K_r'\bigr)_\rho^{(n)}\bigr) = \sum_{j=0}^{n-i} \binom{n-i}{j} W_{i+j}^{(n)}\bigl(K_r'\bigr)\rho^j, \quad i = 0,1,\ldots,n. \tag{3.2}$$

Then combining (3.1) and (3.2) gives

$$\begin{aligned} M_i^{(n)}\left(\partial\left(K_r'\right)_\rho^{(n)}\right) &= \sum_{j=0}^{n-i-1}\binom{n-i-1}{j} M_{i+j}^{(n)}\left(\partial K_r'\right)\rho^j \\ &= \sum_{q=i}^{n-1}\binom{n-i-1}{q-i} M_q^{(n)}\left(\partial K_r'\right)\rho^{q-i}, \end{aligned} \tag{3.3}$$

where in the first step we use the Cauchy formula

$$M_i(\partial K) = nW_{i+1}(K), \quad i = 0,1,\dots,n,$$

for flattened convex bodies.

Now, we are ready to compute the mean curvature integral of $\partial(K_r')_\rho^{(n)}$ from the below three cases.

(1) If $i \geq n-r$, and obviously $q \geq n-r$ in (3.3). Then by Santaló's result (2.16)

$$M_q^{(n)}\left(\partial K_r'\right) = \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_q}{O_{q-n+r}} M_{q-n+r}^{(r)}\left(\partial K_r'\right), \quad \text{for all } q \geq n-r. \tag{3.4}$$

Inserting (3.4) to (3.3), we obtain

$$M_i^{(n)}\left(\partial\left(K_r'\right)_\rho^{(n)}\right) = \sum_{q=i}^{n-1}\binom{n-i-1}{q-i}\frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_q}{O_{q-n+r}} M_{q-n+r}^{(r)}\left(\partial K_r'\right)\rho^{q-i}. \tag{3.5}$$

(2) If $i = n-r-1$, then (3.3) can be rewritten as

$$\begin{aligned} M_{n-r-1}^{(n)}\left(\partial\left(K_r'\right)_\rho^{(n)}\right) &= \sum_{q=n-r-1}^{n-1}\binom{r}{q-n+r+1} M_q^{(n)}\left(\partial K_r'\right)\rho^{q-n+r+1} \\ &= M_{n-r-1}^{(n)}\left(\partial K_r'\right) + \sum_{q=n-r}^{n-1}\binom{r}{q-n+r+1} M_q^{(n)}\left(\partial K_r'\right)\rho^{q-n+r+1} \\ &= \binom{n-1}{n-r-1}^{-1} O_{n-r-1} V_r\left(K_r'\right) \\ &\quad + \sum_{q=n-r}^{n-1}\binom{r}{q-n+r+1}\frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_q}{O_{q-n+r}} M_{q-n+r}^{(r)}\left(\partial K_r'\right)\rho^{q-n+r+1}, \end{aligned} \tag{3.6}$$

where the first equation and the last equation follow from (2.17) and (2.16), respectively.

(3) If $i < n-r-1$, from (2.16) and (2.17), followed by (2.18) and (3.3), then we have

$$\begin{aligned} M_i^{(n)}\left(\partial\left(K_r'\right)_\rho^{(n)}\right) &= \sum_{q=i}^{n-1}\binom{n-i-1}{q-i} M_q^{(n)}\left(\partial K_r'\right)\rho^{q-i} \\ &= \sum_{q=i}^{n-r-2}\binom{n-i-1}{q-i} M_q^{(n)}\left(\partial K_r'\right)\rho^{q-i} + \binom{n-i-1}{n-r-i-1} M_{n-r-1}^{(n)}\left(\partial K_r'\right)\rho^{n-r-i-1} \end{aligned}$$

$$+\sum_{q=n-r}^{n-1}\binom{n-i-1}{q-i}M_q^{(n)}(\partial K_r')\rho^{q-i}$$
$$=\binom{n-i-1}{n-r-i-1}M_{n-r-1}^{(n)}(\partial K_r')\rho^{n-r-i-1}$$
$$+\sum_{q=n-r}^{n-1}\binom{n-i-1}{q-i}M_q^{(n)}(\partial K_r')\rho^{q-i}$$
$$=\binom{n-i-1}{n-r-i-1}\binom{n-1}{n-r-1}^{-1}O_{n-r-1}V_r(K_r')\rho^{n-r-i-1}$$
$$+\sum_{q=n-r}^{n-1}\binom{n-i-1}{q-i}\frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_q}{O_{q-n+r}}M_{q-n+r}^{(r)}(\partial K_r')\rho^{q-i}. \tag{3.7}$$

If we take $i=n-r-1$ in (3.7), then

$$M_{n-r-1}^{(n)}\left(\partial(K_r')_\rho^{(n)}\right)=\binom{n-1}{n-r-1}^{-1}O_{n-r-1}V_r(K_r')$$
$$+\sum_{q=n-r}^{n-1}\binom{r}{q-n+r+1}\frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_q}{O_{q-n+r}}M_{q-n+r}^{(r)}(\partial K_r')\rho^{q-n+r+1}, \tag{3.8}$$

which is in fact (3.6). So combining (3.6) and (3.7) gives (1.2) and completes the proof of Theorem 1. □

Proof of Theorem 2 (1) If $i\ge n-r$, and applying (2.16) and (3.3), then

$$M_i^{(n)}\left(\partial(K_r')_\rho^{(r)}\right)=\frac{\binom{r-1}{i-n+r}}{\binom{n-1}{i}}\frac{O_i}{O_{i-n+r}}M_{i-n+r}^{(r)}\left(\partial(K_r')_\rho^{(r)}\right)$$
$$=\frac{\binom{r-1}{i-n+r}}{\binom{n-1}{i}}\frac{O_i}{O_{i-n+r}}\sum_{q=0}^{2n-i-r-1}\binom{2n-i-r-1}{q}M_{i-n+r+q}^{(r)}\left(\partial(K_r')\right)\rho^q. \tag{3.9}$$

(2) If $i=n-r-1$, then by (2.17),

$$M_{n-r-1}^{(n)}\left(\partial(K_r')_\rho^{(r)}\right)=\binom{n-1}{n-r-1}^{-1}O_{n-r-1}V_r\left((K_r')_\rho^{(r)}\right). \tag{3.10}$$

Next, we turn our attention to the computation of the r-volume $(K_r')_\rho^{(r)}$. By applying the Steiner formula to K_r', we see that

$$V_r\left((K_r')_\rho^{(r)}\right)=\sum_{q=0}^{r}\binom{r}{q}W_q(K_r')\rho^q. \tag{3.11}$$

Hence

$$V_r\left((K_r')_\rho^{(r)}\right)=\sum_{q=0}^{r}\binom{r}{q}W_q(K_r')\rho^q=V_r(K_r')+\sum_{q=0}^{r-1}\frac{\rho^{q+1}}{q+1}\binom{r-1}{q}M_q^{(r)}\left(\partial(K_r')\right). \tag{3.12}$$

Finally, we obtain

$$M^{(n)}_{n-r-1}\bigl(\partial\bigl(K'_r\bigr)^{(r)}_\rho\bigr) = \binom{n-1}{n-r-1}^{-1} O_{n-r-1}\Biggl[V_r\bigl(K'_r\bigr) + \sum_{q=0}^{r-1}\frac{\rho^{q+1}}{q+1}\binom{r-1}{q}M^{(r)}_q\bigl(\partial\bigl(K'_r\bigr)\bigr)\Biggr]. \tag{3.13}$$

(3) If $i < n-r-1$, then by (2.18) we have

$$M^{(n)}_i\bigl(\partial\bigl(K'_r\bigr)^{(r)}_\rho\bigr) = 0. \tag{3.14}$$

□

Based on Theorem 1, we begin to consider the integral of $M^{(n)}_i(\partial(K'_r)^{(n)}_\rho)$ on Grassmann manifold $G_{r,n-r}$, and obtain the following.

Theorem 3 *Let K be a convex body with C^2 boundary ∂K in $\mathbb{R}^n$. Let $(K'_r)^{(n)}_\rho$ be the outer parallel body of K'_r in the distance ρ in $\mathbb{R}^n$, where K'_r is the orthogonal projection of K on the r-dimensional linear subspace $L_{r[O]} \subseteq \mathbb{R}^n$. Denote by $M^{(n)}_i(\partial(K'_r)^{(n)}_\rho)$ ($i = 0, 1, \ldots, n-1$) the mean curvature integrals of $(K'_r)^{(n)}_\rho$ and by $M^{(n)}_i(\partial K)$ ($i = 0, 1, \ldots, n-1$) the mean curvature integrals of K. Then:*

(1) *If $i \ge n-r$, then*

$$\int_{G_{r,n-r}} M^{(n)}_i\bigl(\partial\bigl(K'_r\bigr)^{(n)}_\rho\bigr)\,dL_{r[O]} = \sum_{q=i}^{n-1}\binom{n-i-1}{q-i}\frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_qO_{n-2}\cdots O_{n-r}}{O_{q-n+r}O_{r-2}\cdots O_0}\rho^{q-i}M^{(n)}_q(\partial K).$$

(2) *If $i \le n-r-1$, then*

$$\begin{aligned}\int_{G_{r,n-r}} M^{(n)}_i\bigl(\partial\bigl(K'_r\bigr)^{(n)}_\rho\bigr)\,dL_{r[O]} &= \binom{n-i-1}{n-r-i-1}\binom{n-1}{n-r-1}^{-1}\frac{O_{n-2}\cdots O_{n-r-1}}{rO_{r-2}\cdots O_0}\rho^{n-r-i-1}M^{(n)}_{n-r-1}(\partial K)\\ &\quad+\sum_{q=n-r}^{n-1}\binom{n-i-1}{q-i}\frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_{n-2}\cdots O_{n-r}O_q}{O_{q-n+r}O_{r-2}\cdots O_0}\rho^{q-i}M^{(n)}_q(\partial K).\end{aligned}$$

Proof (1) If $i \ge n-r$, by (1.1) and Lemma 2, the integral of $M^{(n)}_i(\partial(K'_r)^{(n)}_\rho)$ on Grassmann manifold $G_{r,n-r}$ can be obtained as follows:

$$\begin{aligned}\int_{G_{r,n-r}} M^{(n)}_i\bigl(\partial\bigl(K'_r\bigr)^{(n)}_\rho\bigr)\,dL_{r[O]} &= \int_{G_{r,n-r}}\sum_{q=i}^{n-1}\binom{n-i-1}{q-i}\frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_q}{O_{q-n+r}}M^{(r)}_{q-n+r}\bigl(\partial K'_r\bigr)\rho^{q-i}\,dL_{r[O]}\\ &= \sum_{q=i}^{n-1}\binom{n-i-1}{q-i}\frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_q}{O_{q-n+r}}\rho^{q-i}\int_{G_{r,n-r}} M^{(r)}_{q-n+r}\bigl(\partial K'_r\bigr)\,dL_{r[O]}\end{aligned}$$

$$= \sum_{q=i}^{n-1} \binom{n-i-1}{q-i} \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} \frac{O_q}{O_{q-n+r}} \rho^{q-i} \frac{O_{n-2}\cdots O_{n-r}}{O_{r-2}\cdots O_0} M_q^{(n)}(\partial K)$$
$$= \sum_{q=i}^{n-1} \binom{n-i-1}{q-i} \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} \frac{O_q O_{n-2}\cdots O_{n-r}}{O_{q-n+r} O_{r-2}\cdots O_0} \rho^{q-i} M_q^{(n)}(\partial K).$$

(2) If $i \le n-r-1$, then by (1.2) and Lemma 1 we arrive at

$$\begin{aligned}
&\int_{G_{r,n-r}} M_i^{(n)}\bigl(\partial (K_r')_\rho^{(n)}\bigr)\, dL_{r[O]} \\
&= \int_{G_{r,n-r}} \Biggl\{ \binom{n-i-1}{n-r-i-1} \binom{n-1}{n-r-1}^{-1} O_{n-r-1} V_r\bigl(K_r'\bigr) \rho^{n-r-i-1} \\
&\quad + \sum_{q=n-r}^{n-1} \binom{n-i-1}{q-i} \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} \frac{O_q}{O_{q-n+r}} M_{q-n+r}^{(r)}\bigl(\partial K_r'\bigr) \rho^{q-i} \Biggr\}\, dL_{r[O]} \\
&= \binom{n-i-1}{n-r-i-1} \binom{n-1}{n-r-1}^{-1} O_{n-r-1} \rho^{n-r-i-1} \int_{G_{r,n-r}} V_r\bigl(K_r'\bigr)\, dL_{r[O]} \\
&\quad + \sum_{q=n-r}^{n-1} \binom{n-i-1}{q-i} \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} \frac{O_q}{O_{q-n+r}} \rho^{q-i} \int_{G_{r,n-r}} M_{q-n+r}^{(r)}\bigl(\partial K_r'\bigr)\, dL_{r[O]}.
\end{aligned} \tag{3.15}$$

Note that

$$\int_{G_{r,n-r}} V_r\bigl(K_r'\bigr)\, dL_{r[O]} = I_{n-r}(K)$$

and

$$W_{n-r}^{(n)}(K) = \frac{rO_{r-2}\cdots O_0}{nO_{n-2}\cdots O_{n-r}} I_{n-r}(K),$$

therefore we obtain

$$\begin{aligned}
\int_{G_{r,n-r}} V_r\bigl(K_r'\bigr)\, dL_{r[O]} = I_{n-r}(K) \\
&= \frac{nO_{n-2}\cdots O_{n-r}}{rO_{r-2}\cdots O_0} W_{n-r}^{(n)}(K) \\
&= \frac{O_{n-2}\cdots O_{n-r}}{rO_{r-2}\cdots O_0} M_{n-r-1}^{(n)}(\partial K).
\end{aligned} \tag{3.16}$$

Inserting (3.16) to (3.15) and using Lemma 2, we have

$$\begin{aligned}
&\int_{G_{r,n-r}} M_i^{(n)}\bigl(\partial (K_r')_\rho^{(n)}\bigr)\, dL_{r[O]} \\
&= \binom{n-i-1}{n-r-i-1} \binom{n-1}{n-r-1}^{-1} O_{n-r-1} \rho^{n-r-i-1} \frac{O_{n-2}\cdots O_{n-r}}{rO_{r-2}\cdots O_0} M_{n-r-1}^{(n)}(\partial K) \\
&\quad + \sum_{q=n-r}^{n-1} \binom{n-i-1}{q-i} \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} \frac{O_q}{O_{q-n+r}} \rho^{q-i} \frac{O_{n-2}\cdots O_{n-r}}{O_{r-2}\cdots O_0} M_q^{(n)}(\partial K)
\end{aligned}$$

$$= \binom{n-i-1}{n-r-i-1}\binom{n-1}{n-r-1}^{-1}\frac{O_{n-2}\cdots O_{n-r-1}}{rO_{r-2}\cdots O_0}\rho^{n-r-i-1}M^{(n)}_{n-r-1}(\partial K)$$
$$+ \sum_{q=n-r}^{n-1}\binom{n-i-1}{q-i}\frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_{n-2}\cdots O_{n-r}O_q}{O_{q-n+r}O_{r-2}\cdots O_0}\rho^{q-i}M^{(n)}_q(\partial K),$$

we complete the proof of Theorem 3. □

By the Cauchy formula (2.15) and Theorem 3, the following corollary can be obtained.

Corollary 1 *Let K be a convex body with C^2 boundary ∂K in $\mathbb{R}^n$. Let $(K'_r)^{(n)}_\rho$ be the outer parallel body of K'_r in the distance ρ in $\mathbb{R}^n$, where K'_r is the orthogonal projection of K on the r-dimensional linear subspace $L_{r[O]} \subseteq \mathbb{R}^n$. Denote by $W^{(n)}_i((K'_r)^{(n)}_\rho)$ $(i = 1, 2, \ldots, n)$ the Minkowski quermassintegrals of $(K'_r)^{(n)}_\rho$ and by $W^{(n)}_i(K)$ $(i = 1, 2, \ldots, n)$ the Minkowski quermassintegrals of K. Then:*

(1) *If $i \ge n - r + 1$, then*

$$\int_{G_{r,n-r}} W^{(n)}_i\big((K'_r)^{(n)}_\rho\big)\,dL_{r[O]}$$
$$= \sum_{q=i-1}^{n-1}\binom{n-i}{q-i+1}\frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_qO_{n-2}\cdots O_{n-r}}{O_{q-n+r}O_{r-2}\cdots O_0}\rho^{q-i+1}W^{(n)}_{q+1}(K).$$

(2) *If $i \le n - r$, then*

$$\int_{G_{r,n-r}} W^{(n)}_i\big((K'_r)^{(n)}_\rho\big)\,dL_{r[O]}$$
$$= \binom{n-i}{n-r-i}\binom{n-1}{n-r-1}^{-1}\frac{O_{n-2}\cdots O_{n-r-1}}{rO_{r-2}\cdots O_0}\rho^{n-r-i}W^{(n)}_{n-r}(K)$$
$$+ \sum_{q=n-r}^{n-1}\binom{n-i}{q-i+1}\frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_{n-2}\cdots O_{n-r}O_q}{O_{q-n+r}O_{r-2}\cdots O_0}\rho^{q-i+1}W^{(n)}_{q+1}(K).$$

Using $\int_{G_{r,n-r}} M^{(n)}_i(\partial(K'_r)^{(n)}_\rho)\,dL_{r[O]}$ divided by $m(G_{r,n-r})$, and by Theorem 3, we immediately obtain the following corollaries.

Corollary 2 *Let K be a convex body with C^2 boundary ∂K in $\mathbb{R}^n$. Let $(K'_r)^{(n)}_\rho$ be the outer parallel body of K'_r in the distance ρ in $\mathbb{R}^n$, where K'_r is the orthogonal projection of K on the r-dimensional linear space $L_{r[O]}$. Denote by $M^{(n)}_i(\partial(K'_r)^{(n)}_\rho)$ $(i = 0, 1, \ldots, n-1)$ the mean curvature integrals of $(K'_r)^{(n)}_\rho$ and by $M^{(n)}_i(\partial K)$ $(i = 0, 1, \ldots, n-1)$ the mean curvature integrals of K. Then:*

(1) *If $i \ge n - r$,*

$$E\big(M^{(n)}_i\big(\partial\big(K'_r\big)^{(n)}_\rho\big)\big) = \sum_{q=i}^{n-1}\binom{n-i-1}{q-i}\frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}}\frac{O_qO_{r-1}}{O_{q-n+r}O_{n-1}}\rho^{q-i}M^{(n)}_{q+j}(\partial K).$$

(2) *If $i \le n - r - 1$,*

$$E\big(M^{(n)}_i\big(\partial\big(K'_r\big)^{(n)}_\rho\big)\big) = \binom{n-1}{n-r-1}^{-1}\frac{O_{n-r-1}O^2_{r-1}O_{r-2}\cdots O_0}{rO^2_{n-1}O_{n-2}\cdots O_{n-r}}\sum_{i=0}^{r}\binom{r}{i}\rho^iM^{(n)}_{n-r+i-1}(\partial K).$$

Corollary 3 *Let K be a convex body with C^2 boundary ∂K in $\mathbb{R}^n$. Let $(K_r')_\rho^{(n)}$ be the outer parallel body of K_r' in the distance ρ in $\mathbb{R}^n$, where K_r' is the orthogonal projection of K on the r-dimensional linear space $L_{r[O]}$. Denote by $W_i^{(n)}((K_r')_\rho^{(n)})$ $(i = 1, 2, \ldots, n)$ the Minkowski quermassintegrals of $(K_r')_\rho^{(n)}$ and by $W_i^{(n)}(K)$ $(i = 1, 2, \ldots, n)$ the Minkowski quermassintegrals of K. Then*:

(1) *If $i \geq n - r + 1$,*

$$E\big(W_i^{(n)}\big(\big(K_r'\big)_\rho^{(n)}\big)\big) = \sum_{q=i-1}^{n-1} \binom{n-i}{q-i+1} \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} \frac{O_q O_{r-1}}{O_{q-n+r} O_{n-1}} \rho^{q-i+1} W_{q+1}^{(n)}(K).$$

(2) *If $i \leq n - r$,*

$$E\big(W_i^{(n)}\big(\big(K_r'\big)_\rho^{(n)}\big)\big) = \binom{n-i}{n-r-i} \binom{n-1}{n-r-1}^{-1} \frac{O_{r-1} O_{n-r-1}}{r O_{n-1}} \rho^{n-r-i} W_{n-r}^{(n)}(K) + \sum_{q=n-r}^{n-1} \binom{n-i}{q-i+1} \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} \frac{O_{r-1}}{O_{q-n+r} O_{n-1}} \rho^{q-i+1} W_{q+1}^{(n)}(K).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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