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The stability of local strong solutions for a shallow water equation

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Abstract
We establish the L^1 stability of local strong solutions for a shallow water equation which includes the Degasperis-Procesi equation provided that its initial value lies in the Sobolev space $H^s(R)$ with $s > \frac{3}{2}$. The key element in our analysis is that the L^∞ norm of the solutions keeps finite for all finite time t.

MSC: 35G25; 35L05

Keywords: L^1 stability; strong solutions; shallow water equation

1 Introduction

From the propagation of shallow water waves over a flat bed, Constantin and Lannes [1] derived the equation

$$g_t + g_x + \frac{3}{2}\rho g g_x + \mu(\alpha g_{xxx} + \beta g_{txx}) = \rho\mu(\gamma g_x g_{xx} + \delta g g_{xxx}), \tag{1}$$

where the constants α, β, γ, δ, ρ and μ satisfy certain restrictions. As illustrated in [1], using suitable mathematical transformations turns Eq. (1) into the form

$$g_t - g_{txx} + kg_x + mgg_x = ag_x g_{xx} + bgg_{xxx}, \tag{2}$$

where a, b, k and m are constants. We know that the Camassa-Holm and Degasperis-Procesi models are special cases of Eq. (2). Lai and Wu [2] established the well-posedness of local strong solutions and obtained the existence of local weak solutions for Eq. (2).

The aim of this paper is to investigate a special case of Eq. (2). Namely, we study the shallow water equation

$$g_t - g_{txx} + kg_x + mgg_x = 3g_x g_{xx} + gg_{xxx}, \tag{3}$$

where $k \geq 0$ and $m > 0$ are constants. Letting $y = g - \partial_{xx}^2 g$, $v = (m - \partial_{xx}^2)^{-1} g$ and using Eq. (3), we derive the conservation law

$$\int_R yv\,dx = \int_R \frac{1+\xi^2}{m+\xi^2}\left|\hat{g}(t,\xi)\right|^2 d\xi = \int_R \frac{1+\xi^2}{m+\xi^2}\left|\hat{g}_0(\xi)\right|^2 d\xi \sim \|g_0\|_{L^2(R)}, \tag{4}$$

where $g_0 = g(0,x)$ and $\hat{g}(t,\xi)$ is the Fourier transform of $g(t,x)$ with respect to variable x. In fact, the conservation law (4) plays an important role in our further investigations of Eq. (3).

For $m = 4$, $k = 0$, Eq. (3) reduces to the Degasperis-Procesi equation [3]

$$g_t - g_{txx} + 4gg_x = 3g_x g_{xx} + gg_{xxx}. \tag{5}$$

Various dynamic properties for Eq. (5) have been acquired by many scholars. Escher *et al.* [4] and Yin [5] studied the global weak solutions and blow-up structures for Eq. (5), while the blow-up structure for a generalized periodic Degasperis-Procesi equation was obtained in [6]. Lin and Liu [7] established the stability of peakons for Eq. (5) under certain assumptions on the initial value. For other dynamic properties of the Degasperis-Procesi (5) and other shallow water models, the reader is referred to [8–19] and the references therein.

The objective of this work is to establish the $L^1(R)$ stability of local strong solutions for the generalized Degasperis-Procesi equation (3) under the condition that we let the initial value g_0 belong to the space $H^s(R)$ with $s > \frac{3}{2}$. Here we address that the L^1 stability of local strong solutions for Eq. (3) has never been established in the literature. Our main approaches come from those presented in [20].

This paper is organized as follows. Section 2 gives several lemmas. The main result and its proof are presented in Section 3.

2 Several lemmas

The Cauchy problem of Eq. (3) is written in the form

$$\begin{cases} g_t - g_{txx} + kg_x + mgg_x = 3g_x g_{xx} + gg_{xxx}, \\ g(0,x) = g_0(x), \end{cases} \tag{6}$$

which is equivalent to

$$\begin{cases} g_t + gg_x + k\Lambda^{-2}g_x + \frac{m-1}{2}\Lambda^{-2}(g^2)_x = 0, \\ g(0,x) = g_0(x), \end{cases} \tag{7}$$

where $\Lambda^{-2}f = \frac{1}{2}\int_R e^{-|x-y|}f\,dy$ for any $f \in L^2(R)$ or $L^\infty(R)$.

Let $Q_g(t,x) = \frac{m-1}{2}\Lambda^{-2}(g^2) + k\Lambda^{-2}g$ and $J_g = \partial_x(\frac{m-1}{2}\Lambda^{-2}(g^2) + k\Lambda^{-2}g)$, we have

$$g_t + \frac{1}{2}\left(g^2\right)_x + J_g = 0. \tag{8}$$

Lemma 2.1 *For problem* (6) *with* $m > 0$, *it holds that*

$$\int_R yv\,dx = \int_R \frac{1+\xi^2}{m+\xi^2}\left|\hat{g}(t,\xi)\right|^2 d\xi = \int_R \frac{1+\xi^2}{m+\xi^2}\left|\hat{g}_0(\xi)\right|^2 d\xi \sim \|g_0\|_{L^2(R)}. \tag{9}$$

In addition, there exist two positive constants c_1 and c_2 depending only on m such that

$$c_1\|g_0\|_{L^2(R)} \le c_1\|g\|_{L^2(R)} \le c_2\|g_0\|_{L^2(R)}.$$

Proof Letting $y = g - \partial_{xx}^2 g$ and $v = (m - \partial_{xx}^2)^{-1} g$ and using Eq. (3), we have $g = mv - \partial_{xx}^2 v$ and

$$\begin{aligned}
\frac{d}{dt}\int_R yv\,dx &= \int_R y_t v\,dx + \int_R y v_t\,dx = 2\int_R v y_t\,dx \\
&= 2\int_R \biggl[\biggl(-\frac{m}{2}g^2\biggr)_x + kg_x + \frac{1}{2}\partial_{xxx}^3 g^2\biggr] v\,dx \\
&= 2\int_R \biggl[\biggl(-\frac{m}{2}g^2\biggr)_x v + kg_x v + \frac{1}{2}\partial_x g^2 \partial_{xx}^2 v\biggr]\,dx \\
&= \int_R \bigl[\bigl(-mg^2\bigr)_x v + kg_x v + \bigl(g^2\bigr)_x (mv - g)\bigr]\,dx \\
&= -\int_R g\bigl(g^2\bigr)_x\,dx + k\int_R (mv_x - v_{xxx}) v\,dx \\
&= k\int_R v_{xx} v_x\,dx \\
&= 0,
\end{aligned}$$

from which we complete the proof. □

Lemma 2.2 ([2]) *If $g_0 \in H^s(R)$ with $s > \frac{3}{2}$, there exist maximal $T = T(u_0) > 0$ and a unique local strong solution $g(t,x)$ to problem* (6) *such that*

$$g(t,x) \in C\bigl([0,T); H^s(R)\bigr) C^1\bigl([0,T); H^{s-1}(R)\bigr).$$

Firstly, we study the differential equation

$$\begin{cases} p_t = g(t,p), & t \in [0,T), \\ p(0,x) = x. \end{cases} \tag{10}$$

Lemma 2.3 *Let $g_0 \in H^s(R)$, $s > 3$ and let $T > 0$ be the maximal existence time of the solution to problem* (10). *Then problem* (10) *has a unique solution $p \in C^1([0,T) \times R, R)$. Moreover, the map $p(t,\cdot)$ is an increasing diffeomorphism of R with $p_x(t,x) > 0$ for $(t,x) \in [0,T) \times R$.*

Proof From Lemma 2.2, we have $g \in C^1([0,T); H^{s-1}(R))$ and $H^{s-1}(R) \in C^1(R)$. Thus we conclude that both functions $g(t,x)$ and $g_x(t,x)$ are bounded, Lipschitz in space and C^1 in time. Using the existence and uniqueness theorem of ordinary differential equations derives that problem (10) has a unique solution $p \in C^1([0,T) \times R, R)$.

Differentiating (10) with respect to x yields

$$\begin{cases} \frac{d}{dt} p_x = g_x(t,p) p_x, & t \in [0,T), \\ p_x(0,x) = 1, \end{cases} \tag{11}$$

which leads to

$$p_x(t,x) = \exp\biggl(\int_0^t g_x\bigl(\tau, p(\tau,x)\bigr)\,d\tau\biggr). \tag{12}$$

For every $T' < T$, using the Sobolev embedding theorem yields

$$\sup_{(\tau,x)\in[0,T')\times R} \left|g_x(\tau,x)\right| < \infty.$$

It is inferred that there exists a constant $K_0 > 0$ such that $p_x(t,x) \geq e^{-K_0 t}$ for $(t,x) \in [0,T) \times R$. This completes the proof. □

Lemma 2.4 *Assume $g_0 \in H^s(R)$ with $s > \frac{3}{2}$. Let T be the maximal existence time of the solution g to Eq.* (3). *Then we have*

$$\left\|g(t,x)\right\|_{L^\infty} \leq c_0\|g_0\|_{L^2}^2 t + \|g_0\|_{L^\infty} \quad \forall t \in [0,T], \tag{13}$$

where constant c_0 depends on m, k.

Proof Let $Z(x) = \frac{1}{2}e^{-|x|}$, we have $(1-\partial_x^2)^{-1}f = Z \star f$ for all $f \in L^2(R)$ and $g = Z \star y(t,x)$. Using a simple density argument presented in [6], it suffices to consider $s = 3$ to prove this lemma. If T is the maximal existence time of the solution g to Eq. (3) with the initial value $g_0 \in H^3(R)$ such that $g \in C([0,T), H^3(R)) \cap C^1([0,T), H^2(R))$. From (7), we obtain

$$g_t + gg_x = -(m-1)Z \star (gg_x) - kZ \star g_x. \tag{14}$$

Since

$$\begin{aligned} -Z \star (gg_x) &= -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} gg_y\,dy \\ &= -\frac{1}{2}\int_{-\infty}^{x} e^{-x+y} gg_y\,dy - \frac{1}{2}\int_{x}^{+\infty} e^{x-y} gg_y\,dy \\ &= \frac{1}{4}\int_{\infty}^{x} e^{-|x-y|} g^2\,dy - \frac{1}{4}\int_{x}^{\infty} e^{-|x-y|} g^2\,dy \end{aligned} \tag{15}$$

and

$$\begin{aligned} \frac{dg(t,p(t,x))}{dt} &= g_t\big(t,p(t,x)\big) + g_x\big(t,p(t,x)\big)\frac{dp(t,x)}{dt} \\ &= (g_t + gg_x)\big(t,p(t,x)\big), \end{aligned} \tag{16}$$

from (16), we have

$$\frac{dg}{dt} = \frac{m-1}{4}\int_{-\infty}^{p(t,x)} e^{-|p(t,x)-y|} g^2\,dy - \frac{m-1}{4}\int_{p(t,x)}^{\infty} e^{-|p(t,x)-y|} g^2\,dy - kZ \star g_x, \tag{17}$$

from which we get

$$\begin{aligned} \left|\frac{dg(t,p(t,x))}{dt}\right| &\leq \frac{|m-1|}{4}\int_{-\infty}^{\infty} e^{-|p(t,x)-y|} g^2\,dy + |kZ \star g_x| \\ &\leq \frac{|m-1|}{4}\int_{-\infty}^{\infty} g^2\,dy + k\left|\int_{-\infty}^{\infty} \frac{1}{2} e^{-|x-y|} g_y\,dy\right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{|m-1|}{4}\|g\|_{L^2}^2 + k\|g\|_{L^2} \\
&\leq c\|g\|_{L^2(R)} \\
&\leq c\|g_0\|_{L^2(R)}, \qquad (18)
\end{aligned}$$

where c is a positive constant independent of t. Using (18) results in

$$-ct\|g_0\|_{L^2(R)} + g_0 \leq g\bigl(t,p(t,x)\bigr) \leq ct\|g_0\|_{L^2(R)} + g_0. \tag{19}$$

Therefore,

$$\bigl|g\bigl(t,p(t,x)\bigr)\bigr| \leq \bigl\|g\bigl(t,p(t,x)\bigr)\bigr\|_{L^\infty} \leq ct\|g_0\|_{L^2(R)} + \|g_0\|_{L^\infty}. \tag{20}$$

Using the Sobolev embedding theorem to ensure the uniform boundedness of $g_x(s,\eta)$ for $(s,\eta) \in [0,t] \times R$ with $t \in [0,T')$, from Lemma 2.3, for every $t \in [0,T')$, we get a constant $C(t)$ such that

$$e^{-C(t)} \leq p_x(t,x) \leq e^{C(t)}, \quad x \in R.$$

We deduce from the above equation that the function $p(t,\cdot)$ is strictly increasing on R with $\lim_{x\to\pm\infty} p(t,x) = \pm\infty$ as long as $t \in [0,T')$. It follows from (20) that

$$\bigl\|g(t,x)\bigr\|_{L^\infty} = \bigl\|g\bigl(t,p(t,x)\bigr)\bigr\|_{L^\infty} \leq ct\|g_0\|_{L^2(R)} + \|g_0\|_{L^\infty}. \tag{21}$$

□

Lemma 2.5 *Assume $g_0 \in L^2(R)$. Then*

$$\|Q_g\|_{L^\infty(R_+\times R)}, \|J_g\|_{L^\infty(R_+\times R)} \leq c_0\|g_0\|_{L^2}^2, \tag{22}$$

where c_0 is a constant independent of t.

Proof Using (7), we get

$$Q_g(t,x) = \frac{m-1}{4}\int_R e^{-|x-y|}g^2(t,y)\,dy + \frac{k}{2}\int_R e^{-|x-y|}g\,dy, \tag{23}$$

$$\begin{aligned}
J_g(t,x) = {}&\frac{m-1}{4}\int_R e^{-|x-y|}\operatorname{sign}(y-x)g^2(t,y)\,dy \\
&+ \frac{k}{2}\int_R e^{-|x-y|}\operatorname{sign}(y-x)g(t,y)\,dy. \qquad (24)
\end{aligned}$$

It follows from (23)-(24) and Lemma 2.1 that (22) holds. □

Lemma 2.6 *Assume that $g_1(t,x)$ and $g_2(t,x)$ are two local strong solutions of equation* (3) *with initial data $g_{10}, g_{20} \in H^s(R)$, $s > \frac{3}{2}$, respectively. Then, for any $f(t,x) \in C_0^\infty([0,\infty)\times R)$, it holds that*

$$\int_{-\infty}^{\infty}\bigl|J_{g_1}(t,x) - J_{g_2}(t,x)\bigr|\bigl|f(t,x)\bigr|\,dx \leq c_0(1+t)\int_{-\infty}^{\infty}|g_1 - g_2|\,dx, \tag{25}$$

where $c_0 > 0$ depends on t, f, $\|g_{10}\|_{L^2(R)}$, $\|g_{20}\|_{L^2(R)}$, $\|g_{10}\|_{L^\infty(R)}$ and $\|g_{20}\|_{L^\infty(R)}$.

Proof We have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\big|J_{g_1}(t,x)-J_{g_2}(t,x)\big|\big|f(t,x)\big|\,dx\\
&\quad\le \frac{|m-1|}{2}\int_{-\infty}^{\infty}\big|\partial_x\Lambda^{-2}\big(g_1^2-g_2^2\big)\big|\big|f(t,x)\big|\,dx\\
&\qquad+\frac{k}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-|x-y|}\big|\operatorname{sign}(x-y)\big||g_1-g_2|\big|f(t,x)\big|\,dy\,dx\\
&\quad=\frac{|m-1|}{4}\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-|x-y|}\big|\operatorname{sign}(x-y)\big|\big|g_1^2-g_2^2\big|\,dy\big|f(t,x)\big|\,dx\right|\\
&\qquad+c_0\int_{-\infty}^{\infty}|g_1-g_2|\,dy\int_{-\infty}^{\infty}e^{-|x-y|}\big|f(t,x)\big|\,dx\\
&\quad\le\frac{|m-1|}{4}\int_{-\infty}^{\infty}\big|(g_1-g_2)(g_1+g_2)\big|\,dy\left|\int_{-\infty}^{\infty}e^{-|x-y|}\big|f(t,x)\big|\,dx\right|\\
&\qquad+c_0\int_{-\infty}^{\infty}|g_1-g_2|\,dy\\
&\quad\le c_0(1+t)\int_{-\infty}^{\infty}|g_1-g_2|\,dy,
\end{aligned}$$

in which we have used the Tonelli theorem and Lemma 2.4. The proof is completed. □

We define $\delta(\sigma)$ to be a function which is infinitely differentiable on $(-\infty,+\infty)$ such that $\delta(\sigma)\ge 0$, $\delta(\sigma)=0$ for $|\sigma|\ge 1$ and $\int_{-\infty}^{\infty}\delta(\sigma)\,d\sigma=1$. For any number $h>0$, we let $\delta_h(\sigma)=\frac{\delta(h^{-1}\sigma)}{h}$. Then we know that $\delta_h(\sigma)$ is a function in $C^\infty(-\infty,\infty)$ and

$$\begin{cases}\delta_h(\sigma)\ge 0, & \delta_h(\sigma)=0 \quad \text{if } |\sigma|\ge h,\\ |\delta_h(\sigma)|\le \frac{c}{h}, & \int_{-\infty}^{\infty}\delta_h(\sigma)=1.\end{cases}\tag{26}$$

Assume that the function $u(x)$ is locally integrable in $(-\infty,\infty)$. We define an approximation function of u as

$$u^h(x)=\frac{1}{h}\int_{-\infty}^{\infty}\delta\left(\frac{x-y}{h}\right)u(y)\,dy,\quad h>0.\tag{27}$$

We call x_0 a Lebesgue point of the function $u(x)$ if

$$\lim_{h\to 0}\frac{1}{h}\int_{|x-x_0|\le h}\big|u(x)-u(x_0)\big|\,dx=0.$$

At any Lebesgue points x_0 of the function $u(x)$, we have $\lim_{h\to 0}u^h(x_0)=u(x_0)$. Since the set of points which are not Lebesgue points of $u(x)$ has measure zero, we get $u^h(x)\to u(x)$ as $h\to 0$ almost everywhere.

We introduce notation connected with the concept of a characteristic cone. For any $R_0>0$, we define $N>\max_{t\in[0,T]}\|g\|_{L^\infty}<\infty$. Let $\mho$ designate the cone $\{(t,x):|x|<R_0-Nt, 0\le t\le T_0=\min(T,R_0N^{-1})\}$. We let S_τ designate the cross section of the cone $\mho$ by the plane $t=\tau$, $\tau\in[0,T_0]$.

Let $K_{r+2\rho} = \{x : |x| \le r + 2\rho\}$, where $r > 0$, $\rho > 0$ and $\pi_T = [0, T] \times R$ for an arbitrary $T > 0$. The space of all infinitely differentiable functions $f(t,x)$ with compact support in $[0, T] \times R$ is denoted by $C_0^\infty(\pi_T)$.

Lemma 2.7 ([20]) *Let the function $u(t,x)$ be bounded and measurable in cylinder $\Omega_T = [0, T] \times K_r$. If for $\rho \in (0, \min[r, T])$ and any number $h \in (0, \rho)$, then the function*

$$V_h = \frac{1}{h^2} \iiiint_{|\frac{t-\tau}{2}| \le h, \rho \le \frac{t+\tau}{2} \le T-\rho, |\frac{x-y}{2}| \le h, |\frac{x+y}{2}| \le r-\rho} \big|u(t,x) - u(\tau, y)\big| \, dx \, dt \, dy \, d\tau$$

satisfies $\lim_{h\to 0} V_h = 0$.

Lemma 2.8 ([20]) *Let $|\frac{\partial G(u)}{\partial u}|$ be bounded. Then the function*

$$H(u,v) = \operatorname{sign}(u - v)\big(G(u) - G(v)\big)$$

satisfies the Lipschitz condition in u and v, respectively.

Lemma 2.9 *Let g be the strong solution of problem* (7), *$f(t,x) \in C_0^\infty(\pi_T)$ and $f(0,x) = 0$. Then*

$$\iint_{\pi_T} \left\{ |g - k| f_t + \operatorname{sign}(g - k) \frac{1}{2} \big[g^2 - k^2\big] f_x - \operatorname{sign}(g - k) J_g(t,x) f \right\} dx \, dt = 0, \tag{28}$$

where k is an arbitrary constant.

Proof Let $\Phi(g)$ be an arbitrary twice smooth function on the line $-\infty < g < \infty$. We multiply the first equation of problem (7) by the function $\Phi'(g) f(t,x)$, where $f(t,x) \in C_0^\infty(\pi_T)$. Integrating over π_T and transferring the derivatives with respect to t and x to the test function f, for any constant k, we obtain

$$\iint_{\pi_T} \left\{ \Phi(g) f_t + \left[\int_k^g \Phi'(z) z \, dz \right] f_x - \Phi'(g) J_g(t,x) f \right\} dx \, dt = 0, \tag{29}$$

in which we have used $\int_{-\infty}^{\infty} [\int_k^g \Phi'(z) z \, dz] f_x \, dx = -\int_{-\infty}^{\infty} [f \Phi'(g) g g_x] \, dx$.

Integration by parts yields

$$\begin{aligned} \int_{-\infty}^{\infty} \left[\int_k^g \Phi'(z) z \, dz \right] f_x \, dx = \int_{-\infty}^{\infty} \bigg[&\frac{1}{2} \big(g^2 - k^2\big) \Phi'(g) \\ &- \frac{1}{2} \int_k^g \big(z^2 - k^2\big) \Phi''(z) \, dz \bigg] f_x \, dx. \end{aligned} \tag{30}$$

Let $\Phi^h(g)$ be an approximation of the function $|g - k|$ and set $\Phi(g) = \Phi^h(g)$. Using the properties of $\operatorname{sign}(g - k)$, (29), (30) and sending $h \to 0$, we have

$$\iint_{\pi_T} \left\{ |g - k| f_t + \operatorname{sign}(g - k) \frac{1}{2} \big[g^2 - k^2\big] f_x - \operatorname{sign}(g - k) J_g(t,x) f \right\} dx \, dt = 0, \tag{31}$$

which completes the proof. □

In fact, the proof of (28) can also be found in [20].

For $g_{10} \in H^s(R)$ and $g_{20} \in H^s(R)$ with $s > \frac{3}{2}$, using Lemma 2.2, we know that there exists $T > 0$ such that two local strong solutions $g_1(t,x)$ and $g_2(t,x)$ of Eq. (3) satisfy

$$g_1(t,x), g_2(t,x) \in C\big([0,T); H^s(R)\big)C^1\big([0,T); H^{s-1}(R)\big), \quad t \in [0,T). \tag{32}$$

3 Main result

Now, we give the main result of this work.

Theorem 3.1 *Assume that g_1 and g_2 are two local strong solutions of Eq.* (3) *with initial data $g_{10}, g_{20} \in L^1(R) \cap H^s(R)$, $s > \frac{3}{2}$. For $T > 0$ in* (32), *it holds that*

$$\big\|g_1(t,\cdot) - g_2(t,\cdot)\big\|_{L^1(R)} \le ce^{ct} \int_{-\infty}^{\infty} \big|g_{10}(x) - g_{20}(x)\big|\,dx, \quad t \in [0,T], \tag{33}$$

where c depends on $\|g_{10}\|_{L^\infty(R)}$, $\|g_{20}\|_{L^\infty(R)}$, $\|g_{10}\|_{L^2(R)}$, $\|g_{20}\|_{L^2(R)}$ and T.

Proof For arbitrary $T > 0$ and $f(t,x) \in C_0^\infty(\pi_T)$, we assume that $f(t,x) = 0$ outside the cylinder

$$\uplus = \big\{(t,x)\big\} = [\rho, T - 2\rho] \times K_{r-2\rho}, \quad 0 < 2\rho \le \min(T,r). \tag{34}$$

We set

$$\eta = f\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right)\delta_h\left(\frac{t-\tau}{2}\right)\delta_h\left(\frac{x-y}{2}\right) = f(\cdots)\lambda_h(*), \tag{35}$$

where $(\cdots) = (\frac{t+\tau}{2}, \frac{x+y}{2})$ and $(*) = (\frac{t-\tau}{2}, \frac{x-y}{2})$. The function $\delta_h(\sigma)$ is defined in (26). Note that

$$\eta_t + \eta_\tau = f_t(\cdots)\lambda_h(*), \qquad \eta_x + \eta_y = f_x(\cdots)\lambda_h(*). \tag{36}$$

Using the Kruzkov device of doubling the variables [20] and Lemma 2.9, we have

$$\begin{aligned}\iiiint_{\pi_T \times \pi_T} \bigg\{ &\big|g_1(t,x) - g_2(\tau,y)\big|\eta_t \\ &+ \operatorname{sign}\big(g_1(t,x) - g_2(\tau,y)\big)\left(\frac{g_1^2(t,x)}{2} - \frac{g_2^2(\tau,y)}{2}\right)\eta_x \\ &- \operatorname{sign}\big(g_1(t,x) - g_2(\tau,y)\big)J_{g_1}(t,x)\eta\bigg\}\,dx\,dt\,dy\,d\tau = 0.\end{aligned} \tag{37}$$

Similarly, we have

$$\begin{aligned}\iiiint_{\pi_T \times \pi_T} \bigg\{ &\big|g_2(\tau,y) - g_1(t,x)\big|\eta_\tau \\ &+ \operatorname{sign}\big(g_2(\tau,y) - g_1(t,x)\big)\left(\frac{g_2^2(\tau,y)}{2} - \frac{g_1^2(t,x)}{2}\right)\eta_y \\ &- \operatorname{sign}\big(g_2(\tau,y) - g_1(t,x)\big)J_{g_2}(\tau,y)\eta\bigg\}\,dx\,dt\,dy\,d\tau = 0,\end{aligned} \tag{38}$$

from which we obtain

$$\begin{aligned}0 &\le \iiiint_{\pi_T\times\pi_T}\Big\{\big|g_1(t,x)-g_2(\tau,y)\big|(\eta_t+\eta_\tau)\\ &\quad+\operatorname{sign}\big(g_1(t,x)-g_2(\tau,y)\big)\left(\frac{g_1^2(t,x)}{2}-\frac{g_2^2(\tau,y)}{2}\right)(\eta_x+\eta_y)\Big\}\,dx\,dt\,dy\,d\tau\\ &\quad+\left|\iiiint_{\pi_T\times\pi_T}\operatorname{sign}\big(g_1(t,x)-g_2(t,x)\big)\big(J_{g_1}(t,x)-J_{g_2}(\tau,y)\big)\eta\,dx\,dt\,dy\,d\tau\right|\\ &= I_1+I_2+\left|\iiiint_{\pi_T\times\pi_T}I_3\,dx\,dt\,dy\,d\tau\right|.\end{aligned}\tag{39}$$

We will show that

$$\begin{aligned}0 &\le \iint_{\pi_T}\Big\{\big|g_1(t,x)-g_2(t,x)\big|f_t\\ &\quad+\operatorname{sign}\big(g_1(t,x)-g_2(t,x)\big)\left(\frac{g_1^2(t,x)}{2}-\frac{g_2^2(t,x)}{2}\right)f_x\Big\}\,dx\,dt\\ &\quad+\left|\iint_{\pi_T}\operatorname{sign}\big(g_1(t,x)-g_2(t,x)\big)\big[J_{g_1}(t,x)-J_{g_2}(t,x)\big]f\,dx\,dt\right|.\end{aligned}\tag{40}$$

In fact, the first two terms in the integrand of (39) can be represented in the form

$$A_h=A\big(t,x,\tau,y,g_1(t,x),g_2(\tau,y)\big)\lambda_h(*).$$

From Lemma 2.4 and the assumptions on solutions g_1, g_2, we have $\|g_1\|_{L^\infty}<C_T$ and $\|g_2\|_{L^\infty}<C_T$. From Lemma 2.8, we know that A_h satisfies the Lipschitz condition in g_1 and g_2, respectively. By the choice of η, we have $A_h=0$ outside the region

$$\big\{(t,x;\tau,y)\big\}=\left\{\rho\le\frac{t+\tau}{2}\le T-2\rho,\frac{|t-\tau|}{2}\le h,\frac{|x+y|}{2}\le r-2\rho,\frac{|x-y|}{2}\le h\right\}\tag{41}$$

and

$$\begin{aligned}\iiiint_{\pi_T\times\pi_T}A_h\,dx\,dt\,dy\,d\tau &= \iiiint_{\pi_T\times\pi_T}\big[A\big(t,x,\tau,y,g_1(t,x),g_2(\tau,y)\big)\\ &\quad-A\big(t,x,t,x,g_1(t,x),g_2(t,x)\big)\big]\lambda_h(*)\,dx\,dt\,dy\,d\tau\\ &\quad+\iiiint_{\pi_T\times\pi_T}A\big(t,x,t,x,g_1(t,x),g_2(t,x)\big)\lambda_h(*)\,dx\,dt\,dy\,d\tau\\ &= K_{11}(h)+K_{12}.\end{aligned}\tag{42}$$

Considering the estimate $|\lambda(*)|\le\frac{c}{h^2}$ and the expression of function A_h, we have

$$\begin{aligned}\big|K_{11}(h)\big| &\le c\bigg[h+\frac{1}{h^2}\\ &\quad\times\iiiint_{|\frac{t-\tau}{2}|\le h,\rho\le\frac{t+\tau}{2}\le T-\rho,|\frac{x-y}{2}|\le h,|\frac{x+y}{2}|\le r-\rho}|g_2(t,x)-g_2(\tau,y)|\,dx\,dt\,dy\,d\tau\bigg],\end{aligned}\tag{43}$$

where the constant c does not depend on h. Using Lemma 2.7, we obtain $K_{11}(h) \to 0$ as $h \to 0$. The integral K_{12} does not depend on h. In fact, substituting $t = \alpha$, $\frac{t-\tau}{2} = \beta$, $x = \zeta$, $\frac{x-y}{2} = \xi$ and noting that

$$\int_{-h}^{h} \int_{-\infty}^{\infty} \lambda_h(\beta,\xi)\, d\xi\, d\beta = 1, \tag{44}$$

we have

$$\begin{aligned} K_{12} &= 2^2 \iint_{\pi_T} A_h\bigl(\alpha,\zeta,\alpha,\zeta,g_1(\alpha,\zeta),g_2(\alpha,\zeta)\bigr)\left\{\int_{-h}^{h}\int_{-\infty}^{\infty} \lambda_h(\beta,\xi)\, d\xi\, d\beta\right\} d\zeta\, d\alpha \\ &= 4 \iint_{\pi_T} A\bigl(t,x,t,x,g_1(t,x),g_2(t,x)\bigr)\, dx\, dt. \end{aligned} \tag{45}$$

Hence

$$\lim_{h\to 0} \iiiint_{\pi_T\times\pi_T} A_h \, dx\, dt\, dy\, d\tau = 4 \iint_{\pi_T} A\bigl(t,x,t,x,g_1(t,x),g_2(t,x)\bigr)\, dx\, dt. \tag{46}$$

Since

$$I_3 = \operatorname{sign}\bigl(g_1(t,x) - g_2(\tau,y)\bigr)\bigl(J_{g_1}(t,x) - J_{g_2}(\tau,y)\bigr) f \lambda_h(*) = \bar{I}_3(t,x,\tau,y)\lambda_h(*) \tag{47}$$

and

$$\begin{aligned} &\iiiint_{\pi_T\times\pi_T} I_3 \, dx\, dt\, dy\, d\tau \\ &\quad = \iiiint_{\pi_T\times\pi_T} \bigl[\bar{I}_3(t,x,\tau,y) - \bar{I}_3(t,x,t,x)\bigr]\lambda_h(*)\, dx\, dt\, dy\, d\tau \\ &\qquad + \iiiint_{\pi_T\times\pi_T} \bar{I}_3(t,x,t,x)\lambda_h(*)\, dx\, dt\, dy\, d\tau = K_{21}(h) + K_{22}, \end{aligned} \tag{48}$$

we obtain

$$\begin{aligned} \bigl|K_{21}(h)\bigr| \le c\Biggl(h + \frac{1}{h^2} \\ \times \iiiint_{|\frac{t-\tau}{2}|\le h, \rho \le \frac{t+\tau}{2} \le T-\rho, |\frac{x-y}{2}|\le h, |\frac{x+y}{2}|\le r-\rho} \bigl|J_{g_2}(t,x) - J_{g_2}(\tau,y)\bigr|\, dx\, dt\, dy\, d\tau\Biggr). \end{aligned} \tag{49}$$

Using Lemma 2.7, we have $K_{21}(h) \to 0$ as $h \to 0$. Using (44), we have

$$\begin{aligned} K_{22} &= 2^2 \iint_{\pi_T} \bar{I}_3\bigl(\alpha,\zeta,\alpha,\zeta,g_1(\alpha,\zeta),g_2(\alpha,\zeta)\bigr)\left\{\int_{-h}^{h} \lambda_h(\beta,\xi)\, d\xi\, d\beta\right\} d\zeta\, d\alpha \\ &= 4 \iint_{\pi_T} \bar{I}_3\bigl(t,x,t,x,g_1(t,x),g_2(t,x)\bigr)\, dx\, dt \\ &= 4 \iint_{\pi_T} \operatorname{sign}\bigl(g_1(t,x) - g_2(t,x)\bigr)\bigl(J_{g_1}(t,x) - J_{g_2}(t,x)\bigr) f(t,x)\, dx\, dt. \end{aligned} \tag{50}$$

From (42), (46), (48), (49) and (50), we prove that inequality (40) holds.

Let

$$\mu(t) = \int_{-\infty}^{\infty} \left| g_1(t,x) - g_2(t,x) \right| dx. \tag{51}$$

We define

$$\theta_h = \int_{-\infty}^{\sigma} \delta_h(\sigma)\, d\sigma \quad \left(\theta_h'(\sigma) = \delta_h(\sigma) \geq 0\right) \tag{52}$$

and choose two numbers ρ and $\tau \in (0, T_0)$, $\rho < \tau$. In (40), we choose

$$f = \left[\theta_h(t-\rho) - \theta_h(t-\tau)\right]\chi(t,x), \quad h < \min(\rho, T_0 - \tau), \tag{53}$$

where

$$\chi(t,x) = \chi_\varepsilon(t,x) = 1 - \theta_\varepsilon\left(|x| + Nt - R + \varepsilon\right), \quad \varepsilon > 0. \tag{54}$$

We note that the function $\chi(t,x) = 0$ outside the cone $\mho$ and $f(t,x) = 0$ outside the set $\uplus$. For $(t,x) \in \mho$, we have the relations

$$0 = \chi_t + N|\chi_x| \geq \chi_t + N\chi_x. \tag{55}$$

Applying (53)-(55) and (40), we have the inequality

$$\begin{aligned} 0 \leq{} & \iint_{\pi_{T_0}} \left\{ \left[\delta_h(t-\rho) - \delta_h(t-\tau)\right]\chi_\varepsilon \left| g_1(t,x) - g_2(t,x) \right| \right\} dx\, dt \\ & + \int_0^{T_0} \int_{-\infty}^{\infty} \left[\theta_h(t-\rho) - \theta_h(t-\tau)\right] \left| \left[J_{g_1}(t,x) - J_{g_2}(t,x)\right]\chi(t,x) \right| dx\, dt. \end{aligned} \tag{56}$$

Using Lemma 2.6 and letting $\varepsilon \to 0$ and $R_0 \to \infty$, we obtain

$$\begin{aligned} 0 \leq{} & \int_0^{T_0} \left\{ \left[\delta_h(t-\rho) - \delta_h(t-\tau)\right] \int_{-\infty}^{\infty} \left| g_1(t,x) - g_2(t,x) \right| dx \right\} dt \\ & + c_0(1 + T_0) \int_0^{T_0} \left[\theta_h(t-\rho) - \theta_h(t-\tau)\right] \int_{-\infty}^{\infty} \left| g_1(t,x) - g_2(t,x) \right| dx\, dt. \end{aligned} \tag{57}$$

By the properties of the function $\delta_h(\sigma)$ for $h \leq \min(\rho, T_0 - \rho)$, we have

$$\begin{aligned} \left| \int_0^{T_0} \delta_h(t-\rho)\mu(t)\, dt - \mu(\rho) \right| &= \left| \int_0^{T_0} \delta_h(t-\rho) \left| \mu(t) - \mu(\rho) \right| dt \right| \\ &\leq c\frac{1}{h} \int_{\rho-h}^{\rho+h} \left| \mu(t) - \mu(\rho) \right| dt \to 0 \quad \text{as } h \to 0, \end{aligned} \tag{58}$$

where c is independent of h. Letting

$$L(\rho) = \int_0^{T_0} \theta_h(t-\rho)\mu(t)\, dt = \int_0^{T_0} \int_{-\infty}^{t-\rho} \delta_h(\sigma)\, d\sigma\, \mu(t)\, dt, \tag{59}$$

we get

$$L'(\rho) = -\int_0^{T_0} \delta_h(t-\rho)\mu(t)\,dt \to -\mu(\rho), \quad \text{as } h \to 0, \tag{60}$$

from which we obtain

$$L(\rho) \to L(0) - \int_0^{\rho} \mu(\sigma)\,d\sigma \quad \text{as } h \to 0. \tag{61}$$

Similarly, we have

$$L(\tau) \to L(0) - \int_0^{\tau} \mu(\sigma)\,d\sigma \quad \text{as } h \to 0. \tag{62}$$

It follows from (61) and (62) that

$$L(\rho) - L(\tau) \to \int_{\rho}^{\tau} \mu(\sigma)\,d\sigma \quad \text{as } h \to 0. \tag{63}$$

Send $\rho \to 0$, $\tau \to t$, and note that

$$\begin{aligned} \big|g_1(\rho,x) - g_2(\rho,x)\big| &\le \big|g_1(\rho,x) - g_{10}(x)\big| \\ &\quad + \big|g_2(\rho,x) - g_{20}(x)\big| + \big|g_{10}(x) - g_{20}(x)\big|. \end{aligned} \tag{64}$$

Thus, from (57), (58), (63)-(64), we have

$$\begin{aligned} \int_{-\infty}^{\infty} \big|g_1(t,x) - g_2(t,x)\big|\,dx &\le \int_{-\infty}^{\infty} |g_{10} - g_{20}|\,dx \\ &\quad + c_0(1+T_0)\int_0^t \int_{-\infty}^{\infty} \big|g_1(t,x) - g_2(t,x)\big|\,dx\,dt, \end{aligned} \tag{65}$$

from which we complete the proof by using the Gronwall inequality. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The article is a joint work of four authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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