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Generalized hybrid mappings on CAT(κ) spaces

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday.

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Abstract

In this paper, we obtain the demiclosed principle, fixed point theorems, and Δ-convergence theorems for the class of generalized hybrid mappings on CAT(κ) spaces with $\kappa > 0$. Our results extend the results of Lin *et al.* (Fixed Point Theory Appl. 2011:49, 2011) and many others.

Keywords: fixed point; generalized hybrid mapping; Δ-convergence; CAT(κ) space

1 Introduction

For a real number κ, a CAT(κ) space is a geodesic metric space whose geodesic triangle is thinner than the corresponding comparison triangle in a model space with curvature κ. The precise definition is given below. The letters C, A, and T stand for Cartan, Alexandrov, and Toponogov, who have made important contributions to the understanding of curvature via inequalities for the distance function.

Fixed point theory in CAT(κ) spaces was first studied by Kirk [1, 2]. His works were followed by a series of new works by many authors, mainly focusing on CAT(0) spaces (see *e.g.*, [3–18]). Since any CAT(κ) space is a CAT(κ') space for $\kappa' \geq \kappa$, all results for CAT(0) spaces immediately apply to any CAT(κ) space with $\kappa \leq 0$. However, there are only a few articles that contain fixed point results in the setting of CAT(κ) spaces with $\kappa > 0$.

The concept of generalized hybrid mappings was introduced in Hilbert spaces by Kocourek *et al.* [19]. Later on, Lin *et al.* [10] defined a generalized hybrid mapping, which is more general than that of Kocourek *et al.* [19], in a CAT(0) space setting. This class of mappings properly contains the class of nonspreading mappings and the class of hybrid mappings; see [10] for more details. In [10], the authors also obtained the demiclosed principle, fixed point theorems as well as Δ-convergence theorems for generalized hybrid mappings in CAT(0) spaces. In this paper, we extend the results of Lin *et al.* [10] to the general setting of CAT(κ) spaces with $\kappa > 0$.

2 Preliminaries

Let (X,ρ) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $\rho(c(t),c(t')) = |t - t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry and $\rho(x,y) = l$. The image $c([0,l])$ of c is called a *geodesic segment* joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$. This means that $z \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that

$$\rho(x,z) = (1-\alpha)\rho(x,y) \quad \text{and} \quad \rho(y,z) = \alpha\rho(x,y).$$

In this case, we write $z = \alpha x \oplus (1-\alpha)y$. For $D \in (0,+\infty]$, the space X is called a *D-geodesic space* if every two points of X with their distance smaller than D are joined by a geodesic segment. An ∞-geodesic space is simply called a *geodesic space*. The space X is said to be *uniquely geodesic* (*D-uniquely geodesic*) if there is exactly one geodesic segment joining x and y for each $x,y \in X$ (for $x,y \in X$ with $\rho(x,y) < D$). A subset C of X is said to be *convex* if C includes every geodesic segment joining any two of its points. The set C is said to be *bounded* if

$$\operatorname{diam}(C) := \sup\{\rho(x,y) : x,y \in C\} < \infty.$$

Now we present the model spaces M_κ^n, for more details on these spaces the reader is referred to [20]. Let $n \in \mathbb{N}$. We denote by $\mathbb{E}^n$ the metric space $\mathbb{R}^n$ endowed with the usual Euclidean distance. We denote by $(\cdot|\cdot)$ the Euclidean scalar product in $\mathbb{R}^n$, that is,

$$(x|y) = x_1y_1 + \cdots + x_ny_n, \quad \text{where } x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n).$$

Let $\mathbb{S}^n$ denote the *n-dimensional sphere* defined by

$$\mathbb{S}^n = \{x = (x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} : (x|x) = 1\},$$

with metric $d_{\mathbb{S}^n}(x,y) = \arccos(x|y)$, $x,y \in \mathbb{S}^n$.

Let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $u = (u_1,\ldots,u_{n+1})$ and $v = (v_1,\ldots,v_{n+1})$ the real number $\langle u|v\rangle$ is defined by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_iv_i.$$

Let $\mathbb{H}^n$ denote the *hyperbolic n-space* defined by

$$\mathbb{H}^n = \{u = (u_1,\ldots,u_{n+1}) \in \mathbb{E}^{n,1} : \langle u|u\rangle = -1, u_{n+1} > 0\},$$

with metric $d_{\mathbb{H}^n}$ such that

$$\cosh d_{\mathbb{H}^n}(x,y) = -\langle x|y\rangle, \quad x,y \in \mathbb{H}^n.$$

Definition 2.1 Given $\kappa \in \mathbb{R}$, we denote by M_κ^n the following metric spaces:

(i) if $\kappa = 0$ then M_0^n is the Euclidean space $\mathbb{E}^n$;
(ii) if $\kappa > 0$ then M_κ^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $1/\sqrt{\kappa}$;
(iii) if $\kappa < 0$ then M_κ^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by the constant $1/\sqrt{-\kappa}$.

A *geodesic triangle* $\triangle(x,y,z)$ in a geodesic space (X,ρ) consists of three points x, y, z in X (the *vertices* of $\triangle$) and three geodesic segments between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for a geodesic triangle $\triangle(x,y,z)$ in (X,ρ) is a triangle $\overline{\triangle}(\bar{x},\bar{y},\bar{z})$ in M_κ^2 such that

$$\rho(x,y) = d_{M_\kappa^2}(\bar{x},\bar{y}), \qquad \rho(y,z) = d_{M_\kappa^2}(\bar{y},\bar{z}) \quad \text{and} \quad \rho(z,x) = d_{M_\kappa^2}(\bar{z},\bar{x}).$$

If $\kappa \le 0$ then such a comparison triangle always exists in M_κ^2. If $\kappa > 0$ then such a triangle exists whenever $\rho(x,y) + \rho(y,z) + \rho(z,x) < 2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$. A point $\bar{p} \in [\bar{x},\bar{y}]$ is called a *comparison point* for $p \in [x,y]$ if $\rho(x,p) = d_{M_\kappa^2}(\bar{x},\bar{p})$.

A geodesic triangle $\triangle(x,y,z)$ in X is said to satisfy the CAT(κ) *inequality* if for any $p,q \in \triangle(x,y,z)$ and for their comparison points $\bar{p},\bar{q} \in \overline{\triangle}(\bar{x},\bar{y},\bar{z})$, one has

$$\rho(p,q) \le d_{M_\kappa^2}(\bar{p},\bar{q}).$$

Definition 2.2 If $\kappa \le 0$, then X is called a CAT(κ) *space* if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality.

If $\kappa > 0$, then X is called a CAT(κ) *space* if X is D_κ-geodesic and any geodesic triangle $\triangle(x,y,z)$ in X with $\rho(x,y) + \rho(y,z) + \rho(z,x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Now, we recall the concepts of comparison angle and upper (Alexandrov) angle (*cf.* [8]).

Definition 2.3 Let p, q, and r be three points in a geodesic space. The interior angle of $\overline{\triangle}(\bar{p},\bar{q},\bar{r}) \subseteq \mathbb{E}^2$ at $\bar{p}$ is called the *comparison angle* between q and r at p and will be denoted by $\overline{\angle}_p(q,r)$.

Definition 2.4 Let X be a geodesic space and let $c : [0,a] \to X$ and $c' : [0,a'] \to X$ be two geodesic paths with $c(0) = c'(0)$. Given $t \in (0,a]$ and $t' \in (0,a']$, we consider the comparison triangle $\overline{\triangle}(\overline{c(0)},\overline{c(t)},\overline{c'(t')})$ and the comparison angle $\overline{\angle}_{c(0)}(c(t),c'(t'))$ in $\mathbb{E}^2$. The (*Alexandrov*) *angle* or the *upper angle* between the geodesic paths c and c' is the number $\angle(c,c')$ defined by

$$\angle(c,c') := \limsup_{t,t' \to 0^+} \overline{\angle}_{c(0)}\bigl(c(t),c'(t')\bigr).$$

The angle between the geodesic segments $[p,x]$ and $[p,y]$ will be denoted by $\angle_p(x,y)$. Notice that the Alexandrov angle coincides with the spherical angle on $\mathbb{S}^n$ and the hyperbolic angle on $\mathbb{H}^n$.

In a CAT(0) space (X,ρ), if $x,y,z \in X$ then the CAT(0) inequality implies

$$\text{(CN)} \quad \rho^2\left(x, \frac{1}{2}y \oplus \frac{1}{2}z\right) \le \frac{1}{2}\rho^2(x,y) + \frac{1}{2}\rho^2(x,z) - \frac{1}{4}\rho^2(y,z).$$

This is the (CN) *inequality* of Bruhat and Tits [21]. This inequality is extended by Dhompongsa and Panyanak [22] to

$$\bigl(\text{CN}^*\bigr) \quad \rho^2\bigl(x,(1-\alpha)y \oplus \alpha z\bigr) \le (1-\alpha)\rho^2(x,y) + \alpha\rho^2(x,z) - \alpha(1-\alpha)\rho^2(y,z)$$

for all $\alpha \in [0,1]$ and $x, y, z \in X$. In fact, if X is a geodesic space then the following statements are equivalent:

(i) X is a CAT(0) space;
(ii) X satisfies (CN);
(iii) X satisfies (CN*).

Let $R \in (0,2]$. Recall that a geodesic space (X,ρ) is said to be *R-convex* for R (see [23]) if for any three points $x, y, z \in X$, we have

$$\rho^2\big(x,(1-\alpha)y \oplus \alpha z\big) \le (1-\alpha)\rho^2(x,y) + \alpha\rho^2(x,z) - \frac{R}{2}\alpha(1-\alpha)\rho^2(y,z). \tag{1}$$

It follows from (CN*) that a geodesic space (X,ρ) is a CAT(0) space if and only if (X,ρ) is R-convex for $R = 2$. The following lemma is a consequence of Proposition 3.1 in [23].

Lemma 2.5 *Let $\kappa > 0$ and (X,ρ) be a* CAT(κ) *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Then (X,ρ) is R-convex for $R = (\pi - 2\varepsilon)\tan(\varepsilon)$.*

We now collect some elementary facts about CAT(κ) spaces. Most of them are proved in the setting of CAT(1) spaces. For completeness, we state the results in CAT(κ) with $\kappa > 0$.

Lemma 2.6 ([8, Proposition 3.5]) *Let $\kappa > 0$ and (X,ρ) be a complete* CAT(κ) *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Let $x \in X$ and C be a nonempty closed convex subset of X. Then*

(i) *the metric projection $P_C(x)$ of x onto C is a singleton*;
(ii) *if $x \notin C$ and $y \in C$ with $y \neq P_C(x)$, then $\angle_{P_C(x)}(x,y) \ge \pi/2$*;
(iii) *for each $y \in C$, $\rho(P_C(x), P_C(y)) \le \rho(x,y)$.*

Let $\{x_n\}$ be a bounded sequence in a CAT(κ) space (X,ρ). For $x \in X$, we set

$$r\big(x,\{x_n\}\big) = \limsup_{n\to\infty} \rho(x,x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x,\{x_n\}\big) : x \in X\big\},$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\big(\{x_n\}\big) = \big\{x \in X : r\big(x,\{x_n\}\big) = r\big(\{x_n\}\big)\big\}.$$

It is well known from Proposition 4.1 of [8] that in a CAT(κ) space with diameter smaller than $\frac{\pi}{2\sqrt{\kappa}}$, $A(\{x_n\})$ consists of exactly one point. We now give the concept of Δ-convergence and collect some of its basic properties.

Definition 2.7 ([6, 24]) A sequence $\{x_n\}$ in X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.8 *Let $\kappa > 0$ and (X,ρ) be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Then the following statements hold*:

(i) [8, Corollary 4.4] *every sequence in X has a Δ-convergence subsequence*;

(ii) [8, Proposition 4.5] *if $\{x_n\} \subseteq X$ and $\Delta\text{-}\lim_n x_n = x$, then $x \in \bigcap_{k=1}^{\infty} \overline{\operatorname{conv}}\{x_k, x_{k+1}, \dots\}$, where $\overline{\operatorname{conv}}(A) = \bigcap\{B : B \supseteq A$ and B is closed and convex$\}$.*

By the uniqueness of asymptotic centers, we can obtain the following lemma (*cf.* [22, Lemma 2.8]).

Lemma 2.9 *Let $\kappa > 0$ and (X,ρ) be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. If $\{x_n\}$ is a sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{\rho(x_n,u)\}$ converges, then $x = u$.*

Definition 2.10 Let C be a nonempty subset of a $\mathrm{CAT}(\kappa)$ space (X,ρ). A mapping $T : C \to X$ is called a *generalized hybrid mapping* [10] if there exist functions $a_1, a_2, a_3, k_1, k_2 : C \to [0,1)$ such that

(P1) $\rho^2(T(x),T(y)) \le a_1(x)\rho^2(x,y) + a_2(x)\rho^2(T(x),y) + a_3(x)\rho^2(T(y),x) + k_1(x)\rho^2(T(x), x) + k_2(x)\rho^2(T(y),y)$ for all $x,y \in C$;

(P2) $a_1(x) + a_2(x) + a_3(x) \le 1$ for all $x,y \in C$;

(P3) $2k_1(x) < 1 - a_2(x)$ and $k_2(x) < 1 - a_3(x)$ for all $x \in C$.

A point $x \in C$ is called a *fixed point* of T if $x = T(x)$. We denote the set of all fixed points of T with $F(T)$.

3 Main results

3.1 Demiclosed principle

Theorem 3.1 *Let $\kappa > 0$ and (X,ρ) be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to X$ be a generalized hybrid mapping with $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ for all $x \in C$ where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. Let $\{x_n\}$ be a sequence in C with $\Delta\text{-}\lim_n x_n = z$ and $\lim_n \rho(x_n, T(x_n)) = 0$. Then $z \in C$ and $z = T(z)$.*

Proof Since $\Delta\text{-}\lim_n x_n = z$, by Lemma 2.8, $z \in C$. Since T is a generalized hybrid mapping,

$$\begin{aligned}
\rho^2\bigl(T(x_n),T(z)\bigr) &\le a_1(z)\rho^2(z,x_n) + a_2(z)\rho^2\bigl(T(z),x_n\bigr) + a_3(z)\rho^2\bigl(T(x_n),z\bigr) \\
&\quad + k_1(z)\rho^2\bigl(T(z),z\bigr) + k_2(z)\rho^2\bigl(T(x_n),x_n\bigr) \\
&\le a_1(z)\rho^2(z,x_n) + a_2(z)\bigl[\rho\bigl(T(z),T(x_n)\bigr) + \rho\bigl(T(x_n),x_n\bigr)\bigr]^2 \\
&\quad + a_3(z)\bigl[\rho\bigl(T(x_n),x_n\bigr) + \rho(x_n,z)\bigr]^2 + k_1(z)\rho^2\bigl(T(z),z\bigr) \\
&\quad + k_2(z)\rho^2\bigl(T(x_n),x_n\bigr),
\end{aligned}$$

yielding

$$\limsup_{n\to\infty} \rho^2\bigl(T(x_n),T(z)\bigr) \le \limsup_{n\to\infty} \rho^2(z,x_n) + \frac{k_1(z)}{1-a_2(z)}\rho^2\bigl(z,T(z)\bigr).$$

This implies that

$$\begin{aligned}\limsup_{n\to\infty}\rho^2\big(x_n,T(z)\big)&\le\limsup_{n\to\infty}\big[\rho\big(x_n,T(x_n)\big)+\rho\big(T(x_n),T(z)\big)\big]^2\\&\le\limsup_{n\to\infty}\rho^2\big(T(x_n),T(z)\big)\\&\le\limsup_{n\to\infty}\rho^2(z,x_n)+\frac{k_1(z)}{1-a_2(z)}\rho^2\big(z,T(z)\big).\end{aligned}\tag{2}$$

On the other hand, by Lemma 2.5 we have

$$\rho^2\left(x_n,\frac{1}{2}z\oplus\frac{1}{2}T(z)\right)\le\frac{1}{2}\rho^2(x_n,z)+\frac{1}{2}\rho^2\big(x_n,T(z)\big)-\frac{R}{8}\rho^2\big(z,T(z)\big).\tag{3}$$

By (2) and (3), we get

$$\begin{aligned}\limsup_{n\to\infty}\rho^2\left(x_n,\frac{1}{2}z\oplus\frac{1}{2}T(z)\right)&\le\frac{1}{2}\limsup_{n\to\infty}\rho^2(x_n,z)+\frac{1}{2}\limsup_{n\to\infty}\rho^2\big(x_n,T(z)\big)\\&\quad-\frac{R}{8}\rho^2\big(z,T(z)\big)\\&\le\limsup_{n\to\infty}\rho^2(x_n,z)+\frac{k_1(z)}{2(1-a_2(z))}\rho^2\big(z,T(z)\big)\\&\quad-\frac{R}{8}\rho^2\big(z,T(z)\big).\end{aligned}$$

Thus

$$\left(\frac{R}{8}-\frac{k_1(z)}{2(1-a_2(z))}\right)\rho^2\big(z,T(z)\big)\le\limsup_{n\to\infty}\rho^2(x_n,z)-\limsup_{n\to\infty}\rho^2\left(x_n,\frac{1}{2}z\oplus\frac{1}{2}T(z)\right)\le 0.$$

Since $\frac{2k_1(z)}{1-a_2(z)}<\frac{R}{2}$, we get $\frac{k_1(z)}{2(1-a_2(z))}<\frac{R}{8}$ and so $\rho^2(z,T(z))=0$. Hence $z=T(z)$. □

The following corollary shows that how we derive a result for CAT(0) spaces from Theorem 3.1.

Corollary 3.2 *Let (X,ρ) be a complete* CAT(0) *space, C be a nonempty bounded closed convex subset of X, and $T:C\to C$ be a generalized hybrid mapping. Let $\{x_n\}$ be a sequence in C with Δ-$\lim_n x_n=z$ and $\lim_n\rho(x_n,T(x_n))=0$. Then $z\in C$ and $z=T(z)$.*

Proof It is well known that every convex subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space (*cf.* [20]). Then (C,ρ) is a CAT(0) space and hence it is a CAT(κ) space for all $\kappa>0$. Notice also that C is R-convex for $R=2$. Since C is bounded, we can choose $\varepsilon\in(0,\pi/2)$ and $\kappa>0$ so that $\operatorname{diam}(C)\le\frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$. The conclusion follows from Theorem 3.1. □

3.2 Fixed point theorems

Theorem 3.3 *Let $\kappa>0$ and (X,ρ) be a complete* CAT(κ) *space with* $\operatorname{diam}(X)\le\frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon\in(0,\pi/2)$. Let C be a nonempty closed convex subset of X, and $T:C\to C$ be a generalized hybrid mapping with $k_1(x)=k_2(x)=0$ for all $x\in C$. Then T has a fixed point.*

Proof Fix $x \in C$ and define $x_n := T^n(x)$ for $n \in \mathbb{N}$. Suppose that $A(\{x_n\}) = \{z\}$. Then by Lemma 2.8, $z \in C$. Since T is generalized hybrid and $k_1(z) = k_2(z) = 0$,

$$\rho^2\big(x_n, T(z)\big) \leq a_1(z)\rho^2(z, x_{n-1}) + a_2(z)\rho^2\big(T(z), x_{n-1}\big) + a_3(z)\rho^2(x_n, z).$$

Taking the limit superior on both sides, we get

$$\begin{aligned}\limsup_{n\to\infty} \rho^2\big(x_n, T(z)\big) &\leq a_1(z)\limsup_{n\to\infty} \rho^2(z, x_{n-1}) + a_2(z)\limsup_{n\to\infty} \rho^2\big(T(z), x_{n-1}\big) \\ &\quad + a_3(z)\limsup_{n\to\infty} \rho^2(x_n, z) \\ &\leq \big(a_1(z) + a_3(z)\big)\limsup_{n\to\infty} \rho^2(x_n, z) + a_2(z)\limsup_{n\to\infty} \rho^2\big(x_n, T(z)\big).\end{aligned}$$

This implies by (P2) that $\limsup_n \rho^2(x_n, T(z)) \leq \limsup_n \rho^2(x_n, z)$. But, since $A(\{x_n\}) = \{z\}$, it must be the case that $z = T(z)$ and the proof is complete. □

As a consequence of Theorem 3.3, we obtain:

Corollary 3.4 *Let (X, ρ) be a complete* CAT(0) *space, C be a nonempty bounded closed convex subset of X, and $T : C \to C$ be a generalized hybrid mapping with $k_1(x) = k_2(x) = 0$ for all $x \in C$. Then T has a fixed point.*

3.3 Δ-Convergence theorems

We begin this section by proving a crucial lemma.

Lemma 3.5 *Let $\kappa > 0$ and (X, ρ) be a complete* CAT(κ) *space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to X$ be a generalized hybrid mapping with $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ for all $x \in C$ where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. Suppose $\{x_n\}$ is a sequence in C such that $\lim_n \rho(x_n, Tx_n) = 0$ and $\{\rho(x_n, v)\}$ converges for all $v \in F(T)$, then $\omega_w(x_n) \subseteq F(T)$. Here $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Proof Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.8, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in C$. By Theorem 3.1, $v \in F(T)$. By Lemma 2.9, $u = v$. This shows that $\omega_w(x_n) \subseteq F(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subseteq F(T)$, $\{\rho(x_n, u)\}$ converges. Again, by Lemma 2.9, $x = u$. This completes the proof. □

Theorem 3.6 *Let $\kappa > 0$ and (X, ρ) be a complete* CAT(κ) *space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X, and $T : C \to X$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ and define a sequence $\{x_n\}$ in C by*

$$\begin{cases} x_1 \in C & \textit{chosen arbitrary}, \\ x_{n+1} := P_C((1-\alpha_n)x_n \oplus \alpha_n T(x_n)), & n \in \mathbb{N}. \end{cases}$$

Let $R = (\pi - 2\varepsilon)\tan(\varepsilon)$ and suppose that

(i) $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ *for all* $x \in C$,
(ii) $\liminf_n \alpha_n[\frac{(1-\alpha_n)R}{2} - \frac{k_2(z)}{1-a_3(z)}] > 0$ *for all* $z \in F(T)$.

Then $\{x_n\}$ Δ*-converges to an element of* $F(T)$.

Proof Let $z \in F(T)$. Since T is generalized hybrid,

$$\rho^2\big(T(x), z\big) \le \rho^2(z, x) + \frac{k_2(z)}{1-a_3(z)}\rho^2\big(T(x), x\big) \quad \text{for all } x \in C.$$

By Lemmas 2.5 and 2.6, we have

$$\begin{aligned}
\rho^2(x_{n+1}, z) &= \rho^2\big(P_C\big((1-\alpha_n)x_n \oplus \alpha_n T(x_n)\big), z\big) \\
&\le \rho^2\big((1-\alpha_n)x_n \oplus \alpha_n T(x_n), z\big) \\
&\le (1-\alpha_n)\rho^2(x_n, z) + \alpha_n \rho^2\big(T(x_n), z\big) - \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\big(x_n, T(x_n)\big) \\
&\le \rho^2(x_n, z) + \alpha_n\left[\frac{k_2(z)}{1-a_3(z)} - \frac{R(1-\alpha_n)}{2}\right]\rho^2\big(x_n, T(x_n)\big).
\end{aligned} \tag{4}$$

By (ii), there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\alpha_n\left[\frac{(1-\alpha_n)R}{2} - \frac{k_2(z)}{1-a_3(z)}\right] \ge \delta > 0 \quad \text{for all } n \ge N.$$

Without loss of generality, we may assume that

$$\alpha_n\left[\frac{(1-\alpha_n)R}{2} - \frac{k_2(z)}{1-a_3(z)}\right] > 0 \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

It follows from (4) and (5) that $\{\rho(x_n, z)\}$ is a nonincreasing sequence and hence $\lim_n \rho(x_n, z)$ exists. Again, by (4), we have

$$\lim_{n\to\infty} \alpha_n\left[\frac{(1-\alpha_n)R}{2} - \frac{k_2(z)}{1-a_3(z)}\right]\rho^2\big(x_n, T(x_n)\big) = 0.$$

This implies by (ii) that $\lim_n \rho(x_n, T(x_n)) = 0$. By Lemma 3.5, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This shows that $\{x_n\}$ Δ-converges to an element of $F(T)$. □

Theorem 3.7 *Let* $\kappa > 0$ *and* (X, ρ) *be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\mathrm{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some* $\varepsilon \in (0, \pi/2)$. *Let* C *be a nonempty closed convex subset of* X, *and* $T : C \to X$ *be a generalized hybrid mapping with* $F(T) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0,1]$ *and define a sequence* $\{x_n\}$ *in* C *by*

$$\begin{cases}
x_1 \in C \quad \text{chosen arbitrary}, \\
x_{n+1} := P_C((1-\alpha_n)T(x_n) \oplus \alpha_n T(y_n)), \\
y_n := P_C((1-\beta_n)x_n \oplus \beta_n T(x_n)).
\end{cases}$$

Assume that

(i) $k_2(z) = 0$ *for all* $z \in F(T)$,
(ii) $\liminf_n \alpha_n > 0$ *and* $\liminf_n \beta_n(1 - \beta_n) > 0$.

Then $\{x_n\}$ Δ*-converges to an element of* $F(T)$.

Proof Fix $z \in F(T)$. By (i), we have $\rho(T(x), z) \le \rho(x, z)$ for all $x \in C$. Let $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. By Lemmas 2.5 and 2.6, we have

$$\begin{aligned} \rho^2(y_n, z) &= \rho^2\big(P_C\big((1-\beta_n)x_n \oplus \beta_n T(x_n)\big), z\big) \\ &\le \rho^2\big((1-\beta_n)x_n \oplus \beta_n T(x_n), z\big) \\ &\le (1-\beta_n)\rho^2(x_n, z) + \beta_n \rho^2\big(T(x_n), z\big) - \frac{R}{2}\beta_n(1-\beta_n)\rho^2\big(x_n, T(x_n)\big) \\ &\le \rho^2(x_n, z) - \frac{R}{2}\beta_n(1-\beta_n)\rho^2\big(x_n, T(x_n)\big) \\ &\le \rho^2(x_n, z). \end{aligned} \tag{6}$$

This implies that

$$\begin{aligned} \rho^2(x_{n+1}, z) &= \rho^2\big(P_C\big((1-\alpha_n)T(x_n) \oplus \alpha_n T(y_n)\big), z\big) \\ &\le \rho^2\big((1-\alpha_n)T(x_n) \oplus \alpha_n T(y_n), z\big) \\ &\le (1-\alpha_n)\rho^2\big(T(x_n), z\big) + \alpha_n \rho^2\big(T(y_n), z\big) - \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\big(T(x_n), T(y_n)\big) \\ &\le (1-\alpha_n)\rho^2(x_n, z) + \alpha_n \rho^2(y_n, z) - \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\big(T(x_n), T(y_n)\big) \\ &\le \rho^2(x_n, z) - \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\big(T(x_n), T(y_n)\big) \\ &\le \rho^2(x_n, z). \end{aligned}$$

Hence $\lim_n \rho(x_n, z)$ exists and

$$0 \le \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\big(T(x_n), T(y_n)\big) \le \rho^2(x_n, z) - \rho^2(x_{n+1}, z) + \alpha_n\big[\rho^2(y_n, z) - \rho^2(x_n, z)\big].$$

So,

$$\alpha_n\big[\rho^2(x_n, z) - \rho^2(y_n, z)\big] \le \rho^2(x_n, z) - \rho^2(x_{n+1}, z).$$

Since $\liminf_n \alpha_n > 0$, $\limsup_n[\rho^2(x_n, z) - \rho^2(y_n, z)] = 0$. By (6), we have

$$\frac{R}{2}\beta_n(1-\beta_n)\rho^2\big(x_n, T(x_n)\big) \le \rho^2(x_n, z) - \rho^2(y_n, z).$$

This implies by (ii) that $\lim_n \rho(x_n, T(x_n)) = 0$. By Lemma 3.5, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This shows that $\{x_n\}$ Δ-converges to an element of $F(T)$. □

The following lemma is also needed (*cf.* [10, Lemma 4.2]).

Lemma 3.8 *Let $\kappa > 0$ and (X,ρ) be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\mathrm{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X with* $\lim_n \rho(x_n,y_n) = 0$. *If* Δ-$\lim_n x_n = x$ *and* Δ-$\lim_n y_n = y$, *then $x = y$.*

Theorem 3.9 *Let $\kappa > 0$ and (X,ρ) be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\mathrm{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Let C be a nonempty closed convex subset of X, and $T,S : C \to X$ be a two generalized hybrid mappings with $F(T) \cap F(S) \ne \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be a sequence in $[0,1]$ and define a sequence $\{x_n\}$ in C by*

$$\begin{cases} x_1 \in C \quad \textit{chosen arbitrary}, \\ x_{n+1} := P_C((1-\alpha_n)x_n \oplus \alpha_n T(y_n)), \\ y_n := P_C((1-\beta_n)x_n \oplus \beta_n S(x_n)). \end{cases}$$

Let $R = (\pi - 2\varepsilon)\tan(\varepsilon)$ and suppose that
(i) $\liminf_n \alpha_n(1-\alpha_n) > 0$,
(ii) $k_2^T(z) = 0$ *and* $\liminf_n \beta_n[\frac{(1-\beta_n)R}{2} - \frac{k_2^S(z)}{1-a_3^S(z)}] > 0$ *for all $z \in F(T) \cap F(S)$.*
Then $\{x_n\}$ Δ-converges to a common fixed point of S and T.

Proof Let $z \in F(T) \cap F(S)$. Since $k_2^T(z) = 0$, $\rho(T(x),z) \le \rho(x,z)$ for all $x \in C$. By Lemmas 2.5 and 2.6, we have

$$\begin{aligned} \rho^2(y_n,z) &= \rho^2\big(P_C\big((1-\beta_n)x_n \oplus \beta_n S(x_n)\big),z\big) \\ &\le \rho^2\big((1-\beta_n)x_n \oplus \beta_n S(x_n),z\big) \\ &\le (1-\beta_n)\rho^2(x_n,z) + \beta_n\rho^2\big(S(x_n),z\big) - \frac{R}{2}\beta_n(1-\beta_n)\rho^2\big(x_n,S(x_n)\big) \\ &\le (1-\beta_n)\rho^2(x_n,z) + \beta_n\left[\rho^2(x_n,z) + \frac{k_2^S(z)}{1-a_3^S(z)}\rho^2\big(S(x_n),x_n\big)\right] \\ &\quad - \frac{R}{2}\beta_n(1-\beta_n)\rho^2\big(x_n,S(x_n)\big) \\ &\le \rho^2(x_n,z) - \beta_n\left[\frac{(1-\beta_n)R}{2} - \frac{k_2^S(z)}{1-a_3^S(z)}\right]\rho^2\big(S(x_n),x_n\big). \end{aligned} \tag{7}$$

By (ii), there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\beta_n\left[\frac{(1-\beta_n)R}{2} - \frac{k_2^S(z)}{1-a_3^S(z)}\right] \ge \delta > 0 \quad \text{for all } n \ge N.$$

Without loss of generality, we may assume that

$$\beta_n\left[\frac{(1-\beta_n)R}{2} - \frac{k_2^S(z)}{1-a_3^S(z)}\right] > 0 \quad \text{for all } n \in \mathbb{N}.$$

By (7), $\rho(y_n,z) \le \rho(x_n,z)$. Thus

$$\begin{aligned} \rho^2(x_{n+1},z) &= \rho^2\big(P_C\big((1-\alpha_n)x_n \oplus \alpha_n T(y_n)\big),z\big) \\ &\le \rho^2\big((1-\alpha_n)x_n \oplus \alpha_n T(y_n),z\big) \end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n)\rho^2(x_n,z) + \alpha_n\rho^2\big(T(y_n),z\big) - \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\big(x_n,T(y_n)\big) \\
&\leq (1-\alpha_n)\rho^2(x_n,z) + \alpha_n\rho^2(y_n,z) - \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\big(x_n,T(y_n)\big) \\
&\leq \rho^2(x_n,z) - \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\big(x_n,T(y_n)\big) \\
&\leq \rho^2(x_n,z). \qquad (8)
\end{aligned}$$

Hence $\lim_n \rho(x_n,z)$ exists and

$$\lim_{n\to\infty}\alpha_n(1-\alpha_n)\rho^2\big(x_n,T(y_n)\big) = 0.$$

By (i), $\lim_n \rho^2(x_n,T(y_n)) = 0$. It follows from (8) that

$$0 \leq \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2\big(x_n,T(y_n)\big) \leq \rho^2(x_n,z) - \rho^2(x_{n+1},z) + \alpha_n\big[\rho^2(y_n,z) - \rho^2(x_n,z)\big].$$

Thus

$$\alpha_n(1-\alpha_n)\big[\rho^2(x_n,z) - \rho^2(y_n,z)\big] \leq \rho^2(x_n,z) - \rho^2(x_{n+1},z).$$

Again, by (i), $\limsup_n[\rho^2(x_n,z) - \rho^2(y_n,z)] = 0$. By (7), we have

$$\beta_n\left[\frac{(1-\beta_n)R}{2} - \frac{k_2^S(z)}{1-a_3^S(z)}\right]\rho^2\big(x_n,S(x_n)\big) \leq \rho^2(x_n,z) - \rho^2(y_n,z).$$

This implies by (ii) that $\lim_n \rho(x_n,S(x_n)) = 0$. Hence,

$$\begin{aligned}
\limsup_{n\to\infty}\rho(y_n,x_n) &= \limsup_{n\to\infty}\rho\big(P_C\big((1-\beta_n)x_n \oplus \beta_n S(x_n)\big),P_C(x_n)\big) \\
&\leq \limsup_{n\to\infty}\rho\big((1-\beta_n)x_n \oplus \beta_n S(x_n),x_n\big) \\
&= \limsup_{n\to\infty}\beta_n\rho\big(S(x_n),x_n\big) \\
&= 0.
\end{aligned}$$

So, $\lim_n \rho(y_n,T(y_n)) = 0$. By Lemma 3.5, there exist $u,v \in C$ such that $\omega_w(x_n) = \{u\} \subseteq F(S)$ and $\omega_w(y_n) = \{v\} \subseteq F(T)$. This means that Δ-$\lim_n x_n = u$ and Δ-$\lim_n y_n = v$. Hence, by Lemma 3.8, $u = v$ and the proof is complete. □

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
The authors read and approved the final manuscript.

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