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Optimal power mean bounds for Yang mean

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Abstract

In this paper, we prove that the double inequality $M_p(a,b) < U(a,b) < M_q(a,b)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq 2\log 2/(2\log\pi - \log 2) = 0.8684\cdots$ and $q \geq 4/3$, where $U(a,b)$ and $M_r(a,b)$ are the Yang and rth power means of a and b, respectively.

MSC: 26E60

Keywords: Yang mean; power mean; Neuman-Sándor mean

1 Introduction

Let $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$. Then the pth power mean $M_p(a,b)$ of a and b is given by

$$M_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{1/p} \quad (p \neq 0), \qquad M_0(a,b) = \sqrt{ab}.$$

The main properties for the power mean are given in [1]. It is well known that $M_p(a,b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Many classical means are the special cases of the power mean, for example, $M_{-1}(a,b) = 2ab/(a+b) = H(a,b)$ is the harmonic mean, $M_0(a,b) = \sqrt{ab} = G(a,b)$ is the geometric mean, $M_1(a,b) = (a+b)/2 = A(a,b)$ is the arithmetic mean, and $M_2(a,b) = \sqrt{(a^2+b^2)/2} = Q(a,b)$ is the quadratic mean.

Let $L(a,b) = (b-a)/(\log b - \log a)$, $P(a,b) = (a-b)/[2\arcsin((a-b)/(a+b))]$, $M(a,b) = (a-b)/[2\sinh^{-1}((a-b)/(a+b))]$, $I(a,b) = (a^a/b^b)^{1/(a-b)}/e$ and $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$ be the logarithmic, first Seiffert, Neuman-Sándor, identric, and second Seiffert means of two distinct positive real numbers a and b, respectively. Then it is well known that the inequalities

$$\begin{aligned} H(a,b) &< G(a,b) < L(a,b) < P(a,b) \\ &< I(a,b) < A(a,b) < M(a,b) < T(a,b) < Q(a,b) \end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$.

Recently, the bounds for certain bivariate means in terms of the power mean have been the subject of intensive research. Seiffert [2] proved that the inequalities

$$\frac{2}{\pi}M_1(a,b) < P(a,b) < M_1(a,b) < T(a,b) < M_2(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

Jagers [3] proved that the double inequality

$$M_{1/2}(a,b) < P(a,b) < M_{2/3}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

In [4, 5], Hästö established that

$$P(a,b) > M_{\log 2/\log \pi}(a,b), \qquad P(a,b) > \frac{2\sqrt{2}}{\pi} M_{2/3}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Witkowski [6] proved that the double inequality

$$\frac{2\sqrt{2}}{\pi} M_2(a,b) < T(a,b) < \frac{4}{\pi} M_1(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

In [7], Costin and Toader presented the result that

$$M_{\log 2/(\log \pi - \log 2)}(a,b) < T(a,b) < M_{5/3}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Chu and Long [8] proved that the double inequality

$$M_p(a,b) < M(a,b) < M_q(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq \log 2/\log[2\log(1+\sqrt{2})] = 1.224\cdots$ and $q \geq 4/3$.

The following sharp bounds for the logarithmic and identric means in terms of the power means can be found in the literature [9–16]:

$$\begin{aligned} &M_0(a,b) < L(a,b) < M_{1/3}(a,b), \qquad M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b), \\ &M_0(a,b) < L^{1/2}(a,b) I^{1/2}(a,b) < M_{1/2}(a,b), \\ &M_{\log 2/(1+\log 2)}(a,b) < \frac{L(a,b) + I(a,b)}{2} < M_{1/2}(a,b) \end{aligned}$$

for all $a, b > 0$ with $a \neq b$.

Recently, Yang [17] introduced the Yang mean $U(a,b)$ of two distinct positive real numbers a and b as follows:

$$U(a,b) = \frac{a-b}{\sqrt{2}\arctan\frac{a-b}{\sqrt{2ab}}},$$

and he proved that the inequalities

$$\begin{aligned} &P(a,b) < U(a,b) < T(a,b), \qquad \frac{G(a,b)T(a,b)}{A(a,b)} < U(a,b) < \frac{P(a,b)Q(a,b)}{A(a,b)}, \\ &Q^{1/2}(a,b)\left[\frac{2G(a,b) + Q(a,b)}{3}\right]^{1/2} < U(a,b) < Q^{2/3}(a,b)\left[\frac{G(a,b) + Q(a,b)}{2}\right]^{1/3}, \end{aligned}$$

$$\frac{G(a,b)+Q(a,b)}{2} < U(a,b) < \left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^{1/2} + \frac{1}{3}Q^{1/2}(a,b)\right]^2$$

hold for all $a, b > 0$ with $a \neq b$.

In [18], Yang *et al.* presented several sharp bounds for the Yang mean $U(a,b)$ in terms of the geometric mean $G(a,b)$ and quadratic mean $Q(a,b)$.

The main purpose of this article is to find the greatest value p and the least value q such that the double inequality

$$M_p(a,b) < U(a,b) < M_q(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

2 Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 2.1 *Let* $f_1 : (0,1) \times \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f_1(x,p) = \frac{(1-x^2)(1+x^p)}{\sqrt{x}(1+x^2)(1+x^{p-1})} - \sqrt{2}\arctan\frac{1-x}{\sqrt{2x}}. \tag{2.1}$$

Then

(1) $f_1(x,p)$ *is strictly decreasing with respect to* x *on* $(0,1)$ *if and only if* $p \geq 4/3$;
(2) $f_1(x,p)$ *is strictly increasing with respect to* x *on* $(0,1)$ *if and only if* $p \leq 1/2$.

Proof It follows from (2.1) that

$$\frac{\partial f_1(x,p)}{\partial x} = \frac{(1-x)x^{p-1/2}}{2(1+x^2)^2(x+x^p)^2} f_2(x,p), \tag{2.2}$$

where

$$\begin{aligned} f_2(x,p) = {}& x^{1-p}\left(-1+x-5x^2-3x^3\right) + x^p\left(3+5x-x^2+x^3\right) - (2p-1) \\ & + 4x - 4x^3 + (2p-1)x^4. \end{aligned} \tag{2.3}$$

(1) If $f_1(x,p)$ is strictly decreasing with respect to x on $(0,1)$, then (2.2) leads to the conclusion that $f_2(x,p) < 0$ for all $x \in (0,1)$. In particular, from (2.3) we have

$$\lim_{x\to 1^-}\frac{f_2(x,p)}{1-x} = -24\left(p-\frac{4}{3}\right) \leq 0. \tag{2.4}$$

Therefore, $p \geq 4/3$ follows from (2.4).

If $p \geq 4/3$, then it follows from (2.3) that

$$\begin{aligned} \frac{\partial f_2(x,p)}{\partial p} = {}& \left[x^p\left(x^3-x^2+5x+3\right) + x^{1-p}\left(3x^3+5x^2-x+1\right)\right]\log x \\ & - 2\left(1-x^4\right) < 0 \end{aligned} \tag{2.5}$$

for all $x \in (0,1)$.

Equation (2.3) and inequality (2.5) lead to the conclusion that

$$f_2(x,p) \leq f_2\left(x, \frac{4}{3}\right) = -\frac{x^{-1/3}}{3}\left(1-x^{2/3}\right)^3 \times \left(3x^{8/3}+5x^{7/3}+9x^2+12x^{5/3}+6x^{4/3}+12x+9x^{2/3}+5x^{1/3}+3\right) < 0 \tag{2.6}$$

for all $x \in (0,1)$.

Therefore, $f_1(x,p)$ is strictly decreasing with respect to x on $(0,1)$ follows from (2.2) and (2.6).

(2) If $f_1(x,p)$ is strictly increasing with respect to x on $(0,1)$, then (2.2) leads to the conclusion that $f_2(x,p) > 0$ for all $x \in (0,1)$. In particular, we have $f_2(0^+,p) \geq 0$ and we assert that $p \leq 1/2$. Indeed, from (2.3) we clearly see that $f_2(0^+,p) = -\infty$ for $p > 1$, $f_2(0^+,1) = -2$, $f_2(0^+,0) = 4$, $f_2(0^+,p) = \infty$ for $p < 0$, and $f_2(0^+,p) = 1-2p$ for $0 < p < 1$.

If $p \leq 1/2$, then inequality (2.5) holds again. It follows from (2.3) and (2.5) that

$$f_2(x,p) \geq f_2\left(x, \frac{1}{2}\right) = 2x^{1/2}(1-x)\left(x^2+2x^{3/2}+4x+2x^{1/2}+1\right) > 0 \tag{2.7}$$

for all $x \in (0,1)$.

Therefore, $f_1(x,p)$ is strictly increasing with respect to x on $(0,1)$ follows from (2.2) and (2.7). □

Lemma 2.2 *Let* $f_1 : (0,1) \times \mathbb{R} \to \mathbb{R}$ *be defined by* (2.1). *Then*

(1) $f_1(x,p) > 0$ *for all* $x \in (0,1)$ *if and only if* $p \geq 4/3$;

(2) $f_1(x,p) < 0$ *for all* $x \in (0,1)$ *if and only if* $p \leq 1/2$.

Proof (1) If $f_1(x,p) > 0$ for all $x \in (0,1)$, then from (2.1) and the L'Hôpital rules we have

$$\lim_{x\to 1^-} \frac{f_1(x,p)}{(1-x)^3} = \frac{1}{12}(3p-4) \geq 0$$

and $p \geq 4/3$.

If $p \geq 4/3$, then (2.1) and Lemma 2.1(1) lead to the conclusion that $f_1(x,p) > f_1(1,p) = 0$ for all $x \in (0,1)$.

(2) If $f_1(x,p) < 0$ for all $x \in (0,1)$, then $f_1(0^+,p) \leq 0$. We claim that $p \leq 1/2$. Indeed, it follows from (2.1) that $f_1(0^+,p) = +\infty$ if $p > 1/2$.

If $p \leq 1/2$, then (2.1) and Lemma 2.1(2) lead to the conclusion that $f_1(x,p) < f_1(1,p) = 0$ for all $x \in (0,1)$. □

Lemma 2.3 *Let* $f_3 : (0,1) \times \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f_3(x,p) = -x^{1-2p} + x^{2-2p} - 5x^{3-2p} - 3x^{4-2p} + 3 + 5x - x^2 + x^3 - (2p-1)x^{-p} + 4x^{1-p} - 4x^{3-p} + (2p-1)x^{4-p}. \tag{2.8}$$

Then $\partial^4 f_3(x,p)/\partial x^4 < 0$ *for all* $x \in (0,1)$ *if* $p \in (1, 4/3)$.

Proof It follows (2.8) that

$$x^{p+4}\frac{\partial^4 f_3(x,p)}{\partial x^4} = x^{1-p}\left(a_3x^3 + a_2x^2 + a_1x + a_0\right) + b_4x^4 + b_3x^3 + b_1x + b_0, \tag{2.9}$$

where

$$a_3 = -3(2p-1)(2p-2)(2p-3)(2p-4) < 0, \tag{2.10}$$

$$a_2 = -10p(2p-1)(2p-2)(2p-3) > 0, \tag{2.11}$$

$$a_1 = 2p(2p-1)(2p+1)(2p-2) > 0, \tag{2.12}$$

$$a_0 = -2p(2p-1)(2p+1)(2p+2) < 0, \tag{2.13}$$

$$b_4 = (2p-1)(p-1)(p-2)(p-3)(p-4) < 0, \tag{2.14}$$

$$b_3 = -4p(p-1)(p-2)(p-3) < 0, \tag{2.15}$$

$$b_1 = 4p(p-1)(p+1)(p+2) > 0, \tag{2.16}$$

$$b_0 = -p(2p-1)(p+1)(p+2)(p+3) < 0. \tag{2.17}$$

From (2.11)-(2.13) and (2.16) together with (2.17) we get

$$a_2x^2 + a_1x + a_0 < a_2 + a_1 + a_0 = -4p(2p-1)\left(10p^2 - 21p + 17\right) < 0, \tag{2.18}$$

$$b_1x + b_0 < b_1 + b_0 = -p(p+2)(p+1)\left(2p^2 + p + 1\right) < 0 \tag{2.19}$$

for all $x \in (0,1)$.

Therefore, Lemma 2.3 follows easily from (2.9), (2.10), (2.14), (2.15), (2.18), and (2.19). □

Lemma 2.4 *Let* $f_3 : (0,1) \times \mathbb{R} \to \mathbb{R}$ *be defined by* (2.8). *Then* $\partial^2 f_3(x,p)/\partial x^2 < 0$ *for all* $x \in (0,1)$ *if* $p \in (1/2, 4/3)$.

Proof It follows from (2.8) that

$$\begin{aligned} x^{p+2}\frac{\partial^2 f_3(x,p)}{\partial x^2} &= 6x^{p+3} + (2p-1)(p-3)(p-4)x^4 - 2x^{p+2} \\ &\quad - 3(2p-3)(2p-4)x^{4-p} - 5(2p-2)(2p-3)x^{3-p} \\ &\quad + (2p-1)(2p-2)x^{2-p} - 2p(2p-1)x^{1-p} \\ &\quad - 4(p-2)(p-3)x^3 + 4p(p-1)x - p(2p-1)(p+1), \end{aligned} \tag{2.20}$$

$$\left.\frac{\partial^2 f_3(x,p)}{\partial x^2}\right|_{x=1} = -48\left(\frac{4}{3} - p\right)\left(\frac{3}{2} - p\right) < 0, \tag{2.21}$$

$$\left.\frac{\partial^3 f_3(x,p)}{\partial x^3}\right|_{x=1} = 88p^3 - 300p^2 + 380p - 144. \tag{2.22}$$

We divide the proof into two cases.

Case 1. $p \in (1/2, 1]$. Then from

$$(2p-1)(p-3)(p-4) > 0, \qquad -3(2p-3)(2p-4) < 0, \qquad -5(2p-2)(2p-3) \leq 0,$$

$$(2p-1)(2p-2) \le 0, \qquad -2p(2p-1) < 0, \qquad -4(p-2)(p-3) < 0,$$
$$4p(p-1) < 0, \qquad -p(2p-1)(p+1) < 0,$$
$$0 < x^4 \le x^{p+3} < x^{4-p} \le x^3 \le x^{p+2} < x^{3-p} \le x^2 < x^{2-p} \le x < x^{1-p} \le 1$$

and (2.20) we clearly see that

$$\begin{aligned} x^{p+2}\frac{\partial^2 f_3(x,p)}{\partial x^2} &< \bigl[6+(2p-1)(p-3)(p-4)\bigr]x^{4-p} + \bigl[-2-3(2p-3)(2p-4) \\ &\quad - 5(2p-2)(2p-3) + (2p-1)(2p-2) - 2p(2p-1) \\ &\quad - 4(p-2)(p-3) + 4p(p-1) \\ &\quad - p(2p-1)(p+1)\bigr]x^{4-p} \\ &= -8(3p-4)(2p-3)x^{4-p} < 0 \end{aligned}$$

for all $x \in (0,1)$.

Case 2. $p \in (1, 4/3]$. Then (2.22) leads to

$$\frac{\partial^3 f_3(x,p)}{\partial x^3}\bigg|_{x=1} = 88(p-1)\left(p-\frac{53}{44}\right)^2 + \frac{887}{22}(p-1) + 24 > 0. \tag{2.23}$$

It follows from Lemma 2.3 and (2.23) that $\partial^2 f_3(x,p)/\partial x^2$ is strictly increasing with respect to x on $(0,1)$.

Therefore, $\partial^2 f_3(x,p)/\partial x^2 < 0$ for all $x \in (0,1)$ follows from (2.21) and the monotonicity of the $\partial^2 f_3(x,p)/\partial x^2$ with respect to x on the interval $(0,1)$. □

Lemma 2.5 *Let $f_1 : (0,1) \times \mathbb{R} \to \mathbb{R}$ be defined by* (2.1). *Then there exists $\lambda \in (0,1)$ such that $f_1(x,p)$ is strictly decreasing with respect to x on the interval $(0,\lambda]$ and strictly increasing with respect to x on the interval $[\lambda,1)$ if $p \in (1/2, 4/3)$.*

Proof Let $f_2(x,p)$ and $f_3(x,p)$ be defined by (2.3) and (2.8), respectively. Then from (2.8) we clearly see that

$$f_3(1,p) = 0, \qquad f_3(0^+,p) = -\infty, \tag{2.24}$$

$$\frac{\partial f_3(x,p)}{\partial x}\bigg|_{x=1} = 8(3p-4) < 0, \qquad \lim_{x\to 0^+}\frac{\partial f_3(x,p)}{\partial x} = +\infty. \tag{2.25}$$

It follows from Lemma 2.4 and (2.25) that there exists $\lambda_0 \in (0,1)$ such that $f_3(x,p)$ is strictly increasing with respect to x on $(0,\lambda_0]$ and strictly decreasing with respect to x on $[\lambda_0,1)$. This in conjunction with (2.24) leads to the conclusion that there exists $\lambda \in (0,1)$ such that $f_3(x,p) < 0$ for $x \in (0,\lambda)$ and $f_3(x,p) > 0$ for $x \in (\lambda,1)$.

Note that

$$f_2(x,p) = x^p f_3(x,p). \tag{2.26}$$

Therefore, Lemma 2.5 follows from (2.2) and (2.26) together with the piecewise positive and negative of $f_3(x,p)$ on $(0,1)$. □

Lemma 2.6 *Let $f:(0,1)\times\mathbb{R}\to\mathbb{R}$ be defined by*

$$f(x,p)=\log\frac{U(1,x)}{M_p(1,x)}=\log\frac{1-x}{\sqrt{2}\arctan\frac{1-x}{\sqrt{2x}}}-\frac{1}{p}\log\frac{1+x^p}{2}\quad(p\neq 0), \tag{2.27}$$

$$f(x,0)=\lim_{p\to 0}\log\frac{U(1,x)}{M_p(1,x)}=\log\frac{1-x}{\sqrt{2}\arctan\frac{1-x}{\sqrt{2x}}}-\frac{1}{2}\log x. \tag{2.28}$$

Then the following statements are true:

(1) *$f(x,p)$ is strictly increasing with respect to x on $(0,1)$ if and only if $p\geq 4/3$;*
(2) *$f(x,p)$ is strictly decreasing with respect to x on $(0,1)$ if and only if $p\leq 1/2$;*
(3) *If $1/2<p<4/3$, then there exists $\mu\in(0,1)$ such that $f(x,p)$ is strictly increasing with respect to x on $(0,\mu]$ and strictly decreasing with respect to x on $[\mu,1)$.*

Proof It follows from (2.27) and (2.28) that

$$\frac{\partial f(x,p)}{\partial x}=\frac{1+x^{p-1}}{\sqrt{2}(1-x)(1+x^p)\arctan\frac{1-x}{\sqrt{2x}}}f_1(x,p), \tag{2.29}$$

where $f_1(x,p)$ is defined by (2.1).

Therefore, parts (1) and (2) follow from Lemma 2.2 and (2.29).

Next, we prove part (3). If $1/2<p<4/3$, then (2.1) leads to

$$f_1\bigl(0^+,p\bigr)=+\infty,\qquad f_1(1,p)=0. \tag{2.30}$$

From Lemma 2.5 and (2.30) we clearly see that there exists $\mu\in(0,1)$ such that $f_1(x,p)>0$ for $x\in(0,\mu)$ and $f_1(x,p)<0$ for $x\in(\mu,1)$.

Therefore, part (3) follows from (2.29) and the fact that $f_1(x,p)>0$ for $x\in(0,\mu)$ and $f_1(x,p)<0$ for $x\in(\mu,1)$. □

3 Main results

Theorem 3.1 *The double inequality*

$$M_p(a,b)<U(a,b)<M_q(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $p\leq p_0=2\log 2/(2\log\pi-\log 2)=0.8684\cdots$ and $q\geq 4/3$.

Proof Since both the Yang mean $U(a,b)$ and the rth power mean $M_r(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a=1$ and $b=x\in(0,1)$.

We first prove that the inequality $U(1,x)<M_q(1,x)$ holds for all $x\in(0,1)$ if and only if $q\geq 4/3$.

If $q=4/3$, then from (2.27) and Lemma 2.6(1) we get

$$\log\frac{U(1,x)}{M_{4/3}(1,x)}=f\left(x,\frac{4}{3}\right)<f\left(1^-,\frac{4}{3}\right)=0 \tag{3.1}$$

for all $x\in(0,1)$.

Therefore, $U(1,x) < M_q(1,x)$ for all $x \in (0,1)$ and $q \geq 4/3$ follows from (3.1) and the monotonicity of the function $q \to M_q(1,x)$.

If $U(1,x) < M_q(1,x)$, then (2.27) and (2.28) lead to $f(x,q) < 0$ for all $x \in (0,1)$. In particular, we have

$$\lim_{x\to 1^-} \frac{f(x,q)}{(1-x)^2} = \frac{1}{8}\left(\frac{4}{3} - q\right) \leq 0$$

and $q \geq 4/3$.

Next, we prove that the inequality $U(1,x) > M_p(1,x)$ holds for all $x \in (0,1)$ if and only if $p \leq p_0$.

If $U(1,x) > M_p(1,x)$ holds for all $x \in (0,1)$, then (2.27) leads to $f(x,p) > 0$ for all $x \in (0,1)$. In particular, we have

$$f(0^+,p) = \left(\frac{1}{p} + \frac{1}{2}\right)\log 2 - \log\pi \geq 0. \tag{3.2}$$

We claim that $p \leq p_0$. Indeed, $p \leq p_0$ follows from (3.2) if $p > 0$, and $p < p_0$ is obvious if $p < 0$.

If $p = p_0$, then (2.27) leads to

$$f(0^+,p_0) = f(1,p_0) = 0. \tag{3.3}$$

It follows from (2.27) and (3.3) together with Lemma 2.6(3) that

$$\log\frac{U(1,x)}{M_{p_0}(1,x)} = f(x,p_0) > 0 \tag{3.4}$$

for all $x \in (0,1)$.

Therefore, $U(1,x) > M_p(1,x)$ for all $x \in (0,1)$ and $p \leq p_0$ follows from (3.4) and the monotonicity of the function $p \to M_p(1,x)$. □

Theorem 3.2 *Let $a,b > 0$ with $a \neq b$. Then the double inequality*

$$\frac{2^{5/4}}{\pi}M_{4/3}(a,b) < U(a,b) < \frac{2^{5/2}}{\pi}M_{1/2}(a,b)$$

holds with the best possible constants $2^{5/4}/\pi$ and $2^{5/2}/\pi$.

Proof It follows from Lemma 2.6(1) and (2) together with (2.27) that

$$\log\frac{U(1,x)}{M_{1/2}(1,x)} = f\left(x,\frac{1}{2}\right) < f\left(0^+,\frac{1}{2}\right) = \log\frac{2^{5/2}}{\pi} \tag{3.5}$$

and

$$\log\frac{U(1,x)}{M_{4/3}(1,x)} = f\left(x,\frac{4}{3}\right) > f\left(0^+,\frac{4}{3}\right) = \log\frac{2^{5/4}}{\pi} \tag{3.6}$$

for all $x \in (0,1)$.

Therefore, $2^{5/4}/\pi M_{4/3}(1,x) < U(1,x) < 2^{5/2}/\pi M_{1/2}(1,x)$ for all $x \in (0,1)$ follows from (3.5) and (3.6), and the optimality of the parameters $2^{5/4}/\pi$ and $2^{5/2}/\pi$ follows from the monotonicity of the functions $f(x,1/2)$ and $f(x,4/3)$. □

Remark 3.1 For all $a_1, a_2, b_1, b_2 > 0$ with $a_1/b_1 < a_2/b_2 < 1$. Then from Lemma 2.6(1) and (2) together with (2.27) we clearly see that the Ky Fan type inequalities

$$\frac{M_p(a_2,b_2)}{M_p(a_1,b_1)} < \frac{U(a_2,b_2)}{U(a_1,b_1)} < \frac{M_q(a_2,b_2)}{M_q(a_1,b_1)}$$

hold if and only if $p \geq 4/3$ and $q \leq 1/2$.

Let $p \in \mathbb{R}$ and $L_p(a,b) = (a^{p+1} + b^{p+1})/(a^p + b^p)$ be the pth Lehmer mean of two positive real numbers and a and b. Then the function $f_1(x,p)$ defined by (2.1) can be rewritten as

$$f_1(x,p) = (1-x)\left[\frac{A(1,x)L_{p-1}(1,x)}{G(1,x)Q^2(1,x)} - \frac{1}{U(1,x)}\right]. \tag{3.7}$$

From Lemma 2.2 and (3.7) we get Remark 3.2.

Remark 3.2 The double inequality

$$\frac{G(a,b)Q^2(a,b)}{A(a,b)L_{p-1}(a,b)} < U(a,b) < \frac{G(a,b)Q^2(a,b)}{A(a,b)L_{q-1}(a,b)}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \geq 4/3$ and $q \leq 1/2$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Z-HY provided the main idea and carried out the proof of Lemmas 2.1 and 2.2. L-MW carried out the proof of Lemmas 2.3-2.6. Y-MC carried out the proof of Theorems 3.1 and 3.2. All authors read and approved the final manuscript.

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