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Lacunary Δ -statistical convergence in intuitionistic fuzzy n -normed space

Selma Altundağ* and Esra Kamber

*Correspondence:
scaylan@sakarya.edu.tr
Department of Mathematics,
Sakarya University, Sakarya, 54187,
Turkey

Abstract

The concept of lacunary statistical convergence was introduced in intuitionistic fuzzy n -normed spaces in Sen and Debnath (Math. Comput. Model. 54:2978-2985, 2011). In this article, we introduce the notion of lacunary Δ -statistically convergent and lacunary Δ -statistically Cauchy sequences in an intuitionistic fuzzy n -normed space. Also, we give their properties using lacunary density and prove relation between these notions.

MSC: 47H10; 54H25

Keywords: statistical convergence; lacunary sequence; difference sequence; intuitionistic fuzzy n -normed space

1 Introduction

Fuzzy set theory was introduced by Zadeh [1] in 1965. This theory has been applied not only in different branches of engineering such as in nonlinear dynamic systems [2], in the population dynamics [3], in the quantum physics [4], but also in many fields of mathematics such as in metric and topological spaces [5–7], in the theory of functions [8, 9], in the approximation theory [10]. 2-normed and n -normed linear spaces were initially introduced by Gähler [11, 12] and further studied by Kim and Cho [13], Malceski [14] and Gunawan and Mashadi [15]. Vijayabalaji and Narayanan [16] defined fuzzy n -normed linear space. After Saadati and Park [17] introduced the concept of intuitionistic fuzzy normed space, Vijayabalaji *et al.* [18] defined the notion of intuitionistic fuzzy n -normed space. The notion of statistical convergence was investigated by Steinhaus [19] and Fast [20]. Then a lot of authors applied this concept to probabilistic normed spaces [21, 22], random 2-normed spaces [23] and finally intuitionistic fuzzy normed spaces [24, 25]. Fridy and Orhan [26] introduced the idea of lacunary statistical convergence. Using this idea, Mursaleen and Mohiuddine [27], Sen and Debnath [28] investigated lacunary statistical convergence in intuitionistic fuzzy normed spaces and intuitionistic fuzzy n -normed spaces, respectively. The idea of difference sequences was introduced by Kızmaz [29] where $\Delta x = (\Delta x_k) = x_k - x_{k+1}$. Başarır [30] introduced the Δ -statistical convergence of sequences. Bilgin [31] introduced the definition of lacunary strongly Δ -convergence of fuzzy numbers. Hazarika [32] gave the definition of lacunary generalized difference statistical convergence in random 2-normed spaces. Also, the generalized difference sequence spaces were studied by various authors [33–35]. In this article, we shall introduce lacunary Δ -statistical convergence and lacunary Δ -statistically Cauchy sequences in IFnNLS.

2 Preliminaries, background and notation

In this section, we give the basic definitions.

Definition 2.1 ([27]) A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t -norm if it satisfies the following conditions:

- (i) $*$ is associative and commutative,
- (ii) $*$ is continuous,
- (iii) $a * 1 = a$ for all $a \in [0, 1]$,
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 2.2 ([27]) A binary operation \circ : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t -conorm if it satisfies the following conditions:

- (i) \circ is associative and commutative,
- (ii) \circ is continuous,
- (iii) $a \circ 0 = a$ for all $a \in [0, 1]$,
- (iv) $a \circ b \leq c \circ d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 2.3 ([27]) Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$ (here we allow it to be infinite). A real-valued function $\|\bullet, \dots, \bullet\|$ on $X \times \dots \times X = X^n$ satisfying the following four properties:

- (i) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent,
- (ii) x_1, x_2, \dots, x_n are invariant under any permutation,
- (iii) $\|x_1, x_2, \dots, \alpha x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{R}$,
- (iv) $\|x_1, x_2, \dots, x_{n-1}, y + z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$,

is called an n -norm on X and the pair is called an n -normed space.

Definition 2.4 ([28]) An IFnNLS is the five-tuple $(X, \mu, \nu, *, \circ)$ where X is a linear space over a field F , $*$ is a continuous t -norm, \circ is a continuous t -conorm, μ, ν are fuzzy sets on $X^n \times (0, \infty)$, μ denotes the degree of membership and ν denotes the degree of non-membership of $(x_1, x_2, \dots, x_n, t) \in X^n \times (0, \infty)$ satisfying the following conditions for every $(x_1, x_2, \dots, x_n) \in X^n$ and $s, t > 0$:

- (i) $\mu(x_1, x_2, \dots, x_n, t) + \nu(x_1, x_2, \dots, x_n, t) \leq 1$,
- (ii) $\mu(x_1, x_2, \dots, x_n, t) > 0$,
- (iii) $\mu(x_1, x_2, \dots, x_n, t) = 1$ if and only if x_1, x_2, \dots, x_n are linearly dependent,
- (iv) $\mu(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of x_1, x_2, \dots, x_n ,
- (v) $\mu(x_1, x_2, \dots, cx_n, t) = \mu(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ for all $c \neq 0, c \in F$,
- (vi) $\mu(x_1, x_2, \dots, x_n, s) * \mu(x_1, x_2, \dots, x'_n, t) \leq \mu(x_1, x_2, \dots, x_n + x'_n, s + t)$,
- (vii) $\mu(x_1, x_2, \dots, x_n, t) : (0, \infty) \rightarrow [0, 1]$ is continuous in t ,
- (viii) $\lim_{t \rightarrow \infty} \mu(x_1, x_2, \dots, x_n, t) = 1$ and $\lim_{t \rightarrow 0} \mu(x_1, x_2, \dots, x_n, t) = 0$,
- (ix) $\nu(x_1, x_2, \dots, x_n, t) < 1$,
- (x) $\nu(x_1, x_2, \dots, x_n, t) = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent,
- (xi) $\nu(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of x_1, x_2, \dots, x_n ,
- (xii) $\nu(x_1, x_2, \dots, cx_n, t) = \nu(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ for all $c \neq 0, c \in F$,
- (xiii) $\nu(x_1, x_2, \dots, x_n, s) \circ \nu(x_1, x_2, \dots, x'_n, t) \geq \nu(x_1, x_2, \dots, x_n + x'_n, s + t)$
- (xiv) $\nu(x_1, x_2, \dots, x_n, t) : (0, \infty) \rightarrow [0, 1]$ is continuous in t ,
- (xv) $\lim_{t \rightarrow \infty} \nu(x_1, x_2, \dots, x_n, t) = 0$ and $\lim_{t \rightarrow 0} \nu(x_1, x_2, \dots, x_n, t) = 1$.

Example 2.1 ([28]) Let $(X, \|\bullet, \dots, \bullet\|)$ be an n -normed linear space. Also let $a * b = ab$ and $a \circ b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$,

$$\mu(x_1, x_2, \dots, x_n, t) = \frac{t}{t + \|x_1, x_2, \dots, x_n\|} \quad \text{and} \quad \nu(x_1, x_2, \dots, x_n, t) = \frac{\|x_1, x_2, \dots, x_n\|}{t + \|x_1, x_2, \dots, x_n\|}.$$

Then $(X, \mu, \nu, *, \circ)$ is an IFnNLS.

Definition 2.5 ([26]) A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated as q_r . Let $K \subseteq \mathbb{N}$. The number

$$\delta_\theta(K) = \lim_r \frac{1}{h_r} |\{k \in I_r : k \in K\}|$$

is said to be the θ -density of K , provided the limit exists.

Definition 2.6 ([28]) Let θ be a lacunary sequence. A sequence $x = \{x_k\}$ of numbers is said to be lacunary statistically convergent (or S_θ -convergent) to the number L if for every $\varepsilon > 0$, the set $K(\varepsilon)$ has θ -density zero, where

$$K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}.$$

In this case, we write $S_\theta\text{-lim } x = L$.

3 Δ -Convergence and lacunary Δ -statistical convergence in IFnNLS

In this section, we define Δ -convergence and lacunary Δ -statistical convergence in intuitionistic fuzzy n -normed spaces.

Definition 3.1 Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. A sequence $x = \{x_k\}$ in X is said to be Δ -convergent to $L \in X$ with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$ if, for every $\varepsilon > 0, t > 0$ and $y_1, y_2, \dots, y_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \varepsilon$ and $\nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \varepsilon$ for all $k \geq k_0$, where $k \in \mathbb{N}$ and $\Delta x_k = (x_k - x_{k+1})$. It is denoted by $(\mu, \nu)^n\text{-lim } \Delta x = L$ or $\Delta x_k \rightarrow L$ as $k \rightarrow \infty$.

Definition 3.2 Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. A sequence $x = \{x_k\}$ in X is said to be lacunary Δ -statistically convergent or $S_\theta(\Delta)$ -convergent to $L \in X$ with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$ provided that for every $\varepsilon > 0, t > 0$ and $y_1, y_2, \dots, y_{n-1} \in \mathbb{X}$,

$$\delta_\theta(\Delta) (\{k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon \\ \text{or } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon\}) = 0,$$

or, equivalently,

$$\delta_\theta(\Delta) (\{k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \varepsilon \\ \text{and } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \varepsilon\}) = 1.$$

It is denoted by $S_\theta^{(\mu, \nu)^n}(\Delta)\text{-lim } x = L$ or $x_k \rightarrow L(S_\theta(\Delta))$. Using Definition 3.2 and properties of the θ -density, we can easily obtain the following lemma.

Lemma 3.1 *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS and θ be a lacunary sequence. Then, for every $\varepsilon > 0, t > 0$ and $y_1, y_2, \dots, y_{n-1} \in \mathbb{X}$, the following statements are equivalent:*

- (i) $S_\theta^{(\mu, \nu)^n}(\Delta)\text{-lim } x = L$,
- (ii) $\delta_\theta(\Delta)(\{k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon\}) = \delta_\theta(\Delta)(\{k \in \mathbb{N} : \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon\}) = 0$,
- (iii) $\delta_\theta(\Delta)(\{k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \varepsilon$ and $\nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \varepsilon\}) = 1$,
- (iv) $\delta_\theta(\Delta)(\{k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \varepsilon\}) = \delta_\theta(\Delta)(\{k \in \mathbb{N} : \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \varepsilon\}) = 1$,
- (v) $S_\theta\text{-lim } \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) = 1$ and $S_\theta\text{-lim } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) = 0$.

Proceeding exactly in a similar way as in [36], the following theorem can be proved.

Theorem 3.1 *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS and θ be a lacunary sequence. If a sequence $x = \{x_k\}$ in X is lacunary Δ -statistically convergent or $S_\theta(\Delta)$ -convergent to $L \in X$ with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n, S_\theta^{(\mu, \nu)^n}(\Delta)\text{-lim } x$ is unique.*

Theorem 3.2 *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS and θ be a lacunary sequence. If $(\mu, \nu)^n\text{-lim } \Delta x = L$, then $S_\theta^{(\mu, \nu)^n}(\Delta)\text{-lim } x = L$.*

Proof Let $(\mu, \nu)^n\text{-lim } \Delta x = L$. Then, for every $\varepsilon > 0, t > 0$ and $y_1, y_2, \dots, y_{n-1} \in \mathbb{X}$, there exists $k_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \varepsilon$ and $\nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \varepsilon$ for all $k \geq k_0$. Hence the set

$$\{k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon$$

$$\text{or } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon\}$$

has a finite number of terms. Since every finite subset of \mathbb{N} has lacunary density zero,

$$\delta_\theta(\Delta)(\{k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon$$

$$\text{or } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon\}) = 0,$$

that is, $S_\theta^{(\mu, \nu)^n}(\Delta)\text{-lim } x = L$.

It follows from the following example that the converse of Theorem 3.2 is not true in general.

Example 3.1 Consider $X = \mathbb{R}^n$ with

$$\|x_1, x_2, \dots, x_n\| = \text{abs} \left(\begin{array}{ccc} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{array} \right),$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$, and let $a * b = ab$, $a \circ b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. Now, for all $y_1, y_2, \dots, y_{n-1}, x \in \mathbb{R}^n$ and $t > 0$, $\mu(y_1, y_2, \dots, y_{n-1}, x, t) = \frac{t}{t + \|y_1, y_2, \dots, y_{n-1}, x\|}$ and $\nu(y_1, y_2, \dots, y_{n-1}, x, t) = \frac{\|y_1, y_2, \dots, y_{n-1}, x\|}{t + \|y_1, y_2, \dots, y_{n-1}, x\|}$. Then $(\mathbb{R}^n, \mu, \nu, *, \circ)$ is an IFnNLS. Let I_r and h_r be as defined in Definition 2.5. Define a sequence $x = \{x_k\}$ whose terms are given by

$$x_k = \begin{cases} \left(\frac{(n - [\sqrt{h_r}] + 1)(-n + [\sqrt{h_r}])}{2}, 0, \dots, 0 \right) \in \mathbb{R}^n & \text{if } 1 \leq k \leq n - [\sqrt{h_r}], \\ \left(-\frac{1}{2}k^2 + \frac{1}{2}k, 0, \dots, 0 \right) \in \mathbb{R}^n & \text{if } n - [\sqrt{h_r}] + 1 \leq k \leq n, \\ \left(-\frac{1}{2}n^2 - \frac{1}{2}n, 0, \dots, 0 \right) \in \mathbb{R}^n & \text{if } k > n \end{cases}$$

such that

$$\Delta x_k = \begin{cases} (k, 0, \dots, 0) \in \mathbb{N} & \text{if } n - [\sqrt{h_r}] + 1 \leq k \leq n, \\ (0, 0, \dots, 0) \in \mathbb{N} & \text{otherwise.} \end{cases}$$

For every $0 < \varepsilon < 1$ and for any $y_1, y_2, \dots, y_{n-1} \in X$, $t > 0$, let

$$K(\varepsilon, t) = \left\{ k \in I_r : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon \right. \\ \left. \text{or } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \right\}.$$

Now,

$$K(\varepsilon, t) = \left\{ k \in I_r : \|y_1, y_2, \dots, y_{n-1}, \Delta x_k\| \geq \frac{\varepsilon t}{1 - \varepsilon} > 0 \right\} \\ \subseteq \left\{ k \in I_r : \Delta x_k = (k, 0, \dots, 0) \in \mathbb{R}^n \right\}.$$

Thus we have $\frac{1}{h_r} |\{k \in I_r : k \in K(\varepsilon, t)\}| \leq \frac{[\sqrt{h_r}]}{h_r} \rightarrow 0$ as $r \rightarrow \infty$. Hence $S_\theta^{(\mu, \nu)^n}(\Delta)\text{-lim } x = 0$.

On the other hand, $x = \{x_k\}$ in X is not Δ -convergent to 0 with respect to the intuitionistic fuzzy n -norm since

$$\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k, t) = \frac{t}{t + \|y_1, y_2, \dots, y_{n-1}, \Delta x_k\|} \\ = \begin{cases} \frac{t}{t + \|y_1, y_2, \dots, y_{n-1}, \Delta x_k\|} & \text{if } n - [\sqrt{h_r}] + 1 \leq k \leq n, \\ 1, & \text{otherwise,} \end{cases} \\ \leq 1$$

and

$$\nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k, t) = \frac{\|y_1, y_2, \dots, y_{n-1}, \Delta x_k\|}{t + \|y_1, y_2, \dots, y_{n-1}, \Delta x_k\|} \\ = \begin{cases} \frac{\|y_1, y_2, \dots, y_{n-1}, \Delta x_k\|}{t + \|y_1, y_2, \dots, y_{n-1}, \Delta x_k\|} & \text{if } n - [\sqrt{h_r}] + 1 \leq k \leq n, \\ 0, & \text{otherwise} \end{cases} \\ \geq 0.$$

This completes the proof of the theorem. □

Theorem 3.3 *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. Then $S_{\theta}^{(\mu, \nu)^n}(\Delta)$ - $\lim x = L$ if and only if there exists an increasing sequence $K = \{k_n\}$ of the natural numbers such that $\delta_{\theta}(\Delta)(K) = 1$ and $(\mu, \nu)^n$ - $\lim_{k \in K} \Delta x_k = L$.*

Proof Necessity. Suppose that $S_{\theta}^{(\mu, \nu)^n}(\Delta)$ - $\lim x = L$. Then, for every $y_1, y_2, \dots, y_{n-1} \in \mathbb{X}$, $t > 0$ and $j = 1, 2, \dots$,

$$K(j, t) = \left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{j} \right. \\ \left. \text{and } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{j} \right\} \text{ and} \\ M(j, t) = \left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \frac{1}{j} \right. \\ \left. \text{or } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \frac{1}{j} \right\}.$$

Then $\delta_{\theta}(\Delta)(M(j, t)) = 0$ since

$$K(j, t) \supset K(j + 1, t) \tag{3.1}$$

and

$$\delta_{\theta}(\Delta)(K(j, t)) = 1 \tag{3.2}$$

for $t > 0$ and $j = 1, 2, \dots$. Now we have to show that for $k \in K(j, t)$ suppose that for some $k \in K(j, t)$, $x = \{x_k\}$ not Δ -convergent to L with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$. Therefore there is $\alpha > 0$ and a positive integer k_0 such that $\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \alpha$ or $\nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \alpha$ for all $k \geq k_0$. Let $\alpha > \frac{1}{j}$ and

$$K(\alpha, t) = \left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \alpha \right. \\ \left. \text{and } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \alpha \right\}.$$

Then $\delta_{\theta}(\Delta)(K(\alpha, t)) = 0$. Since $\alpha > \frac{1}{j}$, by (3.1) we have $\delta_{\theta}(\Delta)(K(j, t)) = 0$, which contradicts by equation (3.2).

Sufficiency. Suppose that there exists an increasing sequence $K = \{k_n\}$ of the natural numbers such that $\delta_{\theta}(\Delta)(K) = 1$ and $(\mu, \nu)^n$ - $\lim_{k \in K} \Delta x_k = L$, i.e., for every $y_1, y_2, \dots, y_{n-1} \in \mathbb{X}$, $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \varepsilon$ and $\nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \varepsilon$.

Let

$$M(\varepsilon, t) := \left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon \right. \\ \left. \text{or } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \right\} \\ \subseteq -\{k_{n_0+1}, k_{n_0+2}, \dots\}$$

and consequently $\delta_\theta(\Delta)(M(\varepsilon, t)) \leq 1 - 1 = 0$. Hence $S_\theta^{(\mu, \nu)^n}(\Delta)\text{-lim } x = L$. This completes proof of the theorem. \square

Theorem 3.4 *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. Then $S_\theta^{(\mu, \nu)^n}(\Delta)\text{-lim } x = L$ if and only if there exist a convergent sequence $y = \{y_k\}$ and a lacunary Δ -statistically null sequence $z = \{z_k\}$ with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$ such that $(\mu, \nu)^n\text{-lim } y = L$, $\Delta x = y + \Delta z$ and $\delta_\theta(\Delta)(\{k \in \mathbb{N} : \Delta z_k = 0\}) = 1$.*

Proof Necessity. Suppose that $S_\theta^{(\mu, \nu)^n}(\Delta)\text{-lim } x = L$ and

$$K(j, t) = \left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{j} \right. \\ \left. \text{and } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{j} \right\}.$$

Using Theorem 3.3 for any $y_1, y_2, \dots, y_{n-1} \in X$, $t > 0$ and $j \in \mathbb{N}$, we can construct an increasing index sequence $\{r_j\}$ of the natural numbers such that $r_j \in K(j, t)$, $\delta_\theta(\Delta)(K(j, t)) = 1$, and so we can conclude that for all $r > r_j$ ($j \in \mathbb{N}$),

$$\frac{1}{h_r} \left| \left\{ k \in I_r : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{j} \right. \right. \\ \left. \left. \text{and } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{j} \right\} \right| > \frac{j-1}{j}.$$

We define $y = \{y_k\}$ and $z = \{z_k\}$ as follows. If $1 < k < r_1$, we set $y_k = \Delta x_k$ and $z_k = 0$. Now suppose that $j \geq 1$ and $r_j < k \leq r_{j+1}$. If $k \in K(j, t)$, i.e., $\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{j}$ and $\nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{j}$, we set $y_k = \Delta x_k$ and $\Delta z_k = 0$. Otherwise $y_k = L$ and $\Delta z_k = \Delta x_k - L$. Hence it is clear that $\Delta x = y + \Delta z$.

We claim that $(\mu, \nu)^n\text{-lim } y = L$. Let $\varepsilon > \frac{1}{j}$. If $k \in K(j, t)$ for all $k > r_j$, $\mu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) > 1 - \varepsilon$ and $\nu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) < \varepsilon$. Since ε was arbitrary, we have proved the claim.

Next we claim that $z = \{z_k\}$ is a lacunary Δ -statistically null sequence with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$, i.e., $S_\theta^{(\mu, \nu)^n}(\Delta)\text{-lim } z = 0$. It suffices to see that $\delta_\theta(\Delta)(\{k \in \mathbb{N} : \Delta z_k = 0\}) = 1$ to prove the claim. This follows from observing that

$$\left| \{k \in I_r : \Delta z_k = 0\} \right| \\ \leq \left| \left\{ k \in I_r : \mu(y_1, y_2, \dots, y_{n-1}, \Delta z_k, t) > 1 - \varepsilon \text{ and } \nu(y_1, y_2, \dots, y_{n-1}, \Delta z_k, t) < \varepsilon \right\} \right|$$

for any $r \in \mathbb{N}$ and $\varepsilon > 0$.

We show that if $\delta > 0$ and $j \in \mathbb{N}$ such that $\frac{1}{j} < \delta$, then

$$\frac{1}{h_r} \left| \{k \in I_r : \Delta z_k = 0\} \right| > 1 - \delta$$

for all $r > r_j$. Recall from the construction that if $k \in K(j, t)$, then $\Delta z_k = 0$ for $r_j < k \leq r_{j+1}$.

Now, for $t > 0$ and $s \in \mathbb{N}$, let

$$K(s, t) = \left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{s} \right. \\ \left. \text{and } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{s} \right\}.$$

For $s > j$ and $r_s < k \leq r_{s+1}$ by (3.2),

$$K(s, t) = \left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{s} \right. \\ \left. \text{and } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{s} \right\} \\ \subset \{k \in \mathbb{N} : \Delta z_k = 0\}.$$

Consequently, if $r_s < k \leq r_{s+1}$ and $s > j$, then

$$\frac{1}{h_r} \left| \{k \in I_r : \Delta z_k = 0\} \right| \\ \geq \frac{1}{h_r} \left| \left\{ k \in I_r : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) > 1 - \frac{1}{s} \right. \right. \\ \left. \left. \text{and } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) < \frac{1}{s} \right\} \right| \\ > 1 - \frac{1}{s} > 1 - \frac{1}{j} > 1 - \delta.$$

Hence we get $\delta_\theta(\Delta)(\{k \in \mathbb{N} : \Delta z_k = 0\}) = 1$, which establishes the claim.

Sufficiency. Let x , y and z be sequences such that $(\mu, \nu)^n\text{-lim } y = L$, $\Delta x = y + \Delta z$ and $\delta_\theta(\Delta)(\{k \in \mathbb{N} : \Delta z_k = 0\}) = 1$. Then, for any $y_1, y_2, \dots, y_{n-1} \in X$, $\varepsilon > 0$ and $t > 0$, we have

$$\{k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon\} \\ \subseteq \{k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \geq \varepsilon\} \\ \cup \{k \in \mathbb{N} : \Delta z_k \neq 0\}.$$

Therefore

$$\delta_\theta(\Delta)(\{k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon \\ \text{or } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon\}) \\ \leq \delta_\theta(\{k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \geq \varepsilon\}) \\ + \delta_\theta(\Delta)(\{k \in \mathbb{N} : \Delta z_k \neq 0\}).$$

Since $(\mu, \nu)^n\text{-lim } y = L$, the set

$$\{k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \geq \varepsilon\}$$

contains at most finitely many terms and thus

$$\delta_\theta \left(\left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(y_1, y_2, \dots, y_{n-1}, y_k - L, t) \geq \varepsilon \right\} \right).$$

Also by hypothesis, $\delta_\theta(\Delta)(\{k \in \mathbb{N} : \Delta z_k \neq 0\})$. Hence,

$$\delta_\theta(\Delta) \left(\left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \leq 1 - \varepsilon \right. \right. \\ \left. \left. \text{or } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t) \geq \varepsilon \right\} \right) = 0,$$

and consequently $S_\theta^{(\mu, \nu)^n}(\Delta)\text{-lim } x = L$. □

4 Δ -Cauchy and lacunary Δ -statistically Cauchy sequences in IFnNLS

In this section, we introduce the notion of Cauchy sequences and lacunary statistically Cauchy sequences in IFnNLS.

Definition 4.1 Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. A sequence $x = \{x_k\}$ in X is said to be Δ -Cauchy with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$ if, for every $\varepsilon > 0$, $t > 0$ and $y_1, y_2, \dots, y_{n-1} \in \mathbb{X}$, there exists $k_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_m, t) > 1 - \varepsilon$ and $\nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_m, t) < \varepsilon$ for all $k, m \geq k_0$.

Definition 4.2 Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. A sequence $x = \{x_k\}$ in X is said to be lacunary Δ -statistically Cauchy or $S_\theta(\Delta)$ -Cauchy with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$ if, for every $\varepsilon > 0$, $t > 0$ and $y_1, y_2, \dots, y_{n-1} \in \mathbb{X}$, there exists a number $m \in \mathbb{N}$ satisfying

$$\delta_\theta(\Delta) \left(\left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_m, t) \leq 1 - \varepsilon \right. \right. \\ \left. \left. \text{or } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_m, t) \geq \varepsilon \right\} \right) = 0.$$

Theorem 4.1 Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. If a sequence $x = \{x_k\}$ in X is lacunary Δ -statistically convergent with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$ if and only if it is lacunary Δ -statistically Cauchy with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$.

Proof Let $x = \{x_k\}$ be a lacunary Δ -statistically convergent sequence which converges to L . For a given $\varepsilon > 0$, choose $s > 0$ such that $(1 - \varepsilon) * (1 - \varepsilon) > 1 - s$ and $\varepsilon \circ \varepsilon < s$. Let

$$A(\varepsilon, t) = \left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t/2) \leq 1 - \varepsilon \right. \\ \left. \text{or } \nu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t/2) \geq \varepsilon \right\}.$$

Then, for any $t > 0$ and $y_1, y_2, \dots, y_{n-1} \in \mathbb{X}$,

$$\delta_\theta(\Delta)(A(\varepsilon, t)) = 0, \tag{4.1}$$

which implies that $\delta_\theta(\Delta)(A^c(\varepsilon, t)) = 1$.

Let $q \in A^c(\varepsilon, t)$. Then

$$\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_q - L, t/2) > 1 - \varepsilon$$

and

$$v(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t/2) < \varepsilon.$$

Now, let

$$B(s, t) = \left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_q, t) \leq 1 - s \right. \\ \left. \text{or } v(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_q, t) \geq s \right\}.$$

We need to show that $B(s, t) \subset A(\varepsilon, t)$. Let $k \in B(s, t) \cap A^c(\varepsilon, t)$. Hence $\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_q, t) \leq 1 - s$ and $\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t/2) > 1 - \varepsilon$, in particular, $\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_q - L, t/2) > 1 - \varepsilon$. Then

$$1 - s \geq \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_q, t) \\ \geq \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t/2) * \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_q - L, t/2) \\ > (1 - \varepsilon) * (1 - \varepsilon) > 1 - s,$$

which is not possible. On the other hand, $v(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_q, t) \geq s$ and $v(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t/2) < \varepsilon$, in particular, $v(y_1, y_2, \dots, y_{n-1}, \Delta x_q - L, t/2) < \varepsilon$. Hence,

$$s \leq v(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_q, t) \\ \leq v(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t/2) \circ v(y_1, y_2, \dots, y_{n-1}, \Delta x_q - L, t/2) \\ < \varepsilon \circ \varepsilon < s,$$

which is not possible. Hence $B(s, t) \subset A(\varepsilon, t)$ and by (4.1) $\delta_\theta(\Delta)(B(\varepsilon, t)) = 0$. This proves that x is lacunary Δ -statistically Cauchy with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$.

Conversely, let $x = \{x_k\}$ be lacunary Δ -statistically Cauchy but not lacunary Δ -statistically convergent with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$. For a given $\varepsilon > 0$, choose $s > 0$ such that $(1 - \varepsilon) * (1 - \varepsilon) > 1 - s$ and $\varepsilon \circ \varepsilon < s$. Since x is not lacunary Δ -convergent

$$\mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_m, t) \\ \geq \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t/2) * \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_q - L, t/2) \\ > (1 - \varepsilon) * (1 - \varepsilon) > 1 - s, \\ v(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_m, t) \\ \leq v(y_1, y_2, \dots, y_{n-1}, \Delta x_k - L, t/2) \circ v(y_1, y_2, \dots, y_{n-1}, \Delta x_q - L, t/2) \\ < \varepsilon \circ \varepsilon < s.$$

Therefore $\delta_\theta(\Delta)(E^c(s, t)) = 0$, where

$$B(s, t) = \left\{ k \in \mathbb{N} : \mu(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_q, t) \leq 1 - s \right. \\ \left. \text{or } v(y_1, y_2, \dots, y_{n-1}, \Delta x_k - \Delta x_q, t) \geq s \right\}$$

and so $\delta_\theta(\Delta)(E(s, t)) = 1$, which is a contradiction, since x was lacunary Δ -statistically Cauchy with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$. So, x must be lacunary Δ -statistically convergent with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$. \square

Corollary 4.1 *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS and θ be a lacunary sequence. Then, for any sequence $x = \{x_k\}$ in X , the following conditions are equivalent:*

- (i) x is $S_\theta(\Delta)$ -convergent with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$.
- (ii) x is $S_\theta(\Delta)$ -Cauchy with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$.
- (iii) There exists an increasing sequence $K = \{k_n\}$ of the natural numbers such that $\delta_\theta(\Delta)(K) = 1$ and the subsequence $\{x_{k_n}\}$ is $S_\theta(\Delta)$ -Cauchy with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors did not provide this information.

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