On the pth moment estimates for the solution of stochastic differential equations

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Dedicated to Professor Shih-sen Chang on his 80th birthday.

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Abstract

The main aim of this paper is to discuss the L^p-estimate of solution of stochastic differential equations under nonlinear growth condition. Furthermore, almost surely asymptotic estimates for the solutions of these equations are given. More accurately, we shall estimate sample Lyapunov exponent almost surely.

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1 Introduction

Stochastic analysis including Brownian motion processes has played an important role in many areas of science and engineering for a long time. Since the white noise is mathematically represented by a formal derivative of a Brownian motion process, such stochastic analysis is based on various types of stochastic differential equations(SDEs) of Itô type as a stochastic model. After Itô introduced his stochastic calculus, the theory of SDEs has been developed very quickly. SDEs is the most fundamental concept in modern stochastic models.

Consequently, there is an increasing interest in SDEs. The main interest in the field has often referred to the existence and uniqueness of solutions as well as to study of their qualitative and quantitative properties, with a special emphasis on the analysis of various types of L^p-estimate, almost sure estimate, stability and approximation. We refer the reader to monographs [1] by Henderson and Plaschko, [2] by Kolmanovskii and Myshkis, and [3] by Mao, among others, and the literature cited therein.

It is well known that the problems of the solutions to SDEs have received considerable attention from the theoretical point of view. One of the classical results and a subject for inquiry in the study of SDEs is an existence and uniqueness theorem of the solution to SDEs under some special conditions. For the work on the existence and uniqueness theorem of the SDEs, we refer the reader to [4] by Cho *et al.*, [5] by Ren and Xia, [6] by Wei and Wang, and [3] by Mao. Now there is an extensive literature discussing SDEs with pth moment estimate and stability of the solution (see [7] by Govindan, [8] by Kim, [9] by Li and Fu, and [10] by Mao *et al.*).

Further, in the study of the solution for the SDEs, one question arises naturally: Does the pth moment of the solution assure the solution for such SDEs? To the best of our knowl-

edge, there are few results on this problem. It is also worth noting that the pth moment of the solution for such SDEs has not been fully investigated, which remains an interesting research topic.

Our study is essentially based on [3] by Mao referring to the L^p-estimate and almost surely asymptotic estimates of SDEs. In fact, we generalize Mao's results by finding sufficient conditions, which are easy to verify, guaranteeing L^p-estimate and almost surely asymptotic estimates of the solutions of these equations.

2 Preliminary

Let $|\cdot|$ denote an Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T; if A is a matrix, its trace norm is represented by $|A| = \sqrt{\operatorname{trace}(A^T A)}$. Let t_0 be a positive constant and $(\Omega, \mathcal{F}, P)$ throughout this paper, unless otherwise specified, be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ satisfying the usual conditions (*i.e.*, it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets). Assume that $B(t)$ is an m-dimensional Brownian motion defined on a complete probability space, that is, $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))^T$. Let $\mathcal{L}^p([a,b]; R^d)$ denote the family of R^d-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a \leq t \leq b}$ such that $\int_a^b |f(t)|^p \, dt < \infty$ a.s.

Consider the d-dimensional stochastic differential equations of Itô type

$$dx(t) = f(x(t), t) \, dt + g(x(t), t) \, dB(t) \quad \text{on } t_0 \leq t \leq T \tag{2.1}$$

with initial value $x(t_0) = x_0$, $f : R^d \times [t_0, T] \to R^d$, and $g : R^d \times [t_0, T] \to R^{d \times m}$ be Borel measurable.

In [3], the author presented a result stating that, for initial value $x(t_0) = x_0$, the pth moment of the unique solution $x(t)$, $t_0 \leq t \leq T$ of equation (2.1) under Lipschitz condition and linear growth condition will grow at most exponentially with exponent $p\alpha$. We reproduce the result here.

Theorem 2.1 *Let $p \geq 2$ and $x_0 \in L^p(\Omega; R^d)$. Assume that there exists a constant α such that for all $(x, t) \in R^d \times [t_0, T]$,*

$$x^T f(x, t) + \frac{p-1}{2} |g(x, t)|^2 \leq \alpha (1 + |x|^2). \tag{2.2}$$

Then

$$E|x(t)|^p \leq 2^{(p-2)/2} \left[1 + E|x_0|^p \right] \exp\left(p\alpha(t - t_0) \right). \tag{2.3}$$

3 Main results

The topic of our analysis is equation (2.1) with initial data $x(t_0) = x_0$. An $\{\mathcal{F}_t\}$-adapted process $x(t)$ with values in R^d is said to be the solution to equation (2.1) if it satisfies the initial condition and the corresponding stochastic integral equation holds a.s., *i.e.*, for every $t \geq t_0$,

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s) \, ds + \int_{t_0}^t g(x(s), s) \, dB(s) \quad \text{on } t_0 \leq t \leq T.$$

The basic existence and uniqueness theorem based on the Picard method of iterations requires the global or, in a weakened version, Lipschitz condition and linear growth condition for f and g (see [3–6]). Then there exists a unique a.s. continuous and adapted solution $x(t)$ to equation (2.1) satisfying $E|\sup_{t_0 \leq t \leq T} x(t)|^2 < \infty$ for every $T \geq t_0$ under the linear growth condition. Moreover, if $x_0 \in L^p(\Omega; R^d)$, $p \geq 2$, then $E|x(t)|^p < \infty$ under the monotone condition. Since our goal is to study L^p-estimate and almost surely asymptotic estimate problems, we assume that there exists a unique solution $x(t)$ to equation (2.1) under non-Lipschitz condition and nonlinear growth condition (see [5, 6]). We also assume that all the Lebesgue and Itô's integrals employed further are well defined.

We start with the following L^p-estimate.

Theorem 3.1 *Let $p \geq 2$ and $x_0 \in L^p(\Omega; R^d)$. Assume that for all $(x, t) \in R^d \times [t_0, T]$, $\{f(x(t), t)\} \in \mathcal{L}^1([t_0, T]; R^d)$, and $\{g(x(t), t)\} \in \mathcal{L}^2([t_0, T]; R^d)$, it follows that*

$$x^T f\big(x(t), t\big) \vee \frac{p-1}{2}\big|g\big(x(t), t\big)\big|^2 \leq \kappa\big(1 + \big|x(t)\big|^2\big), \tag{3.1}$$

where $\kappa(\cdot)$ is a concave nondecreasing function from R_+ to R_+ such that $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$. Then, for a pair of positive constants a and b such that $\kappa(u) \leq a + bu$, we have

$$E\big|x(t)\big|^p \leq 2^{(p-2)/2}\big[1 + E|x_0|^p\big]\exp\big(2p(a+b)(t - t_0)\big) \tag{3.2}$$

for all $t \in [t_0, T]$.

Proof By Itô's formula, we can derive that for $t \in [t_0, T]$,

$$\big[1 + \big|x(t)\big|^2\big]^{p/2} = \big[1 + \big|x(t_0)\big|^2\big]^{p/2} + p\int_{t_0}^t \big[1 + \big|x(s)\big|^2\big]^{(p-2)/2}x^T(s)f\big(x(s), s\big)\,ds$$

$$+ \frac{p}{2}\int_{t_0}^t \big[1 + \big|x(s)\big|^2\big]^{(p-2)/2}\big|g\big(x(s), s\big)\big|^2\,ds$$

$$+ \frac{p(p-2)}{2}\int_{t_0}^t \big[1 + \big|x(s)\big|^2\big]^{(p-4)/2}\big|x^T(s)g\big(x(s), s\big)\big|^2\,ds$$

$$+ p\int_{t_0}^t \big[1 + \big|x(s)\big|^2\big]^{(p-2)/2}x^T(s)g\big(x(s), s\big)\,dB(s).$$

By condition (3.1), it is easy to see that

$$\big[1 + \big|x(t)\big|^2\big]^{\frac{p}{2}} \leq 2^{\frac{p-2}{2}}\big(1 + |x_0|^p\big) + 2p\int_{t_0}^t \big[1 + \big|x(s)\big|^2\big]^{\frac{p-2}{2}}\kappa\big(1 + \big|x(s)\big|^2\big)\,ds$$

$$+ p\int_{t_0}^t \big[1 + \big|x(s)\big|^2\big]^{(p-2)/2}x^T(s)g\big(x(s), s\big)\,dB(s). \tag{3.3}$$

Given that $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, we can find a pair of positive constants a and b such that $\kappa(u) \leq a + bu$ for all $u \geq 0$. And for each number $n \geq 1$, define the stopping time

$$\tau_n = T \wedge \inf\big\{t \in [t_0, T] : \big|x(t)\big| \geq n\big\}.$$

Obviously, as $n \to \infty$, $\tau_n \uparrow T$ a.s. Moreover, it follows from (3.3) and the property of Itô's integral that

$$E\big(\big[1 + |x(t \wedge \tau_n)|^2\big]^{\frac{p}{2}}\big) \leq 2^{\frac{p-2}{2}}\big[1 + E|x_0|^p\big] + 2p(a+b)E\int_{t_0}^{t \wedge \tau_n}\big[1 + |x(s)|^2\big]^{\frac{p}{2}}\,ds$$

$$\leq 2^{\frac{p-2}{2}}\big[1 + E|x_0|^p\big] + 2p(a+b)\int_{t_0}^{t}E\big(\big[1 + |x(s \wedge \tau_n)|^2\big]^{\frac{p}{2}}\big)\,ds.$$

The Gronwall inequality yields

$$E\big(\big[1 + |x(t \wedge \tau_n)|^2\big]^{\frac{p}{2}}\big) \leq 2^{\frac{p-2}{2}}\big[1 + E|x_0|^p\big]e^{2p(a+b)(t-t_0)}.$$

Letting $n \to \infty$ yields

$$E\big(\big[1 + |x(t)|^2\big]^{\frac{p}{2}}\big) \leq 2^{\frac{p-2}{2}}\big[1 + E|x_0|^p\big]e^{2p(a+b)(t-t_0)} \tag{3.4}$$

and the desired inequality follows. □

Let us now turn to considering the case of $0 < p < 2$. This is rather easy if we note that the Hölder inequality implies

$$E|x(t)|^p \leq \big(E|x(t)|^2\big)^{\frac{p}{2}}.$$

In other words, the estimate for $E|x(t)|^p$ can be done via the estimate for the second moment. For instance, we have the following corollary.

Corollary 3.2 *Let $0 < p < 2$ and $x_0 \in L^p(\Omega; R^d)$. Assume that for all $(x,t) \in R^d \times [t_0, T]$, $\{f(x(t), t)\} \in \mathcal{L}^1([t_0, T]; R^d)$, and $\{g(x(t), t)\} \in \mathcal{L}^2([t_0, T]; R^d)$, it follows that*

$$x^T f\big(x(t), t\big) \vee \frac{1}{2}\big|g\big(x(t), t\big)\big|^2 \leq \kappa\big(1 + |x(t)|^2\big), \tag{3.5}$$

where $\kappa(\cdot)$ is a concave nondecreasing function from R_+ to R_+ such that $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$. Then, for a pair of positive constants a and b such that $\kappa(u) \leq a + bu$, we have

$$E|x(t)|^p \leq \big[1 + E|x_0|^2\big]^{\frac{p}{2}}\exp\big(2p(a+b)(t-t_0)\big) \tag{3.6}$$

for all $t \in [t_0, T]$.

We now verify that if a nonlinear growth condition

$$\big|f\big(x(t), t\big)\big|^2 \vee \big|g\big(x(t), t\big)\big|^2 \leq \kappa_1\big(1 + |x(t)|^2\big), \tag{3.7}$$

with $\kappa_1(\cdot)$ is a concave nondecreasing function from R_+ to R_+ such that $\kappa_1(0) = 0$, $\kappa_1(u) > 0$ for $u > 0$ is fulfilled, then condition (3.1) is satisfied with $\kappa(u) = u\sqrt{\kappa_1(u)} \vee \frac{p-1}{2}\kappa_1(u)$. In fact, using (3.7) and the elementary inequality $2ab \leq a^2 + b^2$, one can derive that for any $\epsilon > 0$,

$$2x^T f(x, t) \leq 2|x|\big|f(x, t)\big| = 2\big(\sqrt{\epsilon}|x|\big)\big(\big|f(x, t)\big|/\sqrt{\epsilon}\big)$$

$$\leq \epsilon|x|^2 + \frac{1}{\epsilon}\big|f(x, t)\big|^2 \leq \epsilon|x|^2 + \frac{1}{\epsilon}\kappa_1\big(1 + |x(t)|^2\big).$$

Letting $\epsilon = \sqrt{\kappa_1(1 + |x(t)|^2)}$ yields

$$2x^T f(x, t) \le \left(1 + |x(t)|^2\right)\sqrt{\kappa_1\left(1 + |x(t)|^2\right)}.$$

Consequently,

$$x^T f(x, t) \vee \frac{(p-1)}{2}|g(x, t)|^2 \le \left(1 + |x|^2\right)\sqrt{\kappa_1\left(1 + |x|^2\right)} \vee \frac{(p-1)}{2}\kappa_1\left(1 + |x|^2\right).$$

We therefore obtain the following useful corollary.

Corollary 3.3 *Let $p \ge 2$ and $x_0 \in L^p(\Omega; R^d)$. Assume that the nonlinear growth condition* (3.7) *holds. Then inequality* (3.1) *holds with*

$$\kappa\left(1 + |x|^2\right) = \left(1 + |x|^2\right)\sqrt{\kappa_1\left(1 + |x|^2\right)} \vee \frac{(p-1)}{2}\kappa_1\left(1 + |x|^2\right).$$

We now apply this result to show one of the important properties of the solution.

Theorem 3.4 *Let $p \ge 2$ and $x_0 \in L^p(\Omega; R^d)$. Assume that the nonlinear growth condition* (3.7) *holds for all $(x, t) \in R^d \times [t_0, T]$. Then*

$$E|x(t) - x(s)|^p \le \left[\frac{1}{2}C(2\alpha)^{\frac{p}{2}} + C2^{p-2}\beta^{\frac{p}{2}}\left[1 + E|x_0|^p\right]e^{2p(a+b)(T-t_0)}\right](t - s)^{\frac{p}{2}},$$

where $C = (2(T - t_0))^{\frac{p}{2}} + \frac{1}{2}(2p(p - 1))^{\frac{p}{2}}$. In particular, the pth moment of the solution is continuous on $[t_0, T]$.

Proof Applying the elementary inequality $|a + b|^p \le 2^{p-1}(|a|^p + |b|^p)$, we can easily see that

$$E|x(t) - x(s)|^p \le 2^{p-1}E\left|\int_s^t f\left(x(r), r\right) dr\right|^p + 2^{p-1}E\left|\int_s^t g\left(x(r), r\right) dB(r)\right|^p.$$

Using the Hölder inequality, the moment inequality ([3], Theorem 1.7.1), and condition (3.7), one can show that

$$E|x(t) - x(s)|^p \le \left(2(t - s)\right)^{p-1}E\int_s^t \left|f\left(x(r), r\right)\right|^p dr$$

$$+ \frac{1}{2}\left(2p(p - 1)\right)^{\frac{p}{2}}(t - s)^{\frac{p-2}{2}}E\int_s^t \left|g\left(x(r), r\right)\right|^p dr$$

$$\le C(t - s)^{\frac{p-2}{2}}E\int_s^t \left(\kappa_1\left(1 + |x(r)|^2\right)\right)^{\frac{p}{2}} dr,$$

where $C = (2(T - t_0))^{\frac{p}{2}} + \frac{1}{2}(2p(p - 1))^{\frac{p}{2}}$. Given that $\kappa_1(\cdot)$ is concave and $\kappa_1(0) = 0$, we can find a pair of positive constants α and β such that $\kappa_1(u) \le \alpha + \beta u$ for all $u \ge 0$. So we have

$$E|x(t) - x(s)|^p \le \frac{1}{2}C(2\alpha)^{\frac{p}{2}}(t - s)^{\frac{p}{2}} + \frac{1}{2}C(2\beta)^{\frac{p}{2}}(t - s)^{\frac{p-2}{2}}\int_s^t E\left(1 + |x(r)|^2\right)^{\frac{p}{2}} dr.$$

Substituting this into (3.4) yields that

$$E\big|x(t) - x(s)\big|^p \leq \left(\frac{1}{2}C(2\alpha)^{\frac{p}{2}} + C2^{p-2}\beta^{\frac{p}{2}}\big[1 + E|x_0|^p\big]e^{2p(a+b)(T-t_0)}\right)(t - s)^{\frac{p}{2}},$$

which is the required inequality. $\square$

Theorem 3.5 *Let $p \geq 2$ and $x_0 \in L^p(\Omega; R^d)$. Assume that the nonlinear growth condition* (3.7) *for all $(x, t) \in R^d \times [t_0, T]$. Then*

$$E\left(\sup_{t_0 \leq s \leq t} \big|x(s)\big|^p\right) \leq C_2 e^{C_1 2^{p-2}\beta^{\frac{p}{2}}(t-t_0)}$$

for all $t_0 \leq t \leq T$, where $C_1 = 3^{p-1}(T - t_0)^{\frac{p-2}{2}}[(T - t_0)^{\frac{p}{2}} + (\frac{p^3}{2(p-1)})^{\frac{p}{2}}]$ and $C_2 = 1 + 3^{p-1}E|x_0|^p + \frac{1}{2}C_1(2\alpha)^{\frac{p}{2}}(T - t_0)$.

Proof Using the Hölder inequality, the moment inequality ([3], Theorem 1.7.2), and condition (3.7), one can show that

$$E\left(\sup_{t_0 \leq s \leq t} \big|x(s)\big|^p\right) \leq 3^{p-1}E|x_0|^p + C_1 E \int_{t_0}^{t} \big(\kappa_1\big(1 + \big|x(s)\big|^2\big)\big)^{\frac{p}{2}} ds,$$

where $C_1 = 3^{p-1}(T - t_0)^{\frac{p-2}{2}}[(T - t_0)^{\frac{p}{2}} + (p^3/2(p-1))^{\frac{p}{2}}]$. From the definition of $\kappa_1(\cdot)$, we can find a pair of positive constants α and β such that $\kappa_1(u) \leq \alpha + \beta u$ for all $u \geq 0$. So we have

$$E\left(\sup_{t_0 \leq s \leq t} \big|x(s)\big|^p\right) \leq 3^{p-1}E|x_0|^p + \frac{1}{2}C_1(2\alpha)^{\frac{p}{2}}(T - t_0) + C_1 2^{p-1}\beta^{\frac{p}{2}}E \int_{t_0}^{t} \big(1 + \big|x(s)\big|^p\big) ds.$$

Hence

$$1 + E\left(\sup_{t_0 \leq s \leq t} \big|x(s)\big|^p\right) \leq C_2 + C_1 2^{p-1}\beta^{\frac{p}{2}} \int_{t_0}^{t} E\left(1 + \sup_{t_0 \leq r \leq s} \big|x(r)\big|^p\right) ds.$$

By the Gronwall inequality one can see that

$$1 + E\left(\sup_{t_0 \leq s \leq t} \big|x(s)\big|^p\right) \leq C_2 e^{C_1 2^{p-1}\beta^{\frac{p}{2}}(t-t_0)},$$

which is the required inequality. $\square$

Let us now consider a d-dimensional stochastic differential equation

$$dx(t) = f\big(x(t), t\big) dt + g\big(x(t), t\big) dB(t) \tag{3.8}$$

on $t \in [t_0, \infty)$ with initial value $x(t_0) = x_0 \in L^2(\Omega, R^d)$. Assume that the equation has a unique global solution $x(t)$ on $t \in [t_0, \infty)$. Moreover, we shall impose the condition: There is a concave nondecreasing function $\kappa(\cdot)$ from R_+ to R_+ such that, for all $(x, t) \in R^d \times [t_0, \infty)$,

$$x^T f\big(x(t), t\big) \vee \frac{1}{2}\big|g\big(x(t), t\big)\big|^2 \leq \kappa\big(1 + \big|x(t)\big|^2\big). \tag{3.9}$$

Let $0 < p \leq 2$. In view of Theorem 3.1 and Corollary 3.2, we know that the pth moment of the solution satisfies

$$E|x(t)|^p \leq \left[1 + E|x_0|^2\right]^{\frac{p}{2}} \exp\left(2p(a+b)(t-t_0)\right)$$

for all $t \geq t_0$. This means that the pth moment will grow at most exponentially with exponent $2p(a+b)$. This can also be expressed as

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(E|x(t)|^p\right) \leq 2p(a+b). \tag{3.10}$$

The left-hand side of (3.10) is called the *pth moment Lyapunov exponent,* and (3.10) shows that the pth moment Lyapunov exponent should not be greater than $2p(a+b)$. In what follows, we shall establish the asymptotic estimate for the solution almost surely. More accurately, we shall estimate

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)|$$

almost surely, which is called the *sample Lyapunov exponent.*

Theorem 3.6 *Under condition* (3.9), *the sample Lyapunov exponent of the solution of equation* (3.8) *should not be greater than* $2(a+b)$, *that is,*

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| \leq 2(a+b)$$

almost surely.

Proof By Itô's formula and condition (3.9), we obtain

$$\log\left(1 + |x(t)|^2\right)$$

$$= \log\left(1 + |x_0|^2\right) + 2\int_{t_0}^t \frac{1}{1+|x(s)|^2}\left(x^T(s)f\left(x(s),s\right) + \frac{1}{2}\left|g\left(x(s),s\right)\right|^2\right) ds$$

$$- 2\int_{t_0}^t \frac{|x^T(s)g(x(s),s)|^2}{(1+|x(s)|^2)^2} ds + M(t)$$

$$\leq \log\left(1 + |x_0|^2\right) + 4\int_{t_0}^t \frac{\kappa(1+|x(s)|^2)}{1+|x(s)|^2} ds - 2\int_{t_0}^t \frac{|x^T(s)g(x(s),s)|^2}{(1+|x(s)|^2)^2} ds + M(t),$$

where

$$M(t) = 2\int_{t_0}^t \frac{x^T(s)g(x(s),s)}{1+|x(s)|^2} dB(s).$$

Furthermore, for every integer $n \geq t_0$, using the exponential martingale inequality (see [3], Theorem 1.7.4) on sees that

$$P\left\{\sup_{t_0 \leq t \leq n}\left[M(t) - 2\int_{t_0}^t \frac{|x^T(s)g(x(s),s)|^2}{(1+|x(s)|^2)^2} ds\right] > 2\log n\right\} \leq \frac{1}{n^2}.$$

An application of the Borel-Cantelli lemma then yields that for almost all $\omega \in \Omega$, there is a random integer $n_0 = n_0(\omega) \geq t_0 + 1$ such that

$$\sup_{t_0 \leq s \leq t} \left[M(t) - 2 \int_{t_0}^{t} \frac{|x^T(s)g(x(s),s)|^2}{(1 + |x(s)|^2)^2} \, ds \right] \leq 2 \log n$$

if $n \geq n_0$. That is,

$$M(t) \leq 2 \log n + 2 \int_{t_0}^{t} \frac{|x^T(s)g(x(s),s)|^2}{(1 + |x(s)|^2)^2} \, ds \tag{3.11}$$

for all $t_0 \leq t \leq n$, $n \geq n_0$ almost surely. From inequality (3.11) and the definition of the function $\kappa(\cdot)$, we deduce that

$$\log\left(1 + |x(t)|^2\right) \leq \log\left(1 + |x_0|^2\right) + 4(a + b)(t - t_0) + 2 \log n$$

for all $t_0 \leq t \leq n$, $n \geq n_0$ almost surely. Therefore, for almost all $\omega \in \Omega$, if $n \geq n_0$, $n - 1 \leq t \leq n$,

$$\frac{1}{t} \log\left(1 + |x(t)|^2\right) \leq \frac{1}{n-1} \left[\log\left(1 + |x_0|^2\right) + 4(a + b)(t - t_0) + 2 \log n \right].$$

This implies

$$\limsup_{t \to \infty} \frac{1}{t} \log\left(1 + |x(t)|\right) \leq \limsup_{t \to \infty} \frac{1}{2t} \log\left(1 + |x(t)|^2\right)$$

$$\leq \limsup_{t \to \infty} \frac{1}{2(n-1)} \left[\log\left(1 + |x_0|^2\right) + 4(a + b)(t - t_0) + 2 \log n \right]$$

$$= 2(a + b)$$

almost surely, which is the required inequality. $\qquad \square$

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