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Optimal consumption of the stochastic Ramsey problem for non-Lipschitz diffusion

Chuandi Liu*

*Correspondence: liuchuandi@swu.edu.cn
School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China

Abstract

The stochastic Ramsey problem is considered in a growth model with the production function of a Cobb-Douglas form. The existence of a unique classical solution is proved for the Hamilton-Jacobi-Bellman equation associated with the optimization problem. A synthesis of the optimal consumption policy in terms of its solution is proposed.

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1 Introduction

We are concerned with the stochastic Ramsey problem in a growth model discussed by Merton [1]. For recent contribution in this direction, we refer to [2]. A firm produces goods according to the Cobb-Douglas production function x^γ for capital x, where $0 < \gamma < 1$ (*cf.* Barro and Sala-i-Martin [3]). The stock of capital X_t at time t is modeled by the stochastic differential equation

$$dX_t = X_t^\gamma \, dt + \sigma X_t \, dB_t, \quad t > 0, X_0 = x > 0, \sigma \neq 0,$$

on a complete probability space $(\Omega, \mathcal{F}, P)$ carrying a standard Brownian motion $\{B_t\}$ endowed with the natural filtration $\mathcal{F}_t$ generated by $\sigma(B_s, s \leq t)$.

The capital stock can be consumed and the flow of consumption at time t is assumed to be written as $c_t X_t$. The rate of consumption $c = \{c_t\}$ per capital stock is called an admissible policy if $\{c_t\}$ is an $\{\mathcal{F}_t\}$-progressively measurable process such that

$$0 \leq c_t \leq 1 \quad \text{for all } t \geq 0 \text{ a.s.} \tag{1.1}$$

We denote by $\mathcal{A}$ the set of admissible policies. Given a policy $c \in \mathcal{A}$, the capital stock process $\{X_t\}$ obeys the equation

$$dX_t = \left[X_t^\gamma - c_t X_t\right] dt + \sigma X_t \, dB_t, \quad X_0 = x > 0. \tag{1.2}$$

The objective is to find an optimal policy $c^* = \{c_t^*\}$ so as to maximize the expected discounted utility of consumption

$$J_x(c) = E\left[\int_0^\infty e^{-\alpha t} U(c_t X_t) \, dt\right] \tag{1.3}$$

over $c \in \mathcal{A}$, where $\alpha > 0$ is a discount rate and $U(x)$ is a utility function in $C^2(0,\infty) \cap C[0,\infty)$, which is assumed to have the following properties:

$$U'(\infty) = U(0) = 0, \qquad U'(0+) = U(\infty) = \infty, \qquad U'' < 0. \tag{1.4}$$

The Hamilton-Jacobi-Bellman (HJB for short) equation associated with this problem is given by

$$\alpha u(x) = \frac{1}{2}\sigma^2 x^2 u''(x) + x^\gamma u'(x) + \tilde{U}\big(x, u'(x)\big), \quad x > 0, \tag{1.5}$$

where

$$\tilde{U}(x,y) = \max_{0\le c\le 1}\big\{U(cx) - cxy\big\} \quad \text{for } x, y > 0. \tag{1.6}$$

This kind of economic growth problem has been studied by Kamien and Schwartz [4] and Sethi and Thompson [5, Chapter 11]. However, the optimization problem is unsolved. It is not guaranteed that (1.2) admits a unique positive solution $\{X_t\}$ and the value function is a viscosity solution of the HJB equation. The main difficulty stems from the fact that (1.5) is a degenerate nonlinear equation of elliptic type with the non-Lipschitz coefficient x^γ in $(0,\infty)$. It is also analytically studied by [6], nevertheless in the finite time horizon. The resulting HJB equation is a parabolic partial differential equation (PDE, for short), which is very different from the elliptic PDE dealt with in the present paper.

In this paper, we provide the existence results on a unique positive solution $\{X_t\}$ to (1.2) and a classical solution u of (1.5) by the theory of viscosity solutions. For the detail of the theory of viscosity solutions, we mention the works [7, 8] and [9]. An optimal policy is shown to exist in terms of u.

This paper is organized as follows. In Section 2, we show that (1.2) admits a unique positive solution. In Section 3, we show the existence of a viscosity solution u of the HJB equation (1.5). Section 4 is devoted to the C^2-regularity of its solution. In Section 5, we present a synthesis of the optimal consumption policy.

2 Preliminaries

In this section, we show the existence of a unique solution $\{X_t\}$ to (1.2).

Proposition 2.1 *There exists a unique positive solution* $\{X_t\} = \{X_t^x\}$ *to* (1.2), *for each* $c \in \mathcal{A}$, *such that*

$$E[X_t] \le \big\{(1-\gamma)t + x^{1-\gamma}\big\}^{1/(1-\gamma)}, \tag{2.1}$$

$$E\big[X_t^2\big] \le e^{\sigma^2 t}\big\{2(1-\lambda)t + x^{2(1-\lambda)}\big\}^{1/(1-\lambda)}, \quad \lambda := (1+\gamma)/2, \tag{2.2}$$

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0 \quad s.t.\ E\big[\big|X_t^x - X_t^y\big|\big] \le C_\varepsilon |x-y| + \varepsilon\big(1 + t^{1/(1-\gamma)} + x + y\big), \quad x, y > 0. \tag{2.3}$$

Proof We set $x_t = X_t^{1-\gamma}$. Then, by Ito's formula and (1.2),

$$\begin{aligned} dx_t &= (1-\gamma)X_t^{-\gamma}\,dX_t + \frac{\sigma^2}{2}(1-\gamma)(-\gamma)X_t^{1-\gamma}\,dt \\ &= (1-\gamma)\left[1 - \left(c_t + \frac{\sigma^2}{2}\gamma\right)x_t\right]dt + (1-\gamma)\sigma x_t\,dB_t, \quad x_0 = x^{1-\gamma}. \end{aligned} \tag{2.4}$$

By linearity, (2.4) has a unique solution $\{x_t\}$. Since

$$d\hat{x}_t = (1-\gamma)\left[-\left(c_t + \frac{\sigma^2}{2}\gamma\right)\hat{x}_t\right]dt + (1-\gamma)\sigma\hat{x}_t\,dB_t, \quad \hat{x}_0 = x^{1-\gamma} \tag{2.5}$$

has a positive solution $\{\hat{x}_t\}$, we see by the comparison theorem [10, Chapter 6, Theorem 1.1] that $x_t \geq \hat{x}_t > 0$ holds almost surely (a.s.). Therefore, (1.2) admits a unique positive solution $\{X_t\}$ of the form $X_t = x_t^{1/(1-\gamma)}$, which satisfies $\sup_{0\leq t\leq T} E[X_t^4] < \infty$ for each $T \geq 0$.

Let θ_t be the right-hand side of (2.1) and $\phi_t = E[X_t]$. Obviously, we see that θ_t is a unique solution of

$$d\theta_t = \theta_t^\gamma\,dt, \quad \theta_0 = x > 0.$$

By (1.2) and Jensen's inequality,

$$d\phi_t = dE[X_t] = E\left[X_t^\gamma - c_t X_t\right]dt \leq \phi_t^\gamma\,dt.$$

Since $\theta_0 = \phi_0 = x$, we deduce $\phi_t \leq \theta_t$, which implies (2.1).

Similarly, let Θ_t be the right-hand side of (2.2) and $\Phi_t = E[X_t^2]$. By substitution, it is easy to see that $\bar{\Theta}_t := e^{-\sigma^2 t}\Theta_t$ is a unique solution of

$$d\bar{\Theta}_t = 2\bar{\Theta}_t^\lambda\,dt, \quad \bar{\Theta}_0 = x^2 > 0.$$

Hence

$$d\Theta_t = e^{\sigma^2 t}\left(2\bar{\Theta}_t^\lambda + \sigma^2\bar{\Theta}_t\right)dt \geq \left(2\Theta_t^\lambda + \sigma^2\Theta_t\right)dt.$$

Furthermore, by (1.2), Ito's formula and Jensen's inequality,

$$\begin{aligned} d\Phi_t &= dE\left[X_t^2\right] \\ &= E\left[2X_t^{2\lambda} - 2c_t X_t^2 + \sigma^2 X_t^2\right]dt \\ &\leq \left(2\Phi_t^\lambda + \sigma^2\Phi_t\right)dt. \end{aligned}$$

Thus, we deduce $\Phi_t \leq \Theta_t$ and $\Phi_0 = \Theta_0$, which implies (2.2).

Next, let $\{Y_t\}$ denote the solution $\{X_t^y\}$ of (1.2) with $Y_0 = y$ and $y_t = Y_t^{1-\gamma}$. Then, by (2.4)

$$d(x_t - y_t) = -(1-\gamma)\left(c_t + \frac{\sigma^2}{2}\gamma\right)(x_t - y_t)\,dt + (1-\gamma)\sigma(x_t - y_t)\,dB_t,$$

which implies

$$x_t - y_t = (x_0 - y_0)\exp\left\{-(1-\gamma)\left(\int_0^t c_s\,ds + \frac{\sigma^2}{2}\gamma t\right) + (1-\gamma)\sigma B_t - \frac{\sigma^2}{2}(1-\gamma)^2 t\right\}.$$

Setting $\beta = 1/(1-\gamma) > 1$, we have

$$\begin{aligned} E\left[|x_t - y_t|^\beta\right] &\leq |x_0 - y_0|^\beta E\left[\exp\left\{\sigma B_t - \frac{\sigma^2}{2}t\right\}\right] \\ &= \left|x^{1-\gamma} - y^{1-\gamma}\right|^{1/(1-\gamma)} \leq |x - y|. \end{aligned} \tag{2.6}$$

By Young's inequality [11], for any $\varepsilon_0 > 0$,

$$\begin{aligned}
\left|x^\beta - y^\beta\right| &\le \beta\left(x^{\beta-1} + y^{\beta-1}\right)|x-y| \\
&\le \beta\left[\frac{1}{\beta}\left(\frac{1}{\varepsilon_0}\right)^\beta |x-y|^\beta + \frac{\beta-1}{\beta}\left\{\varepsilon_0\left(x^{\beta-1} + y^{\beta-1}\right)\right\}^{\beta/(\beta-1)}\right] \\
&\le \left(\frac{1}{\varepsilon_0}\right)^\beta |x-y|^\beta + (\beta-1)(2\varepsilon_0)^{\beta/(\beta-1)}\left(x^\beta + y^\beta\right), \quad x, y \ge 0.
\end{aligned}$$

Hence, for any $\varepsilon > 0$, we choose $C_\varepsilon > 0$ such that

$$\left|x^\beta - y^\beta\right| \le C_\varepsilon |x-y|^\beta + \varepsilon\left(1 + x^\beta + y^\beta\right), \quad x, y \ge 0.$$

Therefore, by (2.1) and (2.6), we have

$$\begin{aligned}
E\left[|X_t - Y_t|\right] &= E\left[\left|x_t^\beta - y_t^\beta\right|\right] \\
&\le C_\varepsilon E\left[|x_t - y_t|^\beta\right] + \varepsilon E\left[1 + x_t^\beta + y_t^\beta\right] \\
&\le C_\varepsilon |x-y| + \varepsilon E[1 + X_t + Y_t] \\
&\le C_\varepsilon |x-y| + \varepsilon\left\{1 + 2^\beta\left(t^\beta + x\right) + 2^\beta\left(t^\beta + y\right)\right\},
\end{aligned}$$

which implies (2.3). □

Remark 2.1 The uniqueness for (1.2) is violated if $x = 0$ and c_t is deterministic since 0 and the limit process $\chi_t := \lim_{x\to 0+} X_t^x$ satisfy (1.2) with $X_0 = 0$, and

$$E\left[\chi_t^{1-\gamma}\right] = E\left[\int_0^t (1-\gamma)\left\{1 - \left(c_s + \frac{\sigma^2}{2}\gamma\right)\chi_s^{1-\gamma}\right\} ds\right] \ne 0. \tag{2.7}$$

3 Viscosity solutions of the HJB equation

Definition 3.1 Let $u \in C(0,\infty)$. Then u is called a viscosity solution of (1.5) if the following relations are satisfied:

$$\begin{aligned}
\alpha u(x) &\le \frac{1}{2}\sigma^2 x^2 q + x^\gamma p + \tilde{U}(x,p), \quad \forall (p,q) \in J^{2,+}u(x), \forall x > 0, \\
\alpha u(x) &\ge \frac{1}{2}\sigma^2 x^2 q + x^\gamma p + \tilde{U}(x,p), \quad \forall (p,q) \in J^{2,-}u(x), \forall x > 0,
\end{aligned}$$

where $J^{2,+}u(x)$ and $J^{2,-}u(x)$ are the second-order superjets and subjets [7].

Define the value function u by

$$u(x) = \sup_{c\in\mathcal{A}} E\left[\int_0^\infty e^{-\alpha t} U(c_t X_t)\, dt\right], \tag{3.1}$$

where the supremum is taken over all systems $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}; \{B_t\}, \{c_t\})$.

In this section, we study the viscosity solution u of the HJB equation (1.5). Due to Proposition 2.1, we can show the value function u with the following properties.

Lemma 3.1 *We assume* (1.4). *Then we have*

$$0 \le u(x) \le \zeta(x) := x + \zeta_0, \quad x > 0 \tag{3.2}$$

for some constant $\zeta_0 > 0$, *and there exists* $C_\rho > 0$ *for any* $\rho > 0$ *such that*

$$\left|u(x) - u(y)\right| \le C_\rho |x - y| + \rho(1 + x + y), \quad x, y > 0. \tag{3.3}$$

Proof Clearly, u is nonnegative. By concavity, there is $\bar{C} > 0$ such that

$$U(x) \le \alpha 2^{-1/(1-\gamma)} x + \bar{C}, \quad x \ge 0.$$

By (1.1) and (2.1), we have

$$\begin{aligned} E\left[\int_0^\infty e^{-\alpha t} U(c_t X_t)\,dt\right] &\le E\left[\int_0^\infty e^{-\alpha t}\left(\alpha 2^{-1/(1-\gamma)} X_t + \bar{C}\right)dt\right] \\ &\le \int_0^\infty e^{-\alpha t}\left\{\alpha\left(t^{1/(1-\gamma)} + x\right) + \bar{C}\right\}dt \\ &= x + \alpha \int_0^\infty e^{-\alpha t} t^{1/(1-\gamma)}\,dt + \bar{C}/\alpha, \end{aligned}$$

which implies (3.2).

Now, let $\rho > 0$ be arbitrary. By (1.4), there is $\delta > 0$ such that $U(x) \le \rho$ for all $x \in [0, \delta]$. Furthermore,

$$\left|U(x) - U(y)\right| \le U'(\delta)|x - y|, \quad x, y \ge \delta.$$

Thus, we obtain a constant $C_\rho > 0$, depending on $\rho > 0$, such that

$$\left|U(x) - U(y)\right| \le C_\rho |x - y| + \rho, \quad \forall x, y \ge 0. \tag{3.4}$$

By (1.1), (2.3) and (3.4), we get

$$\begin{aligned} \left|u(x) - u(y)\right| &\le \sup_{c \in \mathcal{A}} E\left[\int_0^\infty e^{-\alpha t}\left|U(c_t X_t) - U(c_t Y_t)\right|dt\right] \\ &\le \sup_{c \in \mathcal{A}} E\left[\int_0^\infty e^{-\alpha t}\left\{C_\rho |X_t - Y_t| + \rho\right\}dt\right] \\ &\le C_\rho \int_0^\infty e^{-\alpha t}\left\{C_\varepsilon |x - y| + \varepsilon\left(1 + t^{1/(1-\gamma)} + x + y\right)\right\}dt + \rho/\alpha \\ &\le C\left\{C_\rho C_\varepsilon |x - y| + (\varepsilon + \rho)(1 + x + y)\right\}, \quad x, y > 0, \end{aligned} \tag{3.5}$$

where the constant $C > 0$ is independent of ε, $\rho > 0$. Replacing ρ by $\rho/2C$ and choosing sufficiently small $\varepsilon > 0$, we deduce (3.3). □

Remark 3.1 The continuity of u is immediate from (3.3).

Theorem 3.1 *We assume* (1.4). *Then the value function* u *is a viscosity solution of* (1.5).

Proof According to [12], the viscosity property of u follows from the dynamic programming principle for u, that is,

$$u(x) = \sup_{c\in\mathcal{A}} E\left[\int_0^\tau e^{-\alpha t} U(c_t X_t)\,dt + e^{-\alpha\tau} u(X_\tau)\right], \quad x > 0 \tag{3.6}$$

for any bounded stopping time $\tau \geq 0$, where the supremum is taken over all systems $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}; \{B_t\}, \{c_t\})$. Let $\bar{u}(x)$ be the right-hand side of (3.6). Let $\tilde{X}_t = X_{t+r}$ and $\tilde{B}_t = B_{t+r} - B_r$, when $\tau = r$ is non-random. Then we have

$$d\tilde{X}_t = \left[\tilde{X}_t^\gamma - \tilde{c}_t \tilde{X}_t\right] dt + \sigma \tilde{X}_t\, d\tilde{B}_t, \quad \tilde{X}_0 = X_r$$

for the shifted process $\tilde{c} = \{\tilde{c}_t\}$ of c by r, *i.e.*, $\tilde{c}_t = c_{t+r}$. It is easy to see that

$$e^{\alpha r} E\left[\int_r^\infty e^{-\alpha t} U(c_t X_t)\,dt \Big| \mathcal{F}_r\right] = E\left[\int_0^\infty e^{-\alpha t} U(\tilde{c}_t \tilde{X}_t)\,dt \Big| \mathcal{F}_r\right] = J_{X_r}(\tilde{c}) \quad \text{a.s.}$$

with respect to the conditional probability $P(\cdot|\mathcal{F}_r)$. We take $\zeta_1 > 0$ such that $x^\gamma \leq \alpha x + \zeta_1$ and sufficiently large $\zeta_0 > 0$ to obtain

$$-\alpha\zeta + \frac{1}{2}\sigma^2 x^2 \zeta'' + x^\gamma \zeta' \leq -\alpha\zeta_0 + \zeta_1 \leq 0.$$

By (3.2) in Lemma 3.1, Ito's formula and Doob's inequality, we have

$$E\Big[\sup_{0\leq t\leq T} e^{-\alpha t} J_{X_t}(\tilde{c})\Big] \leq E\Big[\sup_{0\leq t\leq T} e^{-\alpha t} \zeta(X_t)\Big] \leq \zeta(x) + C, \quad T > 0$$

for some constant $C > 0$. Hence, approximating τ by a sequence of countably valued stopping times, we see that

$$E\left[e^{-\alpha\tau} J_{X_\tau}(\tilde{c})\right] = E\left[\int_\tau^\infty e^{-\alpha t} U(c_t X_t)\,dt\right].$$

Thus

$$\begin{aligned} J_x(c) &= E\left[\int_0^\tau e^{-\alpha t} U(c_t X_t)\,dt + \int_\tau^\infty e^{-\alpha t} U(c_t X_t)\,dt\right] \\ &\leq E\left[\int_0^\tau e^{-\alpha t} U(c_t X_t)\,dt + e^{-\alpha\tau} u(X_\tau)\right]. \end{aligned}$$

Taking the supremum, we deduce $u \leq \bar{u}$.

We shall show the reverse inequality in the case that $\tau = r$ is constant. For any $\varepsilon > 0$, we consider a sequence $\{S_j : j = 1, \ldots, n+1\}$ of disjoint subsets of $(0, \infty)$ such that

$$\operatorname{diam}(S_j) < \delta, \qquad \bigcup_{j=1}^n S_j = (0, R) \quad \text{and} \quad S_{n+1} = [R, \infty)$$

for $\delta, R > 0$ chosen later. We take $x_j \in S_j$ and $c^{(j)} \in \mathcal{A}$ such that

$$u(x_j) - \varepsilon \leq J_{x_j}\left(c^{(j)}\right), \quad j = 1, \ldots, n+1. \tag{3.7}$$

By the same argument as (3.5), we note that

$$\left|J_x\left(c^{(j)}\right) - J_y\left(c^{(j)}\right)\right| + \left|u(x) - u(y)\right| \leq C_\varepsilon|x-y| + \frac{\varepsilon}{4}(1+x+y), \quad x, y > 0$$

for some constant $C_\varepsilon > 0$. We choose $0 < \delta < 1$ such that $C_\varepsilon\delta < \varepsilon/2$. Then we have

$$\left|J_x\left(c^{(j)}\right) - J_y\left(c^{(j)}\right)\right| + \left|u(x) - u(y)\right| \leq \varepsilon(1+x), \quad x, y \in S_j, j = 1, 2, \ldots, n. \tag{3.8}$$

Hence, by (3.7) and (3.8),

$$\begin{aligned} J_{X_r}\left(c^{(j)}\right) &= J_{X_r}\left(c^{(j)}\right) - J_{x_j}\left(c^{(j)}\right) + J_{x_j}\left(c^{(j)}\right) \\ &\geq -\varepsilon(1+X_r) + u(x_j) - \varepsilon \\ &\geq -2\varepsilon(1+X_r) + u(X_r) - \varepsilon \\ &\geq -3\varepsilon(1+X_r) + u(X_r) \quad \text{if } X_r \in S_j, j = 1, \ldots, n. \end{aligned} \tag{3.9}$$

By definition, we find $c \in \mathcal{A}$ such that

$$\bar{u}(x) - \varepsilon \leq E\left[\int_0^r e^{-\alpha t}U(c_t X_t)\,dt + e^{-\alpha r}u(X_r)\right].$$

In view of [10, Chapter 6, Theorem 1.1], we can take c, $c^{(j)}$ on the same probability space. Define

$$c_t^r = c_t 1_{\{t<r\}} + c_{t-r}^{(j)} 1_{\{r\leq t\}} \quad \text{if } X_r \in S_j, j = 1, \ldots, n+1,$$

where $1_{\{\cdot\}}$ denotes the indicator function. Then $\{c_t^r\}$ belongs to $\mathcal{A}$. Let $\{X_t^r\}$ be the solution of

$$dX_t^r = \left[\left(X_t^r\right)^\gamma - c_t^r X_t^r\right]dt + \sigma X_t^r\,dB_t, \quad X_0^r = x > 0.$$

Clearly, $X_t^r = X_t$ a.s. if $t < r$. Further, for each $j = 1, \ldots, n+1$, we have on $\{X_r \in S_j\}$

$$\begin{aligned} X_{t+r}^r &= X_r + \int_r^{t+r}\left[\left(X_s^r\right)^\gamma - c_s^r X_s^r\right]ds + \int_r^{t+r}\sigma X_s^r\,dB_s \\ &= X_r + \int_0^t\left[\left(X_{s+r}^r\right)^\gamma - c_s^{(j)} X_{s+r}^r\right]ds + \int_0^t \sigma X_{s+r}^r\,d\tilde{B}_s \quad \text{a.s.} \end{aligned}$$

Hence, X_{t+r}^r coincides with the solution $X_t^{(j)}$ of (1.2) for $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_t\}; \{\tilde{B}_t\}, \{c_t^{(j)}\})$ a.s. on $\{X_r \in S_j\}$ with $X_0^{(j)} = X_r$. Thus, we get

$$\begin{aligned} J_{X_r}\left(\tilde{c}^r\right) &= E^{\tilde{P}}\left[\int_0^\infty e^{-\alpha t}U\left(c_{t+r}^r X_{t+r}^r\right)dt \middle| \tilde{\mathcal{F}}_r\right] \\ &= E^{\tilde{P}}\left[\int_0^\infty e^{-\alpha t}U\left(c_t^{(j)} X_t^{(j)}\right)dt \middle| \tilde{\mathcal{F}}_r\right] \\ &= J_{X_r}\left(c^{(j)}\right) \quad \text{a.s. on } \{X_r \in S_j\}, j = 1, \ldots, n+1, \end{aligned} \tag{3.10}$$

where $E^{\tilde{P}}$ denotes the expectation with respect to $\tilde{P}$.

Now, we fix $x > 0$ and choose $R > 0$, by (2.1), (2.2) and (3.1), such that

$$\begin{aligned}\sup_{c\in\mathcal{A}} E\big[u(X_r)1_{\{X_r\ge R\}}\big] &\le \sup_{c\in\mathcal{A}} E\big[\zeta(X_r)1_{\{X_r\ge R\}}\big]\\ &\le \sup_{c\in\mathcal{A}} \frac{1}{R}E\big[X_r^2+\zeta_0X_r\big]\\ &\le \frac{C_0}{R}\big(1+x+x^2\big) < \varepsilon,\end{aligned}\tag{3.11}$$

where the constant $C_0 > 0$ depends only on r, ζ_0. By (3.9), (3.10) and (3.11), we have

$$\begin{aligned}E\bigg[\int_r^\infty e^{-\alpha t}U\big(c_t^rX_t^r\big)\,dt\bigg] &= E\bigg[E\bigg[\int_r^\infty e^{-\alpha t}U\big(c_t^rX_t^r\big)\,dt\bigg|\mathcal{F}_r\bigg]\bigg]\\ &= E\big[e^{-\alpha r}J_{X_r}\big(\tilde{c}^r\big)\big]\\ &= E\Bigg[\sum_{j=1}^{n+1} e^{-\alpha r}J_{X_r}\big(c^{(j)}\big)1_{\{X_r\in S_j\}}\Bigg]\\ &\ge E\Bigg[\sum_{j=1}^{n} e^{-\alpha r}\big\{u(X_r)-3\varepsilon(1+X_r)\big\}1_{\{X_r\in S_j\}}\Bigg]\\ &\ge E\big[e^{-\alpha r}\big\{u(X_r)-u(X_r)1_{\{X_r\ge R\}}\big\}\big]-3\varepsilon E[1+X_r]\\ &\ge E\big[e^{-\alpha r}u(X_r)\big]-\varepsilon-3\varepsilon C(1+x)\end{aligned}$$

for some constant $C > 0$ independent of ε. Thus

$$\begin{aligned}u(x) &\ge E\bigg[\int_0^r e^{-\alpha t}U\big(c_t^rX_t^r\big)\,dt+\int_r^\infty e^{-\alpha t}U\big(c_t^rX_t^r\big)\,dt\bigg]\\ &\ge E\bigg[\int_0^r e^{-\alpha t}U(c_tX_t)\,dt+e^{-\alpha r}u(X_r)\bigg]-\varepsilon-3\varepsilon C(1+x)\\ &\ge \bar{u}(x)-2\varepsilon-3\varepsilon C(1+x).\end{aligned}$$

Letting $\varepsilon\to 0$, we get $\bar{u}\le u$.

In the general case, by the above argument, we note that

$$\begin{aligned}u(X_r)=u(\tilde{X}_0) &\ge E\bigg[\int_0^s e^{-\alpha t}U(\tilde{c}_t\tilde{X}_t)\,dt+e^{-\alpha s}u(\tilde{X}_s)\bigg|\mathcal{F}_r\bigg]\\ &= E\bigg[\int_0^s e^{-\alpha t}U(c_{t+r}X_{t+r})\,dt+e^{-\alpha s}u(X_{s+r})\bigg|\mathcal{F}_r\bigg]\quad \text{a.s. } s,r\ge 0.\end{aligned}$$

Hence $\{e^{-\alpha s}u(X_s)+\int_0^s e^{-\alpha t}U(c_tX_t)\,dt\}$ is a supermartingale. By the optional sampling theorem,

$$u(X_0)\ge E\bigg[\int_0^\tau e^{-\alpha t}U(c_tX_t)\,dt+e^{-\alpha\tau}u(X_\tau)\bigg|\mathcal{F}_0\bigg]\quad\text{a.s.}$$

Taking the expectation and then the supremum over $\mathcal{A}$, we conclude that $\bar{u}\le u$. Noting the continuity of u, we obtain (3.6). □

4 Classical solutions

In this section, using the viscosity solutions technique, we show the C^2-regularity of the viscosity solution u of (1.5). For any fixed $0 < a < b$, we consider the boundary value problem

$$\alpha w = \frac{1}{2}\sigma^2 x^2 w'' + x^\gamma w' + \tilde{U}(x, w') \quad \text{in } (a,b), \tag{4.1}$$

with boundary condition

$$w(a) = u(a), \qquad w(b) = u(b), \tag{4.2}$$

given by u.

Proposition 4.1 *Let $w_i \in C[a,b]$, $i = 1,2$, be two viscosity solutions of* (3.1), (4.2). *Then, under* (1.4), *we have*

$$w_1 = w_2.$$

Proof It is sufficient to show that $w_1 \le w_2$. Suppose that there exists $x_0 \in [a,b]$ such that $w_1(x_0) - w_2(x_0) > 0$. Clearly, by (4.2), $x_0 \ne a, b$, and we find $\bar{x} \in (a,b)$ such that

$$\varrho := \sup_{x\in[a,b]} \{w_1(x) - w_2(x)\} = w_1(\bar{x}) - w_2(\bar{x}) > 0.$$

Define

$$\Psi_k(x,y) = w_1(x) - w_2(y) - \frac{k}{2}|x-y|^2$$

for $k > 0$. Then there exists $(x_k, y_k) \in [a,b]^2$ such that

$$\Psi_k(x_k, y_k) = \sup_{(x,y)\in[a,b]^2} \Psi_k(x,y) \ge \Psi_k(\bar{x},\bar{x}) = \varrho, \tag{4.3}$$

from which

$$\frac{k}{2}|x_k - y_k|^2 < w_1(x_k) - w_2(y_k).$$

Thus

$$|x_k - y_k| \to 0 \quad \text{as } k \to \infty. \tag{4.4}$$

Furthermore, by the definition of (x_k, y_k),

$$\Psi_k(x_k, y_k) \ge \Psi_k(x_k, x_k).$$

Hence, by uniform continuity

$$\begin{aligned}\frac{k}{2}|x_k - y_k|^2 &\le w_2(x_k) - w_2(y_k) \le \sup_{|x-y|\le\rho} |w_2(x) - w_2(y)| \\ &\to 0 \quad \text{as } k \to \infty \text{ and then } \rho \to 0.\end{aligned} \tag{4.5}$$

By (4.3), (4.4) and (4.5), extracting a subsequence, we have

$$(x_k, y_k) \to (\tilde{x}, \tilde{x}) \in (a, b)^2 \quad \text{as } k \to \infty. \tag{4.6}$$

Now, we may consider that $(x_k, y_k) \in (a, b)^2$ for sufficiently large k. Applying Ishii's lemma [7] to $\Psi_k(x, y)$, we obtain $X, Y \in \mathbf{R}$ such that

$$\begin{aligned} &\big(k(x_k - y_k), X\big) \in \bar{J}^{2,+} w_1(x_k), \\ &\big(k(x_k - y_k), Y\big) \in \bar{J}^{2,-} w_2(y_k), \\ &\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3k \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \end{aligned} \tag{4.7}$$

By Definition 3.1,

$$\alpha w_1(x_k) \le \frac{1}{2}\sigma^2 x_k^2 X + x_k^\gamma \mu + \tilde{U}(x_k, \mu),$$
$$\alpha w_2(y_k) \ge \frac{1}{2}\sigma^2 y_k^2 Y + y_k^\gamma \mu + \tilde{U}(y_k, \mu),$$

where $\mu = k(x_k - y_k)$. Putting these inequalities together, we get

$$\begin{aligned} \alpha\big\{w_1(x_k) - w_2(y_k)\big\} &\le \frac{1}{2}\sigma^2\big(x_k^2 X - y_k^2 Y\big) + \big(x_k^\gamma - y_k^\gamma\big)\mu + \big\{\tilde{U}(x_k, \mu) - \tilde{U}(y_k, \mu)\big\} \\ &\equiv I_1 + I_2 + I_3, \quad \text{say.} \end{aligned}$$

By (4.5) and (4.7), it is clear that

$$I_1 = \frac{\sigma^2}{2}\big(x_k^2 X - y_k^2 Y\big) \le \frac{\sigma^2}{2} 3k(x_k - y_k)^2 \to 0 \quad \text{as } k \to \infty.$$

Also, by (4.5)

$$I_2 = k\big(x_k^\gamma - y_k^\gamma\big)(x_k - y_k) \le k\gamma a^{\gamma-1}|x_k - y_k|^2 \to 0 \quad \text{as } k \to \infty.$$

By (1.6), (3.4), (4.5) and (4.6), we have

$$\begin{aligned} I_3 &\le \max_{0 \le c \le 1}\big|U(cx_k) - U(cy_k)\big| + |x_k - y_k||\mu| \\ &\le C_\rho|x_k - y_k| + \rho + k|x_k - y_k|^2 \\ &\to 0 \quad \text{as } k \to \infty \text{ and then } \rho \to 0. \end{aligned}$$

Consequently, by (4.6), we deduce that

$$\alpha\varrho \le \alpha\big\{w_1(\tilde{x}) - w_2(\tilde{x})\big\} \le 0,$$

which is a contradiction. □

Theorem 4.1 *We assume* (1.4). *Then there exists a solution* $u \in C^2(0, \infty)$ *of* (1.5).

Proof For any $0 < a < b$, we recall the boundary value problem (4.1), (4.2). Since

$$U(0) \le U'(x)(0 - x) + U(x), \quad x > 0,$$

we have

$$K_0 := \sup_{0<x\le a} xU'(x) < \infty.$$

Hence, by (1.4)

$$\begin{aligned}\bigl|U(cx_1) - U(cx_2)\bigr| &\le cU'(ca)|x_1 - x_2| \\ &\le \frac{K_0}{a}|x_1 - x_2|, \quad x_1, x_2 \in [a,b], 0 \le c \le 1.\end{aligned}$$

Also, by (1.6)

$$\begin{aligned}\bigl|\tilde{U}(x_1,y_1) - \tilde{U}(x_2,y_2)\bigr| &\le \max_{0\le c\le 1}\bigl|U(cx_1) - U(cx_2)\bigr| + |x_1y_1 - x_2y_2| \\ &\le \frac{K_0}{a}|x_1 - x_2| + |x_1 - x_2||y_1| + b|y_1 - y_2|, \quad y_1, y_2 > 0.\end{aligned}$$

Thus the nonlinear term of (4.1) is Lipschitz. By uniform ellipticity, a standard theory of nonlinear elliptic equations yields that there exists a unique solution $w \in C^2(a,b) \cap C[a,b]$ of (4.1), (4.2). For details, we refer to [13, Theorem 17.18] and [14, Chapter 5, Theorem 3.7]. Clearly, by Theorem 3.1, u is a viscosity solution of (4.1), (4.2). Therefore, by Proposition 4.1, we have $w = u$ and u is smooth. Since a, b are arbitrary, we obtain the assertion. □

5 Optimal consumption

In this section, we give a synthesis of the optimal policy $c^* = \{c_t^*\}$ for the optimization problem (1.4) subject to (1.2). We consider the stochastic differential equation

$$dX_t^* = \bigl[(X_t^*)^\gamma - \eta(X_t^*)X_t^*\bigr]\,dt + \sigma X_t^*\,dB_t, \quad X_0^* = x > 0, \tag{5.1}$$

where $\eta(x) = I(x, u'(x))$ and $I(x,y)$ denotes the maximizer of (1.6) for $x, y > 0$, *i.e.*,

$$I(x,y) = \begin{cases} (U')^{-1}(y)/x & \text{if } U'(x) \le y, \\ 1 & \text{otherwise.} \end{cases} \tag{5.2}$$

Our objective is to prove the following.

Theorem 5.1 *We assume* (1.4). *Then the optimal consumption policy* $\{c_t^*\}$ *is given by*

$$c_t^* = \eta(X_t^*). \tag{5.3}$$

To obtain the optimal consumption policy $\{c_t^*\}$, we should study the properties of the value function u and the existence of strong solution $\{X_t^*\}$ of (5.1). We need the following lemmas.

Lemma 5.1 *Under* (1.4), *the value function u is concave. In addition, we have*

$$u'(x) > 0 \quad \text{for } x > 0, \tag{5.4}$$

$$u'(0+) = \infty. \tag{5.5}$$

Proof Let $x_i > 0$, $i = 1, 2$. For any $\varepsilon > 0$, there exists $c^{(i)} \in \mathcal{A}$ such that

$$u(x_i) - \varepsilon < E\left[\int_0^\infty e^{-\alpha t} U\big(c_t^{(i)} X_t^{(i)}\big)\,dt\right],$$

where $\{X_t^{(i)}\}$ is the solution of (1.2) corresponding to $c^{(i)}$ with $X_0^{(i)} = x_i$. Let $0 \le \xi \le 1$, and we set

$$\bar{c}_t = \frac{\xi c_t^{(1)} X_t^{(1)} + (1-\xi) c_t^{(2)} X_t^{(2)}}{\xi X_t^{(1)} + (1-\xi) X_t^{(2)}},$$

which belongs to $\mathcal{A}$. Define $\{\bar{X}_t\}$ and $\{\tilde{X}_t\}$ by

$$d\bar{X}_t = \big[(\bar{X}_t)^\gamma - \bar{c}_t \bar{X}_t\big]\,dt + \sigma \bar{X}_t\,dB_t, \quad \bar{X}_0 = \xi x_1 + (1-\xi) x_2,$$
$$\tilde{X}_t = \xi X_t^{(1)} + (1-\xi) X_t^{(2)}.$$

By concavity,

$$\tilde{X}_t \le \xi x_1 + (1-\xi) x_2 + \int_0^t \big[(\tilde{X}_s)^\gamma - \bar{c}_s \tilde{X}_s\big]\,ds + \int_0^t \sigma \tilde{X}_s\,dB_s \quad \text{a.s.}$$

By the comparison theorem, we have

$$\tilde{X}_t \le \bar{X}_t \quad \text{for all } t \ge 0 \text{ a.s.}$$

Thus, by (1.4)

$$\begin{aligned}
u\big(\xi x_1 + (1-\xi) x_2\big) &\ge E\left[\int_0^\infty e^{-\alpha t} U(\bar{c}_t \bar{X}_t)\,dt\right] \ge E\left[\int_0^\infty e^{-\alpha t} U(\bar{c}_t \tilde{X}_t)\,dt\right] \\
&= E\left[\int_0^\infty e^{-\alpha t} U\big(\xi c_t^{(1)} X_t^{(1)} + (1-\xi) c_t^{(2)} X_t^{(2)}\big)\,dt\right] \\
&\ge \xi E\left[\int_0^\infty e^{-\alpha t} U\big(c_t^{(1)} X_t^{(1)}\big)\,dt\right] + (1-\xi) E\left[\int_0^\infty e^{-\alpha t} U\big(c_t^{(2)} X_t^{(2)}\big)\,dt\right] \\
&\ge \xi u(x_1) + (1-\xi) u(x_2) - \varepsilon.
\end{aligned}$$

Therefore, letting $\varepsilon \to 0$, we obtain the concavity of u.

To prove (5.4), by Theorem 4.1, we recall that u is smooth. Furthermore, we get $u'(x) \ge 0$ for $x > 0$. If not, then $u'(a_0) < 0$ for some $a_0 > 0$. By concavity,

$$0 \le u(x) \le u'(a_0)(x - a_0) + u(a_0) \to -\infty \quad \text{as } x \to \infty,$$

which is a contradiction. Suppose that $u'(z) = 0$ for some $z > 0$. Then, by concavity, we have $u'(x) = 0$ for all $x \geq z$. Hence, by (1.5) and (1.6),

$$\alpha u(z) = \alpha u(x) = \tilde{U}(x, 0) = U(x), \quad x \geq z.$$

This is contrary to (1.4). Thus, we obtain (5.4).

Next, by definition, we have

$$0 < E\left[\int_0^\infty e^{-\alpha t} U(\check{X}_t)\, dt\right] \leq u(x), \quad x > 0,$$

where $\{\check{X}_t\}$ is the solution of (1.2) corresponding to $c_t = 1$. As in (2.7), the limit process $\check{\chi}_t := \lim_{x\to 0+} \check{X}_t$ is different from 0. Hence

$$0 < E\left[\int_0^\infty e^{-\alpha t} U(\check{\chi}_t)\, dt\right] \leq u(0+).$$

Suppose that $u'(0+) < \infty$. By (1.5) and concavity, we get $u(0+) = 0$, which is a contradiction. This implies (5.5). □

Lemma 5.2 *Under* (1.4), *there exists a unique positive strong solution* $\{X_t^*\}$ *of* (5.1).

Proof Let $\{N_t\}$ be the solution of (1.2) corresponding to $c_t = 0$. We can take the Brownian motion $\{B_t\}$ on the canonical probability space [4, p.71]. Since $0 \leq \eta \leq 1$, the probability measure $\hat{P}$ is defined by

$$d\hat{P}/dP = \exp\left\{-\int_0^t \eta(N_s)/\sigma\, dB_s - \frac{1}{2}\int_0^t \bigl(\eta(N_s)/\sigma\bigr)^2\, ds\right\}$$

for every $t \geq 0$. Girsanov's theorem yields that

$$\hat{B}_t := B_t + \int_0^t \eta(N_s)/\sigma\, ds \quad \text{is a Brownian motion under } \hat{P}.$$

Hence

$$dN_t = \bigl[(N_t)^\gamma - \eta(N_t)N_t\bigr]\, dt + \sigma N_t\, d\hat{B}_t \quad \text{under } \hat{P}.$$

Thus, (5.1) admits a weak solution.

Now, by (5.2), we have

$$\eta(x)x = \min\bigl\{\bigl(U'\bigr)^{-1} \circ u'(x), x\bigr\}.$$

Hence, by (1.4) and concavity,

$$\frac{d}{dx}\bigl(U'\bigr)^{-1} \circ u'(x) = \frac{u''(x)}{U'' \circ (U')^{-1} \circ u'(x)} \geq 0.$$

Thus, $\eta(x)x$ is nondecreasing on $(0, \infty)$. We rewrite (5.1) as the form of (2.4) to obtain $X_t^* > 0$ a.s. Then we see that the pathwise uniqueness holds for (5.1). Therefore, by the

Yamada-Watanabe theorem [10], we deduce that (5.1) admits a unique strong solution $\{X_t^*\}$. □

Proof of Theorem 5.1 Since $\{c_t^*\}$ satisfies (1.1), it belongs to $\mathcal{A}$. By Lemma 5.2, we note that

$$0 < u'(x)x \leq u(x) - u(0+) < u(x), \quad x > 0.$$

Hence, by (2.2) and (3.2),

$$\begin{aligned} E\left[\int_0^t \{e^{-\alpha s}u'(X_s^*)X_s^*\}^2\,ds\right] &\leq E\left[\int_0^t \{e^{-\alpha s}u(X_s^*)\}^2\,ds\right] \\ &\leq E\left[\int_0^t e^{-\alpha s}\zeta(X_s^*)^2\,ds\right] < \infty. \end{aligned}$$

This yields that $\{\int_0^t e^{-\alpha s}u'(X_s^*)X_s^*\,dB_s\}$ is a martingale. By (1.6), (5.3) and Ito's formula,

$$\begin{aligned} E\left[e^{-\alpha t}u(X_t^*)\right] &= u(x) + E\left[\int_0^t e^{-\alpha s}\left\{-\alpha u(X_s^*) + (X_s^*)^\gamma u'(X_s^*)\right.\right. \\ &\qquad \left.\left. - c_s^* X_s^* u'(X_s^*) + \frac{1}{2}\sigma^2(X_s^*)^2 u''(X_s^*)\right\}ds\right] \\ &= u(x) - E\left[\int_0^t e^{-\alpha s}U(c_s^*X_s^*)\,ds\right]. \end{aligned}$$

By (2.1) and (3.2), it is clear that

$$\begin{aligned} E\left[e^{-\alpha t}u(X_t^*)\right] &\leq E\left[e^{-\alpha t}\zeta(X_t^*)\right] \\ &\leq e^{-\alpha t}\left\{(1-\gamma)t + x^{(1-\gamma)}\right\}^{1/(1-\gamma)} + e^{-\alpha t}\zeta_0 \to 0 \quad \text{as } t \to \infty. \end{aligned}$$

Letting $t \to \infty$, we deduce

$$E\left[\int_0^\infty e^{-\alpha t}U(c_t^*X_t^*)\,dt\right] = u(x).$$

By the same calculation as above, we obtain

$$E\left[\int_0^\infty e^{-\alpha t}U(c_tX_t)\,dt\right] \leq u(x)$$

for any $c \in \mathcal{A}$. The proof is complete. □

Remark 5.1 From the proof of Theorem 5.1, it follows that the solution u of the HJB equation (1.5) coincides with the value function. This implies that the uniqueness holds for (1.5).

Competing interests
The author declares that they have no competing interests.

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