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On some fixed-point theorems for ψ-contraction on metric space involving a graph

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Abstract
In this paper, we introduce the (G,ψ)-contraction and the (G,ψ)-graphic contraction in a metric space by using a graph. We explain some conditions for a mapping which is a (G,ψ)-contraction to have a unique fixed point and also we give conditions as regards the existence of a fixed point for (G,ψ)-graphic contraction by applying the connectivity of the graph in both cases. Moreover, we give examples to show that our results are a substantial improvement of some known results in the literature.
MSC: 47H10; 54H25

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1 Introduction

The metric fixed-point theory has been researched extensively in the past two decades such as in a metric space endowed with a partial ordering, and many results appeared giving sufficient conditions for a mapping to be a Picard operator. For these concepts have been given two main theorems, which are the Banach Contraction Principle and the Knaster-Tarski Theorem [1].

Recently Jachymski [2] and Gwóźdź-Lukawska and Jachymski [3] have given an interesting concept in fixed-point theory with some general structures by using the context of metric spaces endowed with a graph. Jachymski [2] has proved some generalizations of the Banach Contraction Principle to mappings on a metric space endowed with a graph and also has presented its applications to the Kelisky-Rivlin Theorem on iterates of the Bernstein operators on the space $C[0,1]$. Afterwards different contractions have been studied by various authors. In [4] the contraction principle for set-valued mappings, in [5–7] Kannan type, Reich type contractions, and φ-contractions have been investigated, respectively. Some new fixed-point results for graphic contractions on a complete metric space with a graph have been presented in [8]; also they gave a particular case of almost contractions.

In this paper, motivated by the work of Jachymski [2] and Petruşel [8], we introduce new contractions for the mappings on complete metric space and prove some fixed-point theorems. Our results generalize and unify some results by the above-mentioned authors.

2 Basic facts and definitions

Let (X,d) be a metric space and Δ denote the diagonal of the Cartesian product $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$. Assume that G has no parallel edges, so one can identify G with the pair $(V(G), E(G))$.

The conversion of a graph G is denoted by G^{-1} and this is a graph obtained from G by reversing the direction of the edges. Hence

$$E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}.$$

By $\tilde{G}$ we denote the undirected graph obtained from G by omitting the direction of the edges. Indeed, it is more convenient to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric, and under this convention, we have

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

A subgraph of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let x and y be vertices in a graph G. A path from x to y of length N ($N \in \mathbf{N} \cup \{0\}$) is a sequence $(x_i)_{i=0}^N$ of $N+1$ distinct vertices such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, N$. The number edges in G forming the path is called the length of the path. A graph G is connected if there is a path between any two vertices. If a graph G is not connected then it is called disconnected and its different paths are called the components of G. Every component of G is a subgraph of it. Furthermore, G is weakly connected if $\tilde{G}$ is connected. Let G_x be the component of G which consists of all edges and vertices contained in some path in G beginning at x. Suppose that G is such that $E(G)$ is symmetric; then $V(G) = [x]_G$ where $[x]_G$ denotes the equivalence class of relations $\Re$ defined on $V(G)$ by the rule

$$y \Re z \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

Some basic notations related to connectivity of graphs can be found in [9].

If $f : X \to X$ is an operator, then we denote by

$$F(f) = \{x \in X : x = fx\}$$

the set of all fixed points of f.

Definition 1 [2] A mapping $f : X \to X$ is a Banach G-contraction or simply G-contraction if f preserves edges of G;

$$(x,y) \in E(G) \quad \Rightarrow \quad (fx, fy) \in E(G), \tag{1}$$

for all $x, y \in X$, and f decreases weights of edges of G: for all $x, y \in X$ there exists $\alpha \in (0,1)$ such that

$$(x,y) \in E(G) \quad \Rightarrow \quad d(fx, fy) <\leq \alpha d(x,y). \tag{2}$$

Definition 2 [8] The mapping $f : X \to X$ is a G-graphic contraction

(i) if f preserves edges of G;

$$(x,y) \in E(G) \quad \Rightarrow \quad (fx, fy) \in E(G), \tag{3}$$

for all $x, y \in X$;

(ii) there exists $\alpha \in (0,1)$ such that

$$(x,y) \in E(G) \quad \Rightarrow \quad d\big(fx, f^2x\big) \le \alpha d(x, fx), \tag{4}$$

for all $x, y \in X_f$.

Definition 3 [2] A mapping $f : X \to X$ is called orbitally continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbf{N}}$ of positive integers,

$$f^{k_n}x \to y \quad \text{implies} \quad f\big(f^{k_n}x\big) \to fy \quad \text{as } n \to \infty.$$

Definition 4 [2] A mapping $f : X \to X$ is called orbitally G-continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbf{N}}$ of positive integers,

$$f^{k_n}x \to y, \quad \big(f^{k_n}x, f^{k_{n+1}}x\big) \in E(G) \quad \text{imply} \quad f\big(f^{k_n}x\big) \to fy \quad \text{as } n \to \infty.$$

Now, we give a definition of the class Ψ which is used in several well-known papers to obtain some fixed-point results [10–13].

Definition 5 Let us define the class $\Psi = \{\psi : \mathbf{R}^+ \to \mathbf{R}^+ \mid \psi \text{ is nondecreasing}\}$ which satisfies the following conditions:

(i) $\psi(\omega) = 0$ if and only if $\omega = 0$;
(ii) for every $(\omega_n) \in \mathbf{R}^+$, $\psi(\omega_n) \to 0$ if and only if $\omega_n \to 0$;
(iii) for every $\omega_1, \omega_2 \in \mathbf{R}^+$, $\psi(\omega_1 + \omega_2) \le \psi(\omega_1) + \psi(\omega_2)$.

3 (G, ψ)-Contraction and related fixed-point theorems

We establish some fixed-point theorems in metric space with a graph by defining the (G, ψ)-contraction.

Definition 6 We say that a mapping $f : X \to X$ is a (G, ψ)-contraction if the following hold;

(i) f preserves edges of G, *i.e.* $((x,y) \in E(G) \Rightarrow (fx, fy) \in E(G))$, $\forall x, y \in X$;
(ii) f decreases the weight of edges of G, that is, there exists $c \in (0,1)$ such that

$$(x,y) \in E(G) \quad \Rightarrow \quad \psi\big(d(fx, fy)\big) \le c\psi\big(d(x,y)\big),$$

for all $x, y \in X$.

Lemma 1 *If $f : X \to X$ is a (G, ψ)-contraction, then f is both a (G^{-1}, ψ)-contraction and a $(\tilde{G}, \psi)$-contraction.*

Proof The proof can be obtained by the symmetry of d and the definition of the $(\tilde{G}, \psi)$-contraction. □

Lemma 2 *Let $f : X \to X$ be a (G, ψ)-contraction with constant $c \in (0,1)$; for a given $x \in X$ and $y \in [x]_{\tilde{G}}$, there exists $r(x,y) \ge 0$ such that*

$$\psi\big(d\big(f^n x, f^n y\big)\big) \le c^n r(x,y). \tag{5}$$

Proof Let $x \in X$ and $y \in [x]_{\tilde{G}}$. Then there is a path $(x_i)_{i=0}^N$ in $\tilde{G}$ from x to y, which means $x_0 = x$, $x_N = y$, and $(x_{i-1}, x_i) \in E(\tilde{G})$ for $i = 1, 2, \dots, N$. By Lemma 1, f is a $(\tilde{G}, \psi)$-contraction. With an easy induction, we have $(f^n x_{i-1}, f^n x_i) \in E(\tilde{G})$ and

$$\begin{aligned} \psi\bigl(d\bigl(f^n x_{i-1}, f^n x_i\bigr)\bigr) &\leq c\psi\bigl(d\bigl(f^{n-1} x_{i-1}, f^{n-1} x_i\bigr)\bigr) \\ &\leq c\bigl(c\psi\bigl(d\bigl(f^{n-2} x_{i-1}, f^{n-2} x_i\bigr)\bigr)\bigr) \leq \cdots \leq c^n \psi\bigl(d(x_{i-1}, x_i)\bigr) \end{aligned}$$

for all $n \in \mathbf{N}$ and $i = 1, 2, \dots, N$.

Hence using the triangle inequality, we get

$$\psi\bigl(d\bigl(f^n x, f^n y\bigr)\bigr) \leq \sum_{i=1}^{N} \psi\bigl(d\bigl(f^n x_{i-1}, f^n x_i\bigr)\bigr) \leq c^n \sum_{i=1}^{N} \psi\bigl(d(x_{i-1}, x_i)\bigr).$$

So it qualifies to set $r(x, y) := \sum_{i=1}^{N} \psi(d(x_{i-1}, x_i))$. □

Lemma 3 *Let (X, d) be a complete metric space endowed with a graph G and $f : X \to X$ be a (G, ψ)-contraction for which there exists $x_0 \in X$ such that $fx_0 \in [x_0]_{\tilde{G}}$. Let $\tilde{G}_{x_0}$ be the component of $\tilde{G}$ containing x_0. Then $[x_0]_{\tilde{G}}$ is f-invariant and $f|_{[x]_{\tilde{G}}}$ is a $(\tilde{G}_{x_0}, \psi)$-contraction. Furthermore, $x, y \in [x_0]_{\tilde{G}}$, and the sequences $(f^n x)_{n\in\mathbf{N}}$ and $(f^n y)_{n\in\mathbf{N}}$ are Cauchy equivalent.*

Proof The proof of this lemma can obtained by using similar arguments as given in [7]. So we omit the proof. □

The following result shows that there is a close relation between convergence of an iteration sequence which can be obtained by using a (G, ψ)-contraction mapping and connectivity of the graph.

Theorem 1 *Let (X, d) be a metric space endowed with a graph G and $f : X \to X$ be a (G, ψ)-contraction, then the following statements are equivalent*:

(i) *G is weakly connected*;
(ii) *for given $x, y \in X$, the sequences $(f^n x)_{n\in\mathbf{N}}$ and $(f^n y)_{n\in\mathbf{N}}$ are Cauchy equivalent*;
(iii) $\operatorname{card} F(f) \leq 1$.

Proof (i) $\Rightarrow$ (ii) Let f be a (G, ψ)-contraction and $x, y \in X$. By hypothesis, $[x]_{\tilde{G}} = X$, so $fx \in [x]_{\tilde{G}}$. By Lemma 2, we get

$$\psi\bigl(d\bigl(f^n x, f^{n+1} x\bigr)\bigr) \leq c^n r(x, fx)$$

for all $n \in \mathbf{N}$. Hence

$$\sum_{n=0}^{\infty} \psi\bigl(d\bigl(f^n x, f^{n+1} x\bigr)\bigr) < \infty$$

and if we use a standard argument, then $(f^n x)_{n\in\mathbf{N}}$ is obtained as a Cauchy sequence. Since also $y \in [x]_{\tilde{G}}$, Lemma 2 leads to $\psi(d(f^n x, f^n y)) \leq c^n r(x, y)$. Therefore, $(f^n x)_{n\in\mathbf{N}}$ and $(f^n y)_{n\in\mathbf{N}}$ are equivalent. Clearly, because $(f^n x)_{n\in\mathbf{N}}$ is a Cauchy sequence, so is $(f^n y)_{n\in\mathbf{N}}$.

(ii) ⇒ (iii) Let f be a (G,ψ)-contraction and $x,y \in F(f)$. By (ii), $(f^n x)_{n\in\mathbf{N}}$ and $(f^n y)_{n\in\mathbf{N}}$ are equivalent, which yields $x = y$.

(iii) ⇒ (ii) Suppose, to the contrary, G is not weakly connected, that is, $\tilde{G}$ is disconnected. Let $x_0 \in X$. Then the sets $[x_0]_{\tilde{G}}$ and $X - [x_0]_{\tilde{G}}$ both are nonempty. Let $y_0 \in X - [x_0]_{\tilde{G}}$ and define

$$fx = \begin{cases} x_0, & \text{if } x \in [x_0]_{\tilde{G}}, \\ y_0, & \text{if } x \in X - [x_0]_{\tilde{G}}. \end{cases}$$

Obviously, $F(f) = \{x_0, y_0\}$. We show f is a (G,ψ)-contraction. Let $(x,y) \in E(G)$. Then $[x]_{\tilde{G}} = [y]_{\tilde{G}}$, so either $x,y \in [x_0]_{\tilde{G}}$ or $x,y \in X - [x_0]_{\tilde{G}}$. Hence in both cases $fx = fy$, so $(fx,fy) \in E(G)$ as $E(G) \supseteq \Delta$, and $\psi(d(fx,fy)) = 0$. Thereby, f is a (G,ψ)-contraction having two fixed points which violates the assumption. □

The following result is an easy consequence of Theorem 1.

Corollary 1 *Let (X,d) be a complete metric space endowed with a graph G and $f : X \to X$ be a (G,ψ)-contraction, then the following statements are equivalent*:

(i) *G is weakly connected*;
(ii) *there is $x^* \in X$ such that* $\lim_{n\to\infty} f^n x = x^*$, *for all $x \in X$.*

Now, we give an example of f being a (G,ψ)-contraction and this example shows that we could not add that x^* is a fixed point of f in Corollary 1.

Example 1 Let $X = [0,1]$ be endowed with the usual metric. Take

$$E(G) = \{(0,0)\} \cup \{(0,1)\} \cup \{(x,y) \in (0,1] \times (0,1] : x \geq y\},$$

and $f : X \to X$ as follows:

$$fx = \begin{cases} \frac{x}{3}, & \text{if } x \in (0,1], \\ \frac{1}{2}, & \text{if } x = 0. \end{cases}$$

Then f is a (G,ψ)-contraction where $\psi(\omega) = \frac{\omega}{\omega+1}$.

Proof It can be easily seen that G is a weakly connected graph and f is a (G,ψ)-contraction where $\psi(\omega) = \frac{\omega}{\omega+1}$. It is a fact that $(f^n x) \to 0$, for all $x \in X$ but f has no fixed point. □

For any mapping which satisfies the condition of Corollary 1 to have a fixed point we need to add condition (6), which is given in the following theorem.

Theorem 2 *Let (X,d) be a complete metric space and the triple (X,d,G) have the following condition*:

$$\begin{aligned} &\textit{for any } (x_n)_{n\in\mathbf{N}} \textit{ in } X, \textit{ if } x_n \to x \textit{ and } (x_n, x_{n+1}) \in E(G) \textit{ for } n \in \mathbf{N}, \\ &\textit{then there is a subsequence } (x_{k_n})_{n\in\mathbf{N}} \textit{ with } (x_{k_n}, x) \in E(G) \textit{ for } n \in \mathbf{N}. \end{aligned} \tag{6}$$

Let $f : X \to X$ be a (G,ψ)-contraction, and $X_f = \{x \in X : (x,fx) \in E(G)\}$. Then the following statements hold.

(i) $\operatorname{card} F(f) = \operatorname{card}\{[x]_{\tilde{G}} : x \in X_f\}$.
(ii) $F(f) \neq \emptyset$ *iff* $X_f \neq \emptyset$.
(iii) f *has a unique fixed point iff there exists* $x_0 \in X_f$ *such that* $X_f \subseteq [x_0]_{\tilde{G}}$.
(iv) *For any* $x \in X_f$, $f|_{[x]_{\tilde{G}}}$ *is a Picard operator.*
(v) *If* $X_f \neq \emptyset$ *and* G *is weakly connected, then* f *is a Picard operator.*
(vi) *If* $X' := \bigcup\{[x]_{\tilde{G}} : x \in X_f\}$, *then* $f|_{X'}$ *is a weakly Picard operator.*
(vii) *If* $f \subseteq E(G)$, *then* f *is a weakly Picard operator.*

Proof Initially, we prove the items (iv) and (v). Take $x \in X_f$ and then $fx \in [x]_{\tilde{G}}$, so by Lemma 3, if $y \in [x]_{\tilde{G}}$, then $(f^n x)_{n\in\mathbf{N}}$ and $(f^n y)_{n\in\mathbf{N}}$ are Cauchy equivalent. Since X is complete, $(f^n x)_{n\in\mathbf{N}}$ converges to some $x^* \in X$. It is obvious that $\lim_{n\to\infty} f^n y = x^*$. Then by using induction we get

$$\left(f^n x, f^{n+1} x\right) \in E(G) \tag{7}$$

for all $n \in \mathbf{N}$, since $(x, fx) \in E(G)$. By (6), there is a subsequence $(f^{k_n} x)_{n\in\mathbf{N}}$ such that $(f^{k_n} x, x^*) \in E(G)$ for all $n \in \mathbf{N}$. If we use (7), we conclude that $(x, fx, f^2 x, \ldots, f^{k_1}, x^*)$ is a path in G and also in $\tilde{G}$ from x to x^*, and this means that $x^* \in [x]_{\tilde{G}}$. Since f is a (G,ψ)-contraction we have

$$\psi\left(d\left(f^{k_{n+1}} x, fx^*\right)\right) \leq c\psi\left(d\left(f^{k_n} x, x^*\right)\right),$$

for all $n \in \mathbf{N}$. By taking the limit as $n \to \infty$, we deduce $fx^* = x^*$. Thereby, $f|_{[x]_{\tilde{G}}}$ is a Picard operator. Also, we conclude that f is a Picard operator, when $[x]_{\tilde{G}} = X$, since there is weakly connectedness of G.

(vi) is obvious from (iv). For proof of (vii), if $f \subseteq E(G)$ then $X_f = X$ and so $X' = X$ holds. Thus f is a weakly Picard operator because of (vi).

Let us define a mapping to prove (i): $\rho(x) = [x]_{\tilde{G}}$ for all $x \in F(f)$. It is sufficient to show that $\rho : F(f) \to C = \{[x]_{\tilde{G}} : x \in X_f\}$ is a bijection. Because $E(G) \supseteq \Delta$, we deduce $F(f) \subseteq X_f$ and then $\rho(F(f)) \subseteq C$. Beside, if $x \in X_f$, then by (iv), $\lim_{n\to\infty} f^n x \in [x]_{\tilde{G}} \cap F(f)$, which implies $\rho(\lim_{n\to\infty} f^n x) = [x]_{\tilde{G}}$ and so ρ is a surjective mapping. We show that f is injective. Take $x_1, x_2 \in F(f)$ which are such that $\rho(x_1) = \rho(x_2) \Rightarrow [x_1]_{\tilde{G}} = [x_2]_{\tilde{G}}$, then $x_2 \in [x_1]_{\tilde{G}}$ and so, by (i),

$$\lim_{n\to\infty} f^n x_2 \in [x_1]_{\tilde{G}} \cap F(f) = \{x_1\},$$

which gives $x_1 = x_2$. Thus, f is injective and this is the desired result. Finally, one can see that (ii) and (iii) are easy consequences of (i). □

Corollary 2 *Let* (X,d) *be complete metric space and* (X,d,G) *obey condition* (6). *The following are equivalent*:
(i) G *is weakly connected*;
(ii) *every* (G,ψ)*-contraction* $f : X \to X$ *such that* $(x_0, fx_0) \in E(G)$, *for some* $x_0 \in X$, *is a Picard operator*;
(iii) *for any* (G,ψ)*-contraction,* $\operatorname{card} F(f) \leq 1$.

Proof (i) $\Rightarrow$ (ii): This can be obtained directly from Theorem 2(v).

(ii) $\Rightarrow$ (iii): Let $f : X \to X$ be a (G, ψ)-contraction. If X_f is empty, so is $F(f)$, because $F(f)$ is a subset of X_f. If X_f is nonempty, then by (ii), $F(f)$ is singleton. In these two cases, $\operatorname{card} F(f) \leq 1$.

(iii) $\Rightarrow$ (i): This implication follows from Theorem 1. □

Remark 1 In the above results by taking $\psi(\omega) = \omega$, we obtain Corollary 3.2, which is given in [2].

4 (G, ψ)-Graphic contraction and fixed-point theorems

Now, we define (G, ψ)-graphic contraction and give some results and examples.

Definition 7 Let (X, d) be a metric space and G be a graph. The mapping $f : X \to X$ is called a (G, ψ)-graphic contraction if the following conditions hold:

(i) $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ (f is edge preserving);

(ii) there exists a $\psi \in \Psi$ with constants $c \in [0, 1)$ such that

$$\psi\big(d\big(fx, f^2x\big)\big) \leq c\psi\big(d(x, fx)\big)$$

for all $x \in X^f$, where $X^f := \{x \in X : (x, fx) \in E(G) \text{ or } (fx, x) \in E(G)\}$.

Firstly, we give the following lemmas which can be proved as in the above section.

Lemma 4 *If $f : X \to X$ is a (G, ψ)-graphic contraction, then f is both a (G^{-1}, ψ)-graphic contraction and a $(\tilde{G}, \psi)$-graphic contraction.*

Lemma 5 *Let $f : X \to X$ be a (G, ψ)-graphic contraction with constant $c \in [0, 1)$. Then, given $x \in X^f$, there exists $r(x) \geq 0$ such that*

$$\psi\big(d\big(f^n x, f^{n+1}x\big)\big) \leq c^n r(x), \tag{8}$$

for all $n \in \mathbf{N}$, where $r(x) := \psi(d(x, fx))$.

Lemma 6 *Suppose that $f : X \to X$ is a (G, ψ)-graphic contraction. Then for each $x \in X^f$, there exists $x^* \in X$ such that the sequence $(f^n x)_{n\in\mathbf{N}}$ converges to x^* as $n \to \infty$.*

Proof Take an arbitrary element x in X^f. By Lemma 5, we obtain

$$\psi\big(d\big(f^n x, f^{n+1}x\big)\big) \leq c^n r(x),$$

for all $n \in \mathbf{N}$. Therefore, $\sum_{n=0}^{\infty} \psi(d(f^n x, f^{n+1}x)) < \infty$ and so $\psi(d(f^n x, f^{n+1}x)) \to 0$; consequently using the property of ψ we have $d(f^n x, f^{n+1}x) \to 0$. Then we say that $(f^n x)_{n\in\mathbf{N}}$ is a Cauchy sequence. By the completeness of X, there exists $x^* \in X$ such that $(f^n x)_{n\in\mathbf{N}}$ converges as $n \to \infty$. □

Lemma 7 *The self-mapping f is a (G, ψ)-graphic contraction for which there exists $x_0 \in X$ such that $fx_0 \in [x_0]_{\tilde{G}}$. Then the set $[x_0]_{\tilde{G}}$ invariant with respect to f and $f|_{[x_0]_{\tilde{G}}}$ is a $(\tilde{G}_{x_0}, \psi)$-graphic contraction, where $\tilde{G}_{x_0}$ is the component of $\tilde{G}$ containing x_0.*

Proof Let x be an element in $[x_0]_{\tilde{G}}$. Then there exist $(x_i)_{i=0}^N$ in $\tilde{G}$ from x_0 to x, *i.e.*, $x_N = x$ and $(x_{i-1}, x_i) \in E(\tilde{G})$ for $i = 1, 2, \ldots, N$. Since f is a (G, ψ)-graphic contraction we get $(fx_{i-1}, fx_i) \in E(\tilde{G})$ for $i = 1, 2, \ldots, N$. So we have a path from fx_0 to fx. Therefore $fx \in [fx_0]_{\tilde{G}} = [x_0]_{\tilde{G}}$ since $fx_0 \in [x_0]_{\tilde{G}}$. Consequently $[x_0]_{\tilde{G}}$ is invariant with respect to f.

Take $(x, y) \in E(\tilde{G}_{x_0})$; then there is a path $(x_i)_{i=0}^N$ in $\tilde{G}$ from x_0 to y such that $x_{N-1} = x$. Also let $(y_i)_{i=0}^M$ be a path in $\tilde{G}$ from x_0 to fx_0. Then we realize

$$(y_0, y_1, \ldots, y_M, fx_1, fx_2, \ldots, fx_{N-1} = fx, fx_N = fy)$$

is a path in $\tilde{G}$ from x_0 to fy such that $(fx, fy) \in E(\tilde{G}_{x_0})$. Furthermore, f is a $(\tilde{G}_{x_0}, \psi)$-graphic contraction because $E(\tilde{G}_{x_0}) \subseteq E(\tilde{G})$ and f is a $(\tilde{G}, \psi)$-graphic contraction. □

Theorem 3 *Let (X, d) be a complete metric space and let the triple (X, d, G) have the following condition*:

$$\begin{aligned}
&\textit{for any } (x_n)_{n\in\mathbf{N}} \textit{ in } X, \textit{ if } x_n \to x \textit{ and } (x_n, x_{n+1}) \in E(G) \\
&\bigl(\textit{or, respectively}, (x_{n+1}, x_n) \in E(G)\bigr) \\
&\textit{ for all } n \in \mathbf{N}, \textit{ then there is a subsequence } (x_{k_n})_{n\in\mathbf{N}} \textit{ with } (x_{k_n}, x) \in E(G) \\
&\bigl(\textit{or, respectively}, (x, x_{k_n}) \in E(G)\bigr) \textit{ for all } n \in \mathbf{N}.
\end{aligned} \tag{9}$$

Let $f : X \to X$ be a (G, ψ)-graphic contraction and f is orbitally G-continuous. Then the following statements hold:

(i) *$F(f) \neq \emptyset$ if and only if $X^f \neq \emptyset$.*

(ii) *If $X^f \neq \emptyset$ and G is weakly connected, then f is a weakly Picard operator.*

(iii) *For any $x \in X^f$, we see that $f|_{[x]_{\tilde{G}}}$ is a weakly Picard operator.*

Proof We begin with the statement (iii). Let $x \in X^f$; by Lemma 6, there exists $x^* \in X$ such that $\lim_{n\to\infty} f^n x = x^*$. Since $x \in X^f$, then $f^n x \in X^f$ for every $n \in \mathbf{N}$. Now assume that $(x, fx) \in E(G)$. (A similar deduction can be made if $(fx, x) \in E(G)$.) By condition (9), there is a subsequence $(f^{k_n} x)_{n\in\mathbf{N}}$ of $(f^n x)_{n\in\mathbf{N}}$ such that $(f^{k_n} x, x^*) \in E(G)$ for each $n \in \mathbf{N}$. A path in G can be formed by using the points $x, fx, \ldots, f^{k_1} x, x^*$ and hence $x^* \in [x]_{\tilde{G}}$. Since f is orbitally G-continuous, we see that x^* is a fixed point for $f|_{[x]_{\tilde{G}}}$.

To prove (i), using (iii) we have $F(f) \neq \emptyset$ if $X^f \neq \emptyset$. Suppose that $F(f) \neq \emptyset$. By using the assumption that $\Delta \subseteq E(G)$, we immediately obtain $X^f \neq \emptyset$. Hence (i) holds.

For proving (ii) let $x \in X^f$. If we use weak connectivity of G, we have $X = [x]_{\tilde{G}}$ and by applying (iii) we obtain the desired result. □

The next example illustrates that f must be orbitally G-continuous in order to obtain statements which are given in Theorem 3.

Example 2 Let $X = [0, 1]$ be endowed with the usual metric. Consider

$$E(G) = \bigl\{(0, 0)\bigr\} \cup \bigl\{(0, x) : x \geq 1/2\bigr\} \cup \bigl\{(x, y) : x, y \in (0, 1]\bigr\},$$

and $f : X \to X$,

$$fx = \begin{cases} \frac{x}{2}, & \text{if } x \in (0,1]; \\ \frac{1}{2}, & \text{if } x = 0. \end{cases}$$

Then G is weakly connected, X^f is nonempty and f is a (G,ψ)-graphic contraction where $\psi(\omega) = \frac{\omega}{3}$, but it is not orbitally G-continuous. Thus, f does not have a fixed point.

Remark 2 In Theorem 3, by replacing the condition that the triple (X,d,G) satisfies (9) and f is orbitally G-continuous with the mapping f is orbitally continuous, we have the above result, too.

The following example demonstrates that the (G,ψ)-graphic contraction is more general than the (G,ψ)-contraction.

Example 3 Let $X = [0,1]$ be endowed with the usual metric. Take

$$E(G) = \{(0,0)\} \cup \{(0,1)\} \cup \{(x,y) \in (0,1] \times (0,1] : x \geq y\},$$

and $f : X \to X$ as follows:

$$fx = \begin{cases} \frac{x}{2}, & \text{if } x \in (0,1], \\ \frac{3}{4}, & \text{if } x = 0. \end{cases}$$

Then G is weakly connected and X^f is nonempty and f is a (G,ψ)-graphic contraction with $\psi(\omega) = \frac{\omega}{2}$ which is not a (G,ψ)-contraction.

Proof It is clear that G is weakly connected, $X^f \neq \emptyset$, and with simple calculations it can be easily seen that f is a (G,ψ)-graphic contraction. Take

$$\psi\left(d\left(f0, f\frac{1}{2}\right)\right) \leq c\psi\left(d\left(0, \frac{1}{2}\right)\right) \quad \Rightarrow \quad \frac{1}{4} \leq c\frac{1}{4},$$

which is a contradiction since $c \in [0,1)$. Thus, f is not (G,ψ)-contraction. □

Remark 3 In Theorem 3, if we take $\psi(\omega) = \omega$, then we get Theorem 2.1, which is given in [8].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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