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Barnes-type Narumi of the second kind and Barnes-type Peters of the second kind hybrid polynomials

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Abstract

In this paper, by considering Barnes-type Narumi polynomials of the second kind and Barnes-type Peters polynomials of the second kind, we define and investigate the hybrid polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

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1 Introduction

In this paper, we consider the polynomials

$$\widehat{NS}_n(x) = \widehat{NS}_n(x|a; \lambda; \mu) = \widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s)$$

called Barnes-type Narumi of the second kind and Barnes-type Peters of the second kind hybrid polynomials, whose generating function is given by

$$\prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^x = \sum_{n=0}^{\infty} \widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \frac{t^n}{n!}, \tag{1}$$

where $a_1, \dots, a_r, \lambda_1, \dots, \lambda_s, \mu_1, \dots, \mu_s \in \mathbb{C}$ with $a_1, \dots, a_r, \lambda_1, \dots, \lambda_s \neq 0$. When $x = 0$, $\widehat{NS}_n = \widehat{NS}_n(0) = \widehat{NS}_n(0|a; \lambda; \mu) = \widehat{NS}_n(0|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s)$ are called Barnes-type Narumi of the second kind and Barnes-type Peters of the second kind hybrid numbers.

Recall that the Barnes-type Narumi polynomials of the second kind, denoted by $\widehat{N}_n(x|a_1, \dots, a_r)$, are given by the generating function as

$$\prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) (1+t)^x = \sum_{n=0}^{\infty} \widehat{N}_n(x|a_1, \dots, a_r) \frac{t^n}{n!}.$$

If $x = 0$, we write $\widehat{N}_n(a_1, \dots, a_r) = \widehat{N}_n(0|a_1, \dots, a_r)$. Narumi polynomials were mentioned in [1, p.127]. In addition, the Barnes-type Peters polynomials of the second kind, denoted by



$\hat{s}_n(x|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s)$, are given by the generating function as

$$\prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^x = \sum_{n=0}^{\infty} \hat{s}_n(x|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \frac{t^n}{n!}.$$

If $s = 1$, then $\hat{s}_n(x|\lambda; \mu)$ are the Peters polynomials of the second kind. Peters polynomials were mentioned in [1, p.128] and have been investigated in e.g. [2].

In this paper, by considering Barnes-type Narumi polynomials of the second kind and Barnes-type Peters polynomials of the second kind, we define and investigate the hybrid polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2 Umbral calculus

Let \mathbb{C} be the complex number field and let \mathcal{F} be the set of all formal power series in the variable t :

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \mid a_k \in \mathbb{C} \right\}. \tag{2}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let \mathbb{P}^* be the vector space of all linear functionals on \mathbb{P} . $\langle L|p(x) \rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on \mathbb{P}^* are defined by $\langle L + M|p(x) \rangle = \langle L|p(x) \rangle + \langle M|p(x) \rangle$, $\langle cL|p(x) \rangle = c\langle L|p(x) \rangle$, where c is a complex constant in \mathbb{C} . For $f(t) \in \mathcal{F}$, let us define the linear functional on \mathbb{P} by setting

$$\langle f(t)|x^n \rangle = a_n \quad (n \geq 0). \tag{3}$$

In particular,

$$\langle t^k|x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0), \tag{4}$$

where $\delta_{n,k}$ is Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k \rangle}{k!} t^k$, we have $\langle f_L(t)|x^n \rangle = \langle L|x^n \rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from \mathbb{P}^* onto \mathcal{F} . Henceforth, \mathcal{F} denotes both the algebra of formal power series in t and the vector space of all linear functionals on \mathbb{P} , and so an element $f(t)$ of \mathcal{F} will be thought of as both a formal power series and a linear functional. We call \mathcal{F} the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O(f(t))$ of a power series $f(t) (\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$, then $f(t)$ is called a *delta series*; if $O(f(t)) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k|s_n(x) \rangle = n! \delta_{n,k}$, for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$.

For $f(t), g(t) \in \mathcal{F}$, and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x) \rangle = \langle f(t)|g(t)p(x) \rangle = \langle g(t)|f(t)p(x) \rangle \tag{5}$$

and

$$\begin{aligned}
 f(t) &= \sum_{k=0}^{\infty} (f(t)|x^k) \frac{t^k}{k!}, \\
 p(x) &= \sum_{k=0}^{\infty} (t^k|p(x)) \frac{x^k}{k!}
 \end{aligned}
 \tag{6}$$

[1, Theorem 2.2.5]. Thus, by (6), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt} p(x) = p(x+y).
 \tag{7}$$

Sheffer sequences are characterized by the generating function of [1, Theorem 2.3.4].

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if*

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim (g(t), f(t))$, we have the following equations [1, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]:

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0),
 \tag{8}$$

$$s_n(x) = \sum_{j=0}^n \frac{1}{j!} (g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n) x^j,
 \tag{9}$$

$$s_n(x+y) = \sum_{j=0}^n \binom{n}{j} s_j(x) p_{n-j}(y),
 \tag{10}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$. Then the transfer formula [1, Corollary 3.8.2] is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim (g(t), f(t))$, and $r_n(x) \sim (h(t), l(t))$, assume that

$$s_n(x) = \sum_{m=0}^n C_{n,m} r_m(x) \quad (n \geq 0).$$

Then we have [1, p.132]

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m \middle| x^n \right\rangle.
 \tag{11}$$

3 Main results

For convenience, we introduce an Appell sequence of polynomials $F_n(a_1, \dots, a_r)$ ($a_1, \dots, a_r \neq 0$), defined by

$$\prod_{i=1}^r \left(\frac{e^{a_i t} - 1}{t} \right) e^{xt} = \sum_{n=0}^{\infty} F_n(x|a_1, \dots, a_r) \frac{t^n}{n!}.$$

If $x = 0$, we write $F_n(a_1, \dots, a_r) = F_n(0|a_1, \dots, a_r)$. By [1, Theorem 2.5.8] with $y = 0$, we have

$$F_n(x|a_1, \dots, a_r) = \sum_{m=0}^n \binom{n}{m} F_{n-m}(a_1, \dots, a_r) x^m.$$

More precisely, one can show that

$$F_n(x|a_1, \dots, a_r) = \sum_{i=0}^n \sum_{l_1 + \dots + l_r = i} \frac{i!}{(i+r)!} \binom{n}{i} \binom{i+r}{l_1+1, \dots, l_r+1} a_1^{l_1+1} \dots a_r^{l_r+1} x^{n-i},$$

where

$$\binom{i+r}{l_1+1, \dots, l_r+1} = \frac{(i+r)!}{(l_1+1)! \dots (l_r+1)!}.$$

Remember that the generalized Barnes-type Euler polynomials $E_n(x|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s)$ are defined by the generating function

$$\prod_{j=1}^s \left(\frac{2}{1 + e^{\lambda_j t}} \right)^{\mu_j} e^{xt} = \sum_{n=0}^{\infty} E_n(x|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \frac{t^n}{n!}.$$

If $\mu_1 = \dots = \mu_s = 1$, then $E_n(x|\lambda_1, \dots, \lambda_s) = E_n(x|\lambda_1, \dots, \lambda_s; 1, \dots, 1)$ are called the Barnes-type Euler polynomials. If further $\lambda_1 = \dots = \lambda_s = 1$, then $E_n^{(r)}(x) = E_n(x|1, \dots, 1; 1, \dots, 1)$ are called the Euler polynomials of order s . If $x = 0$, we write $E_n(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) = E_n(0|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s)$. By [1, Theorem 2.5.8] with $y = 0$, we have

$$E_n(x|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) = \sum_{m=0}^n \binom{n}{m} E_{n-m}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) x^m.$$

We note that

$$\begin{aligned} tF_n(x|a_1, \dots, a_r) &= \frac{d}{dx} F_n(x|a_1, \dots, a_r) = nF_{n-1}(x|a_1, \dots, a_r), \\ tE_n(x|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) &= \frac{d}{dx} E_n(x|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\ &= nE_{n-1}(x|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s). \end{aligned}$$

From the definition (1), $\widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s)$ is the Sheffer sequence for the pair

$$g(t) = \prod_{i=1}^r \left(\frac{te^{a_i t}}{e^{a_i t} - 1} \right) \prod_{j=1}^s \left(\frac{1 + e^{\lambda_j t}}{e^{\lambda_j t}} \right)^{\mu_j} \quad \text{and} \quad f(t) = e^t - 1.$$

So,

$$\widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \sim \left(\prod_{i=1}^r \left(\frac{te^{a_i t}}{e^{a_i t} - 1} \right) \prod_{j=1}^s \left(\frac{1 + e^{\lambda_j t}}{e^{\lambda_j t}} \right)^{\mu_j}, e^t - 1 \right). \quad (12)$$

3.1 Explicit expressions

Let $(n)_j = n(n-1)\cdots(n-j+1)$ ($j \geq 1$) with $(n)_0 = 1$. The (signed) Stirling numbers of the first kind $S_1(n, m)$ are defined by

$$(x)_n = \sum_{m=0}^n S_1(n, m)x^m.$$

Theorem 1

$$\begin{aligned} &\widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\ &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m (-1)^m \binom{m}{l} S_1(n, m) E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\ &\quad \times F_l(-x|a_1, \dots, a_r) \end{aligned} \quad (13)$$

$$\begin{aligned} &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m \binom{m}{l} S_1(n, m) E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\ &\quad \times F_l \left(x + \sum_{j=1}^s \lambda_j \mu_j - \sum_{i=1}^r a_i \mid a_1, \dots, a_r \right) \end{aligned} \quad (14)$$

$$\begin{aligned} &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m (-1)^m \binom{m}{l} S_1(n, m) E_{m-l}(a_1, \dots, a_r) \\ &\quad \times E_l(-x|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \end{aligned} \quad (15)$$

$$= \sum_{j=0}^n \sum_{l=j}^n \binom{n}{l} S_1(l, j) \widehat{NS}_{n-l}(x^j) \quad (16)$$

$$= \sum_{l=0}^n \binom{n}{l} \widehat{s}_{n-l}(\lambda_1, \dots, \lambda_r; \mu_1, \dots, \mu_r) \widehat{N}_l(x|a_1, \dots, a_r), \quad (17)$$

$$= \sum_{l=0}^n \binom{n}{l} \widehat{N}_{n-l}(a_1, \dots, a_r) \widehat{s}_l(x|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s). \quad (18)$$

Proof Since

$$\prod_{i=1}^r \left(\frac{te^{a_i t}}{e^{a_i t} - 1} \right) \prod_{j=1}^s \left(\frac{1 + e^{\lambda_j t}}{e^{\lambda_j t}} \right)^{\mu_j} \widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \sim (1, e^t - 1) \quad (19)$$

and

$$(x)_n \sim (1, e^t - 1), \quad (20)$$

we have

$$\begin{aligned}
 & \widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= \prod_{i=1}^r \left(\frac{e^{a_i t} - 1}{t e^{a_i t}} \right) \prod_{j=1}^s \left(\frac{e^{\lambda_j t}}{1 + e^{\lambda_j t}} \right)^{\mu_j} (x)_n \\
 &= \sum_{m=0}^n S_1(n, m) \prod_{i=1}^r \left(\frac{e^{a_i t} - 1}{t e^{a_i t}} \right) \prod_{j=1}^s \left(\frac{e^{\lambda_j t}}{1 + e^{\lambda_j t}} \right)^{\mu_j} x^m \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n S_1(n, m) \prod_{i=1}^r \left(\frac{e^{-a_i t} - 1}{-t} \right) \prod_{j=1}^s \left(\frac{2}{1 + e^{-\lambda_j t}} \right)^{\mu_j} x^m \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n S_1(n, m) \prod_{i=1}^r \left(\frac{e^{-a_i t} - 1}{-t} \right) (-1)^m E_m(-x|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n S_1(n, m) \prod_{i=1}^r \left(\frac{e^{-a_i t} - 1}{-t} \right) (-1)^m \\
 &\quad \times \sum_{l=0}^m \binom{m}{l} E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) (-x)^l \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \\
 &\quad \times \sum_{l=0}^m (-1)^l \binom{m}{l} E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \prod_{i=1}^r \left(\frac{e^{-a_i t} - 1}{-t} \right) x^l \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \\
 &\quad \times \sum_{l=0}^m (-1)^l \binom{m}{l} E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) (-1)^l F_l(-x|a_1, \dots, a_r).
 \end{aligned}$$

So, we get (13).

We can obtain an alternative expression (14) for $\widehat{NS}_n(x)$ as follows:

$$\begin{aligned}
 \widehat{NS}_n(x) &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n S_1(n, m) e^{(\sum_{j=1}^s \lambda_j \mu_j - \sum_{i=1}^r a_i) t} \\
 &\quad \times \prod_{i=1}^r \left(\frac{e^{a_i t} - 1}{t} \right) \prod_{j=1}^s \left(\frac{2}{1 + e^{\lambda_j t}} \right)^{\mu_j} x^m \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n S_1(n, m) e^{(\sum_{j=1}^s \lambda_j \mu_j - \sum_{i=1}^r a_i) t} \\
 &\quad \times \prod_{i=1}^r \left(\frac{e^{a_i t} - 1}{t} \right) E_m(x|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n S_1(n, m) e^{(\sum_{j=1}^s \lambda_j \mu_j - \sum_{i=1}^r a_i) t} \\
 &\quad \times \prod_{i=1}^r \left(\frac{e^{a_i t} - 1}{t} \right) \sum_{l=0}^m \binom{m}{l} E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) x^l
 \end{aligned}$$

$$\begin{aligned}
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m \binom{m}{l} S_1(n, m) E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &\quad \times e^{(\sum_{j=1}^s \lambda_j \mu_j - \sum_{i=1}^r a_i) t} \prod_{i=1}^r \left(\frac{e^{a_i t} - 1}{t} \right) x^l \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m \binom{m}{l} S_1(n, m) E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &\quad \times F_l \left(x + \sum_{j=1}^s \lambda_j \mu_j - \sum_{i=1}^r a_i \mid a_1, \dots, a_r \right).
 \end{aligned}$$

From the proof of (13),

$$\begin{aligned}
 &\widehat{NS}_n(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n S_1(n, m) \prod_{j=1}^s \left(\frac{2}{1 + e^{-\lambda_j t}} \right)^{\mu_j} \prod_{i=1}^r \left(\frac{e^{-a_i t} - 1}{-t} \right) x^m \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n S_1(n, m) \prod_{j=1}^s \left(\frac{2}{1 + e^{-\lambda_j t}} \right)^{\mu_j} (-1)^m F_m(-x \mid a_1, \dots, a_r) \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \prod_{j=1}^s \left(\frac{2}{1 + e^{-\lambda_j t}} \right)^{\mu_j} \sum_{l=0}^m \binom{m}{l} F_{m-l}(a_1, \dots, a_r) (-x)^l \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \sum_{l=0}^m \binom{m}{l} F_{m-l}(a_1, \dots, a_r) (-1)^l \prod_{j=1}^s \left(\frac{2}{1 + e^{-\lambda_j t}} \right)^{\mu_j} x^l \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \sum_{l=0}^m \binom{m}{l} F_{m-l}(a_1, \dots, a_r) E_l(-x \mid \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m (-1)^m \binom{m}{l} S_1(n, m) F_{m-l}(a_1, \dots, a_r) E_l(-x \mid \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s),
 \end{aligned}$$

which is the identity (15).

By (12), we have

$$\begin{aligned}
 &(g(\bar{f}(t))^{-1} \bar{f}(t)^j \mid x^n) \\
 &= \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (\ln(1+t))^j \mid x^n \right\rangle \\
 &= \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \left| j! \sum_{l=j}^{\infty} S_1(l, j) \frac{t^l}{l!} x^n \right. \right\rangle \\
 &= j! \sum_{l=j}^n \binom{n}{l} S_1(l, j) \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \mid x^{n-l} \right\rangle \\
 &= j! \sum_{l=j}^n \binom{n}{l} S_1(l, j) \left\langle \sum_{i=0}^{\infty} \widehat{NS}_i \frac{t^i}{i!} \mid x^{n-l} \right\rangle = j! \sum_{l=j}^n \binom{n}{l} S_1(l, j) \widehat{NS}_{n-l}.
 \end{aligned}$$

By (9), we get the identity (16).

Next,

$$\begin{aligned}
 & \widehat{NS}_n(y|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= \left\langle \sum_{i=0}^{\infty} \widehat{NS}_i(y|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \frac{t^i}{i!} \middle| x^n \right\rangle \\
 &= \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^y \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \middle| \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) (1+t)^y x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \middle| \sum_{l=0}^{\infty} \widehat{N}_l(y|a_1, \dots, a_r) \frac{t^l}{l!} x^n \right\rangle \\
 &= \sum_{l=0}^n \binom{n}{l} \widehat{N}_l(y|a_1, \dots, a_r) \left\langle \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \middle| x^{n-l} \right\rangle \\
 &= \sum_{l=0}^n \binom{n}{l} \widehat{N}_l(y|a_1, \dots, a_r) \left\langle \sum_{i=0}^{\infty} \widehat{s}_i(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
 &= \sum_{l=0}^n \binom{n}{l} \widehat{N}_l(y|a_1, \dots, a_r) \widehat{s}_{n-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s).
 \end{aligned}$$

Thus, we obtain (17).

Finally, we obtain

$$\begin{aligned}
 & \widehat{NS}_n(y|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= \left\langle \sum_{i=0}^{\infty} \widehat{NS}_i(y|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \frac{t^i}{i!} \middle| x^n \right\rangle \\
 &= \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^y \middle| x^n \right\rangle \\
 &= \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \middle| \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^y x^n \right\rangle \\
 &= \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \middle| \sum_{l=0}^{\infty} \widehat{s}_l(y|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \frac{t^l}{l!} x^n \right\rangle \\
 &= \sum_{l=0}^n \widehat{s}_l(y|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \binom{n}{l} \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \middle| x^{n-l} \right\rangle \\
 &= \sum_{l=0}^n \widehat{s}_l(y|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \binom{n}{l} \left\langle \sum_{i=0}^{\infty} \widehat{N}_i(a_1, \dots, a_r) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
 &= \sum_{l=0}^n \binom{n}{l} \widehat{s}_l(y|\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \widehat{N}_{n-l}(a_1, \dots, a_r).
 \end{aligned}$$

Thus, we get the identity (18). □

3.2 Sheffer identity

Theorem 2

$$\begin{aligned} & \widehat{NS}_n(x+y|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\ &= \sum_{l=0}^n \binom{n}{l} \widehat{NS}_l(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s)(y)_{n-l}. \end{aligned} \tag{21}$$

Proof By (12) with

$$\begin{aligned} p_n(x) &= \prod_{i=1}^r \left(\frac{te^{a_i t}}{e^{a_i t} - 1} \right) \prod_{j=1}^s \left(\frac{1 + e^{\lambda_j t}}{e^{\lambda_j t}} \right)^{\mu_j} \widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\ &= (x)_n \sim (1, e^t - 1), \end{aligned}$$

using (10), we have (21). □

3.3 Difference relations

Theorem 3

$$\begin{aligned} & \widehat{NS}_n(x+1|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) - \widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\ &= n \widehat{NS}_{n-1}(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s). \end{aligned} \tag{22}$$

Proof By (8) with (12), we get

$$\begin{aligned} & (e^t - 1) \widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\ &= n \widehat{NS}_{n-1}(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s). \end{aligned}$$

By (7), we have (22). □

3.4 Recurrence

Theorem 4

$$\begin{aligned} & \widehat{NS}_{n+1}(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\ &= \left(x + \sum_{j=1}^s \mu_j \lambda_j - \sum_{i=1}^r a_i \right) \widehat{NS}_n(x-1|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\ &+ 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m \frac{(-1)^{m+1} \binom{m}{l}}{l+1} S_1(n, m) E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\ &\times \left(\sum_{i=1}^r a_i F_{l+1}(1-x|a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r) - r F_{l+1}(1-x|a_1, \dots, a_r) \right) \\ &+ 2^{-1-\sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m (-1)^{m+1} \binom{m}{l} S_1(n, m) F_{m-l}(a_1, \dots, a_r) \\ &\times \sum_{i=1}^s \mu_i \lambda_i E_l(1-x|\lambda; \mu + e_i). \end{aligned} \tag{23}$$

Proof By applying

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right) \frac{1}{f'(t)} s_n(x) \tag{24}$$

[1, Corollary 3.7.2] with (12), we get

$$\widehat{NS}_{n+1}(x) = x \widehat{NS}_n(x-1) - e^{-t} \frac{g'(t)}{g(t)} \widehat{NS}_n(x).$$

Observe that

$$\begin{aligned} \frac{g'(t)}{g(t)} &= (\ln g(t))' \\ &= \left(r \ln t + \left(\sum_{i=1}^r a_i \right) t - \sum_{i=1}^r \ln(e^{a_i t} - 1) + \sum_{j=1}^s \mu_j \ln(1 + e^{\lambda_j t}) - \left(\sum_{j=1}^s \mu_j \lambda_j \right) t \right)' \\ &= \frac{r}{t} + \sum_{i=1}^r a_i - \sum_{i=1}^r \frac{a_i e^{a_i t}}{e^{a_i t} - 1} + \sum_{j=1}^s \frac{\mu_j \lambda_j e^{\lambda_j t}}{1 + e^{\lambda_j t}} - \sum_{j=1}^s \mu_j \lambda_j \\ &= \frac{r - \sum_{i=1}^r \frac{a_i t e^{a_i t}}{e^{a_i t} - 1}}{t} + \sum_{i=1}^r a_i + \sum_{j=1}^s \frac{\mu_j \lambda_j e^{\lambda_j t}}{1 + e^{\lambda_j t}} - \sum_{j=1}^s \mu_j \lambda_j, \end{aligned}$$

where

$$r - \sum_{i=1}^r \frac{a_i t e^{a_i t}}{e^{a_i t} - 1} = -\frac{1}{2} \left(\sum_{i=1}^r a_i \right) t + \dots$$

has the order at least one. Since from the proofs of (13) and (15)

$$\begin{aligned} \widehat{NS}_n(x) &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \\ &\quad \times \sum_{l=0}^m (-1)^l \binom{m}{l} E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \prod_{i=1}^r \left(\frac{e^{-a_i t} - 1}{-t} \right) x^l \\ &= 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \\ &\quad \times \sum_{l=0}^m \binom{m}{l} E_{m-l}(a_1, \dots, a_r) (-1)^l \prod_{j=1}^s \left(\frac{2}{1 + e^{-\lambda_j t}} \right)^{\mu_j} x^l, \end{aligned}$$

we have

$$\begin{aligned} \frac{g'(t)}{g(t)} \widehat{NS}_n(x) &= \left(\sum_{i=1}^r a_i - \sum_{j=1}^s \mu_j \lambda_j \right) \widehat{NS}_n(x) \end{aligned}$$

$$\begin{aligned}
 &+ 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \sum_{l=0}^m (-1)^l \binom{m}{l} E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &\times \prod_{i=1}^r \left(\frac{e^{-a_i t} - 1}{-t} \right) \frac{r - \sum_{i=1}^r \frac{a_i t}{1 - e^{-a_i t}}}{t} x^l \\
 &+ 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \sum_{l=0}^m \binom{m}{l} F_{m-l}(a_1, \dots, a_r) (-1)^l \\
 &\times \sum_{i=1}^s \frac{\mu_i \lambda_i}{1 + e^{-\lambda_i t}} \prod_{j=1}^s \left(\frac{2}{1 + e^{-\lambda_j t}} \right)^{\mu_j} x^l.
 \end{aligned}$$

The third term is equal to

$$\begin{aligned}
 &2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m (-1)^m \binom{m}{l} S_1(n, m) F_{m-l}(a_1, \dots, a_r) (-1)^l \\
 &\times \sum_{i=1}^s \frac{\mu_i \lambda_i}{2} (-1)^l E_l(-x | \lambda; \mu + e_i) \\
 &= 2^{-1 - \sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m (-1)^m \binom{m}{l} S_1(n, m) F_{m-l}(a_1, \dots, a_r) \\
 &\times \sum_{i=1}^s \mu_i \lambda_i E_l(-x | \lambda; \mu + e_i),
 \end{aligned}$$

where $\lambda = (\lambda_1, \dots, \lambda_s)$, $\mu = (\mu_1, \dots, \mu_s)$, and $e_i = (\underbrace{0, \dots, 0}_{i-1}, 1, \underbrace{0, \dots, 0}_{s-i})$. The second term is

$$\begin{aligned}
 &2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \sum_{l=0}^m \frac{(-1)^l}{l+1} \binom{m}{l} E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &\times \prod_{i=1}^r \left(\frac{e^{-a_i t} - 1}{-t} \right) \left(r - \sum_{i=1}^r \frac{a_i t}{1 - e^{-a_i t}} \right) x^{l+1} \\
 &= r 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \sum_{l=0}^m \frac{(-1)^l}{l+1} \binom{m}{l} E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &\times \prod_{i=1}^r \left(\frac{e^{-a_i t} - 1}{-t} \right) x^{l+1} \\
 &- 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \sum_{l=0}^m \frac{(-1)^l}{l+1} \binom{m}{l} E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &\times \sum_{i=1}^r a_i \frac{-t}{e^{-a_i t} - 1} \prod_{i=1}^r \left(\frac{e^{-a_i t} - 1}{-t} \right) x^{l+1} \\
 &= r 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \sum_{l=0}^m \frac{(-1)^l}{l+1} \binom{m}{l} E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &\times (-1)^{l+1} F_{l+1}(-x | a_1, \dots, a_r) \\
 &- 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n (-1)^m S_1(n, m) \sum_{l=0}^m \frac{(-1)^l}{l+1} \binom{m}{l} E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s)
 \end{aligned}$$

$$\begin{aligned}
 & \times \sum_{i=1}^r a_i (-1)^{l+1} F_{l+1}(-x|a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r) \\
 & = 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m \frac{(-1)^{m+1} \binom{m}{l}}{l+1} S_1(n, m) E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 & \quad \times \left(r F_{l+1}(-x|a_1, \dots, a_r) - \sum_{i=1}^r a_i F_{l+1}(-x|a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r) \right).
 \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
 & \widehat{NS}_{n+1}(x) \\
 & = \left(x + \sum_{j=1}^s \mu_j \lambda_j - \sum_{i=1}^r a_i \right) \widehat{NS}_n(x-1) \\
 & \quad + 2^{-\sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m \frac{(-1)^{m+1} \binom{m}{l}}{l+1} S_1(n, m) E_{m-l}(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 & \quad \times \left(\sum_{i=1}^r a_i F_{l+1}(1-x|a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r) - r F_{l+1}(1-x|a_1, \dots, a_r) \right) \\
 & \quad + 2^{-1-\sum_{j=1}^s \mu_j} \sum_{m=0}^n \sum_{l=0}^m (-1)^{m+1} \binom{m}{l} S_1(n, m) F_{m-l}(a_1, \dots, a_r) \\
 & \quad \times \sum_{i=1}^s \mu_i \lambda_i E_l(1-x|\lambda; \mu + e_i),
 \end{aligned}$$

which is the identity (23). □

3.5 Differentiation

Theorem 5

$$\begin{aligned}
 & \frac{d}{dx} \widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 & = n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} \widehat{NS}_l(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s).
 \end{aligned} \tag{25}$$

Proof We shall use

$$\frac{d}{dx} s_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} (\bar{f}(t)|x^{n-l}) s_l(x)$$

(cf. [1, Theorem 2.3.12]). Since

$$\begin{aligned}
 (\bar{f}(t)|x^{n-l}) & = (\ln(1+t)|x^{n-l}) \\
 & = \left\langle \sum_{m=1}^{\infty} \frac{(-1)^{m-1} t^m}{m} \middle| x^{n-l} \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{m=1}^{n-l} \frac{(-1)^{m-1}}{m} \langle t^m | x^{n-l} \rangle \\
 &= \sum_{m=1}^{n-l} \frac{(-1)^{m-1}}{m} (n-l)! \delta_{m,n-l} \\
 &= (-1)^{n-l-1} (n-l-1)!
 \end{aligned}$$

with (12), we have

$$\begin{aligned}
 &\frac{d}{dx} \widehat{\text{NS}}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= \sum_{l=0}^{n-1} \binom{n}{l} (-1)^{n-l-1} (n-l-1)! \widehat{\text{NS}}_l(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} \widehat{\text{NS}}_l(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s),
 \end{aligned}$$

which is the identity (25). □

3.6 One more relation

The classical Cauchy numbers of the first kind c_n are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$$

(see e.g. [3, 4]).

Theorem 6

$$\begin{aligned}
 &\widehat{\text{NS}}_n(x|a; \lambda; \mu) \\
 &= \left(x - \sum_{i=1}^r a_i \right) \widehat{\text{NS}}_{n-1}(x-1|a; \lambda; \mu) + \sum_{j=1}^s \lambda_j \mu_j \widehat{\text{NS}}_{n-1}(x-\lambda_j-1|a; \lambda; \mu + e_j) \\
 &\quad + \frac{1}{n} \sum_{l=0}^n \binom{n}{l} c_l \left(\sum_{i=1}^r a_i \widehat{\text{NS}}_{n-l}(x-1|a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r; \lambda; \mu) \right. \\
 &\quad \left. - r \widehat{\text{NS}}_{n-l}(x-1|a; \lambda; \mu) \right). \tag{26}
 \end{aligned}$$

Proof For $n \geq 1$, we have

$$\begin{aligned}
 &\widehat{\text{NS}}_n(y|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= \left\langle \sum_{l=0}^{\infty} \widehat{\text{NS}}_l(y|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \frac{t^l}{l!} \middle| x^n \right\rangle \\
 &= \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^y \middle| x^n \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 &= \left\langle \partial_t \left(\prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^y \right) \middle| x^{n-1} \right\rangle \\
 &= \left\langle \left(\partial_t \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^y \middle| x^{n-1} \right\rangle \\
 &\quad + \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \left(\partial_t \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \right) (1+t)^y \middle| x^{n-1} \right\rangle \\
 &\quad + \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (\partial_t (1+t)^y) \middle| x^{n-1} \right\rangle.
 \end{aligned}$$

The third term is

$$\begin{aligned}
 &y \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{y-1} \middle| x^{n-1} \right\rangle \\
 &= y \widehat{\text{NS}}_{n-1}(y-1 | a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s).
 \end{aligned}$$

Since

$$\begin{aligned}
 &\partial_t \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \\
 &= \frac{1}{1+t} \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \frac{\sum_{v=1}^r \left(\frac{a_v t(1+t)^{a_v}}{(1+t)^{a_v-1}} - \frac{t}{\ln(1+t)} \right)}{t} \\
 &\quad - \frac{1}{1+t} \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \left(\sum_{l=1}^r a_l \right),
 \end{aligned}$$

with

$$\sum_{v=1}^r \left(\frac{a_v t(1+t)^{a_v}}{(1+t)^{a_v-1}} - \frac{t}{\ln(1+t)} \right)$$

having order ≥ 1 , the first term is

$$\begin{aligned}
 &\left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{y-1} \middle| \frac{\sum_{v=1}^r \left(\frac{a_v t(1+t)^{a_v}}{(1+t)^{a_v-1}} - \frac{t}{\ln(1+t)} \right)}{t} x^{n-1} \right\rangle \\
 &\quad - \left(\sum_{l=1}^r a_l \right) \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{y-1} \middle| x^{n-1} \right\rangle \\
 &= - \left(\sum_{l=1}^r a_l \right) \widehat{\text{NS}}_{n-1}(y-1) + \frac{1}{n} \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \right. \\
 &\quad \left. \times (1+t)^{y-1} \middle| \sum_{v=1}^r \left(\frac{a_v t(1+t)^{a_v}}{(1+t)^{a_v-1}} - \frac{t}{\ln(1+t)} \right) x^n \right\rangle \\
 &= - \left(\sum_{l=1}^r a_l \right) \widehat{\text{NS}}_{n-1}(y-1)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{n} \left(\sum_{v=1}^r a_v \left\langle \frac{(1+t)^{a_v} \ln(1+t)}{(1+t)^{a_v} - 1} \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \right. \right. \\
 & \times (1+t)^{y-1} \left. \left| \frac{t}{\ln(1+t)} x^n \right\rangle \right. \\
 & \left. - r \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{y-1} \left| \frac{t}{\ln(1+t)} x^n \right\rangle \right) \\
 & = - \left(\sum_{l=1}^r a_l \right) \widehat{NS}_{n-1}(y-1) \\
 & + \frac{1}{n} \left(\sum_{v=1}^r a_v \left\langle \frac{(1+t)^{a_v} \ln(1+t)}{(1+t)^{a_v} - 1} \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \right. \right. \\
 & \times (1+t)^{y-1} \left. \left| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^n \right\rangle \right. \\
 & \left. - r \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{y-1} \left| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^n \right\rangle \right) \\
 & = - \left(\sum_{l=1}^r a_l \right) \widehat{NS}_{n-1}(y-1) \\
 & + \frac{1}{n} \left(\sum_{v=1}^r \sum_{l=0}^n \binom{n}{l} a_v c_l \widehat{NS}_{n-l}(y-1 | a_1, \dots, a_{v-1}, a_{v+1}, \dots, a_r; \lambda; \mu) \right. \\
 & \left. - r \sum_{l=0}^n \binom{n}{l} c_l \widehat{NS}_{n-l}(y-1 | a; \lambda; \mu) \right) \\
 & = - \left(\sum_{v=1}^r a_v \right) \widehat{NS}_{n-1}(y-1) \\
 & + \frac{1}{n} \sum_{l=0}^n \binom{n}{l} c_l \left(\sum_{v=1}^r a_v \widehat{NS}_{n-l}(y-1 | a_1, \dots, a_{v-1}, a_{v+1}, \dots, a_r; \lambda; \mu) \right. \\
 & \left. - r \widehat{NS}_{n-l}(y-1 | a; \lambda; \mu) \right).
 \end{aligned}$$

Since

$$\partial_t \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} = \sum_{l=1}^s \lambda_l \mu_l (1+t)^{-\lambda_l-1} \left(\frac{(1+t)^{\lambda_l}}{1 + (1+t)^{\lambda_l}} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j},$$

the second term is

$$\begin{aligned}
 & \sum_{l=1}^s \lambda_l \mu_l \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \left(\frac{(1+t)^{\lambda_l}}{1 + (1+t)^{\lambda_l}} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{y-\lambda_l-1} \left| x^{n-1} \right\rangle \right. \\
 & \left. = \sum_{l=1}^s \lambda_l \mu_l \widehat{NS}_{n-1}(y - \lambda_l - 1 | a; \lambda; \mu + e_l). \right.
 \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & \widehat{NS}_n(x|a; \lambda; \mu) \\ &= \left(x - \sum_{i=1}^r a_i \right) \widehat{NS}_{n-1}(x-1|a; \lambda; \mu) + \sum_{j=1}^s \lambda_j \mu_j \widehat{NS}_{n-1}(x - \lambda_j - 1|a; \lambda; \mu + e_j) \\ & \quad + \frac{1}{n} \sum_{l=0}^n \binom{n}{l} c_l \left(\sum_{i=1}^r a_i \widehat{NS}_{n-l}(x-1|a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r; \lambda; \mu) \right. \\ & \quad \left. - r \widehat{NS}_{n-l}(x-1|a; \lambda; \mu) \right), \end{aligned}$$

which is the identity (26). □

3.7 A relation involving the Stirling numbers of the first kind

Theorem 7 For $n-1 \geq m \geq 1$, we have

$$\begin{aligned} & \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) \widehat{NS}_l(a; \lambda; \mu) \\ &= - \left(\sum_{i=1}^r a_i \right) \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \widehat{NS}_l(-1|a; \lambda; \mu) \\ & \quad + \frac{1}{n} \sum_{k=0}^{n-m} \sum_{l=k}^{n-m} \binom{n}{l} \binom{l}{k} S_1(n-l, m) c_{l-k} \\ & \quad \times \left(\sum_{i=1}^r a_i \widehat{NS}_k(-1|a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r; \lambda; \mu) - r \widehat{NS}_k(-1|a; \lambda; \mu) \right) \\ & \quad + \sum_{j=1}^s \lambda_j \mu_j \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \widehat{NS}_l(-\lambda_j - 1|a; \lambda; \mu + e_j) \\ & \quad + \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) \widehat{NS}_l(-1|a; \lambda; \mu). \end{aligned} \tag{27}$$

Proof We shall compute

$$\left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (\ln(1+t))^m \middle| x^n \right\rangle$$

in two different ways. On the one hand, it is equal to

$$\begin{aligned} & \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \middle| (\ln(1+t))^m x^n \right\rangle \\ &= \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \middle| m! \sum_{l=m}^{\infty} S_1(l, m) \frac{t^l}{l!} x^n \right\rangle \\ &= m! \sum_{l=m}^n \binom{n}{l} S_1(l, m) \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \middle| x^{n-l} \right\rangle \end{aligned}$$

$$\begin{aligned}
 &= m! \sum_{l=m}^n \binom{n}{l} S_1(l, m) \left\langle \sum_{i=0}^{\infty} \widehat{NS}_i(a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \frac{t^i}{i!} \Big| x^{n-l} \right\rangle \\
 &= m! \sum_{l=m}^n \binom{n}{l} S_1(l, m) \widehat{NS}_{n-l}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= m! \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) \widehat{NS}_l(a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s).
 \end{aligned}$$

On the other hand, it is equal to

$$\begin{aligned}
 &\left\langle \partial_t \left(\prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (\ln(1+t))^m \right) \Big| x^{n-1} \right\rangle \\
 &= \left\langle \left(\partial_t \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (\ln(1+t))^m \Big| x^{n-1} \right\rangle \\
 &\quad + \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \left(\partial_t \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \right) (\ln(1+t))^m \Big| x^{n-1} \right\rangle \\
 &\quad + \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (\partial_t (\ln(1+t))^m) \Big| x^{n-1} \right\rangle. \tag{28}
 \end{aligned}$$

The third term of (28) is equal to

$$\begin{aligned}
 &m \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{-1} \Big| (\ln(1+t))^{m-1} x^{n-1} \right\rangle \\
 &= m \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \right. \\
 &\quad \left. \times (1+t)^{-1} \Big| (m-1)! \sum_{l=m-1}^{\infty} S_1(l, m-1) \frac{t^l}{l!} x^{n-1} \right\rangle \\
 &= m! \sum_{l=m-1}^{n-1} \binom{n-1}{l} S_1(l, m-1) \\
 &\quad \times \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{-1} \Big| x^{n-1-l} \right\rangle \\
 &= m! \sum_{l=m-1}^{n-1} \binom{n-1}{l} S_1(l, m-1) \\
 &\quad \times \widehat{NS}_{n-1-l}(-1|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) \\
 &\quad \times \widehat{NS}_l(-1|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s).
 \end{aligned}$$

Since

$$\begin{aligned} & \partial_t \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \\ &= \frac{1}{1+t} \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \frac{\sum_{v=1}^r \left(\frac{a_v t (1+t)^{a_v}}{(1+t)^{a_v} - 1} - \frac{t}{\ln(1+t)} \right)}{t} \\ & \quad - \frac{1}{1+t} \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \left(\sum_{v=1}^r a_v \right), \end{aligned}$$

the first term of (28) is equal to

$$\begin{aligned} & \frac{1}{n} \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \right. \\ & \quad \times (1+t)^{-1} \sum_{v=1}^r \left(\frac{a_v t (1+t)^{a_v}}{(1+t)^{a_v} - 1} - \frac{t}{\ln(1+t)} \right) \left| (\ln(1+t))^m x^n \right\rangle \\ & \quad - \left(\sum_{v=1}^r a_v \right) \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \right. \\ & \quad \times (1+t)^{-1} \left| (\ln(1+t))^m x^{n-1} \right\rangle \\ &= \frac{1}{n} \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{-1} \right. \\ & \quad \times \sum_{v=1}^r \left(\frac{a_v t (1+t)^{a_v}}{(1+t)^{a_v} - 1} - \frac{t}{\ln(1+t)} \right) \left| m! \sum_{l=m}^{\infty} S_1(l, m) \frac{t^l}{l!} x^n \right\rangle \\ & \quad - \left(\sum_{v=1}^r a_v \right) \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \right. \\ & \quad \times (1+t)^{-1} \left| m! \sum_{l=m}^{\infty} S_1(l, m) \frac{t^l}{l!} x^{n-1} \right\rangle \\ &= \frac{m!}{n} \sum_{l=m}^n \binom{n}{l} S_1(l, m) \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \right. \\ & \quad \times (1+t)^{-1} \sum_{v=1}^r \left(\frac{a_v t (1+t)^{a_v}}{(1+t)^{a_v} - 1} - \frac{t}{\ln(1+t)} \right) \left| x^{n-l} \right\rangle \\ & \quad - m! \left(\sum_{v=1}^r a_v \right) \sum_{l=m}^{n-1} \binom{n-1}{l} S_1(l, m) \\ & \quad \times \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{-1} \left| x^{n-l-1} \right\rangle \right. \\ &= -m! \left(\sum_{v=1}^r a_v \right) \sum_{l=m}^{n-1} \binom{n-1}{l} S_1(l, m) \widehat{NS}_{n-l-1}(-1) \end{aligned}$$

$$\begin{aligned}
 & + \frac{m!}{n} \sum_{l=m}^n \binom{n}{l} S_1(l, m) \left(\left(\sum_{v=1}^r a_v \right) \left\langle \frac{(1+t)^{a_v} \ln(1+t)}{(1+t)^{a_v} - 1} \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \right. \right. \\
 & \times \left. \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{-1} \left| \frac{t}{\ln(1+t)} x^{n-l} \right. \right) \\
 & - r \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{-1} \left| \frac{t}{\ln(1+t)} x^{n-l} \right. \right) \\
 & = -m! \left(\sum_{v=1}^r a_v \right) \sum_{l=m}^{n-1} \binom{n-1}{l} S_1(l, m) \widehat{NS}_{n-l-1}(-1) \\
 & + \frac{m!}{n} \sum_{l=m}^n \binom{n}{l} S_1(l, m) \sum_{k=0}^{n-l} \binom{n-l}{k} c_k \\
 & \times \left(\left(\sum_{v=1}^r a_v \right) \widehat{NS}_{n-l-k}(-1|a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r; \lambda; \mu) - r \widehat{NS}_{n-l-k}(-1|a; \lambda; \mu) \right) \\
 & = -m! \left(\sum_{v=1}^r a_v \right) \sum_{l=0}^{n-1-m} \binom{n-1}{l} S_1(n-l-1, m) \widehat{NS}_l(-1) \\
 & + \frac{m!}{n} \sum_{k=0}^{n-m} \sum_{l=k}^{n-m} \binom{n}{l} \binom{l}{k} S_1(n-l, m) c_{l-k} \\
 & \times \left(\left(\sum_{v=1}^r a_v \right) \widehat{NS}_k(-1|a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r; \lambda; \mu) - r \widehat{NS}_k(-1|a; \lambda; \mu) \right).
 \end{aligned}$$

The second term of (28) is equal to

$$\begin{aligned}
 & \sum_{v=1}^s \lambda_v \mu_v \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \left(\frac{(1+t)^{\lambda_v}}{1 + (1+t)^{\lambda_v}} \right) \right. \\
 & \times \left. \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{-\lambda_v-1} \left| (\ln(1+t))^m x^{n-1} \right. \right) \\
 & = \sum_{v=1}^s \lambda_v \mu_v \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \left(\frac{(1+t)^{\lambda_v}}{1 + (1+t)^{\lambda_v}} \right) \right. \\
 & \times \left. \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{-\lambda_v-1} \left| m! \sum_{l=m}^{\infty} S_1(l, m) \frac{t^l}{l!} x^{n-1} \right. \right) \\
 & = m! \sum_{v=1}^s \lambda_v \mu_v \sum_{l=m}^{n-1} \binom{n-1}{l} S_1(l, m) \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \left(\frac{(1+t)^{\lambda_v}}{1 + (1+t)^{\lambda_v}} \right) \right. \\
 & \times \left. \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{-\lambda_v-1} \left| x^{n-l-1} \right. \right) \\
 & = m! \sum_{v=1}^s \lambda_v \mu_v \sum_{l=m}^{n-1} \binom{n-1}{l} S_1(l, m) \widehat{NS}_{n-l-1}(-\lambda_v - 1|a; \lambda; \mu + e_v) \\
 & = m! \sum_{v=1}^s \lambda_v \mu_v \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \widehat{NS}_l(-\lambda_v - 1|a; \lambda; \mu + e_v).
 \end{aligned}$$

Therefore, we get, for $n - 1 \geq m \geq 1$,

$$\begin{aligned}
 & m! \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) \widehat{NS}_l(a; \lambda; \mu) \\
 &= -m! \left(\sum_{i=1}^r a_i \right) \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \widehat{NS}_l(-1|a; \lambda; \mu) \\
 &\quad + \frac{m!}{n} \sum_{k=0}^{n-m} \sum_{l=k}^{n-m} \binom{n}{l} \binom{l}{k} S_1(n-l, m) c_{l-k} \\
 &\quad \times \left(\sum_{i=1}^r a_i \widehat{NS}_k(-1|a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r; \lambda; \mu) - r \widehat{NS}_k(-1|a; \lambda; \mu) \right) \\
 &\quad + m! \sum_{j=1}^s \lambda_j \mu_j \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \widehat{NS}_l(-\lambda_j - 1|a; \lambda; \mu + e_j) \\
 &\quad + m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) \widehat{NS}_l(-1|a; \lambda; \mu).
 \end{aligned}$$

Dividing both sides by $m!$, we obtain, for $n - 1 \geq m \geq 1$,

$$\begin{aligned}
 & \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) \widehat{NS}_l(a; \lambda; \mu) \\
 &= - \left(\sum_{i=1}^r a_i \right) \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \widehat{NS}_l(-1|a; \lambda; \mu) \\
 &\quad + \frac{1}{n} \sum_{k=0}^{n-m} \sum_{l=k}^{n-m} \binom{n}{l} \binom{l}{k} S_1(n-l, m) c_{l-k} \\
 &\quad \times \left(\sum_{i=1}^r a_i \widehat{NS}_k(-1|a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r; \lambda; \mu) - r \widehat{NS}_k(-1|a; \lambda; \mu) \right) \\
 &\quad + \sum_{j=1}^s \lambda_j \mu_j \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \widehat{NS}_l(-\lambda_j - 1|a; \lambda; \mu + e_j) \\
 &\quad + \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) \widehat{NS}_l(-1|a; \lambda; \mu).
 \end{aligned}$$

Thus, we get (27). □

3.8 A relation with the falling factorials

Theorem 8

$$\begin{aligned}
 & \widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= \sum_{m=0}^n \binom{n}{m} \widehat{NS}_{n-m}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s)(x)_m.
 \end{aligned} \tag{29}$$

Proof For (12) and (20), assume that

$$\widehat{\text{NS}}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) = \sum_{m=0}^n C_{n,m}(x)_m.$$

By (11), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{1}{\prod_{i=1}^r \left(\frac{\ln(1+t)e^{a_i \ln(1+t)}}{e^{a_i \ln(1+t)} - 1} \right) \prod_{j=1}^s \left(\frac{1+e^{\lambda_j \ln(1+t)}}{e^{\lambda_j \ln(1+t)}} \right)^{\mu_j}} t^m \middle| x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \middle| t^m x^n \right\rangle \\ &= \binom{n}{m} \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \middle| x^{n-m} \right\rangle \\ &= \binom{n}{m} \widehat{\text{NS}}_{n-m}. \end{aligned}$$

Thus, we get the identity (29). □

3.9 A relation with higher-order Frobenius-Euler polynomials

For $\alpha \in \mathbb{C}$ with $\alpha \neq 1$, the Frobenius-Euler polynomials of order r , $H_n^{(r)}(x|\alpha)$ are defined by the generating function

$$\left(\frac{1-\alpha}{e^t-\alpha} \right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\alpha) \frac{t^n}{n!}$$

(see e.g. [5]).

Theorem 9

$$\begin{aligned} &\widehat{\text{NS}}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\ &= \sum_{m=0}^n \left(\sum_{j=0}^{n-m} \sum_{l=0}^{n-m-j} \binom{\sigma}{j} \binom{n-j}{l} (n)_j (1-\alpha)^{-j} S_1(n-j-l, m) \widehat{\text{NS}}_l \right) H_m^{(\sigma)}(x|\alpha). \end{aligned} \tag{30}$$

Proof For (12) and

$$H_n^{(\sigma)}(x|\alpha) \sim \left(\left(\frac{e^t - \alpha}{1 - \alpha} \right)^\sigma, t \right), \tag{31}$$

assume that $\widehat{\text{NS}}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) = \sum_{m=0}^n C_{n,m} H_m^{(\sigma)}(x|\alpha)$. By (11), similarly to the proof of (27), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{\left(\frac{e^{\ln(1+t)} - \alpha}{1 - \alpha} \right)^\sigma}{\prod_{i=1}^r \left(\frac{\ln(1+t)e^{a_i \ln(1+t)}}{e^{a_i \ln(1+t)} - 1} \right) \prod_{j=1}^s \left(\frac{1+e^{\lambda_j \ln(1+t)}}{e^{\lambda_j \ln(1+t)}} \right)^{\mu_j}} (\ln(1+t))^m \middle| x^n \right\rangle \\ &= \frac{1}{m!(1-\alpha)^\sigma} \end{aligned}$$

$$\begin{aligned}
 & \times \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (\ln(1+t))^m (1-\alpha+t)^\sigma \Big| x^n \right\rangle \\
 &= \frac{1}{m!(1-\alpha)^\sigma} \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} \right. \\
 & \quad \times (\ln(1+t))^m \left. \sum_{v=0}^{\min\{\sigma, n\}} \binom{\sigma}{v} (1-\alpha)^{\sigma-v} t^v x^n \right\rangle \\
 &= \frac{1}{m!(1-\alpha)^\sigma} \sum_{v=0}^{n-m} \binom{\sigma}{v} (1-\alpha)^{\sigma-v} (n)_v \\
 & \quad \times \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i} - 1}{(1+t)^{a_i} \ln(1+t)} \right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1 + (1+t)^{\lambda_j}} \right)^{\mu_j} (\ln(1+t))^m \Big| x^{n-v} \right\rangle \\
 &= \frac{1}{m!(1-\alpha)^\sigma} \sum_{v=0}^{n-m} \binom{\sigma}{v} (1-\alpha)^{\sigma-v} (n)_v \sum_{l=0}^{n-m-v} m! \binom{n-v}{l} S_1(n-v-l, m) \widehat{NS}_l \\
 &= \sum_{v=0}^{n-m} \sum_{l=0}^{n-m-v} \binom{\sigma}{v} \binom{n-v}{l} (n)_v (1-\alpha)^{-v} S_1(n-v-l, m) \widehat{NS}_l.
 \end{aligned}$$

Thus, we get the identity (30). □

3.10 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} \frac{\mathfrak{B}_n^{(r)}(x)}{n!} t^n$$

(see e.g. [1, Section 2.2]). In addition, the Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)} \right)^r = \sum_{n=0}^{\infty} \frac{\mathfrak{C}_n^{(r)}}{n!} t^n$$

(see e.g. [6, equation (2.1)], [7, equation (6)]).

Theorem 10

$$\begin{aligned}
 & \widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) \\
 &= \sum_{m=0}^n \left(\sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i} \binom{n-i}{l} \mathfrak{C}_i^{(\sigma)} S_1(n-i-l, m) \widehat{NS}_l \right) \mathfrak{B}_m^{(\sigma)}(x). \tag{32}
 \end{aligned}$$

Proof For (12) and

$$\mathfrak{B}_n^{(\sigma)}(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^\sigma, t \right), \tag{33}$$

assume that $\widehat{NS}_n(x|a_1, \dots, a_r; \lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_s) = \sum_{m=0}^n C_{n,m} \mathfrak{B}_m^{(s)}(x)$. By (11), similarly to the proof of (27), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t)}\right)^\sigma}{\prod_{i=1}^r \left(\frac{\ln(1+t)e^{a_i \ln(1+t)}}{e^{a_i \ln(1+t)}-1}\right) \prod_{j=1}^s \left(\frac{1+e^{\lambda_j \ln(1+t)}}{e^{\lambda_j \ln(1+t)}}\right)^{\mu_j}} (\ln(1+t))^m \middle| x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i} \ln(1+t)}\right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} (\ln(1+t))^m \middle| \left(\frac{t}{\ln(1+t)}\right)^\sigma x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i} \ln(1+t)}\right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} (\ln(1+t))^m \middle| \sum_{i=0}^{\infty} \mathfrak{C}_i^{(\sigma)} \frac{t^i}{i!} x^n \right\rangle \\ &= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(\sigma)} \binom{n}{i} \left\langle \prod_{i=1}^r \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i} \ln(1+t)}\right) \prod_{j=1}^s \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} (\ln(1+t))^m \middle| x^{n-i} \right\rangle \\ &= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(\sigma)} \binom{n}{i} \sum_{l=0}^{n-m-i} m! \binom{n-i}{l} S_1(n-i-l, m) \widehat{NS}_l \\ &= \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i} \binom{n-i}{l} \mathfrak{C}_i^{(\sigma)} S_1(n-i-l, m) \widehat{NS}_l. \end{aligned}$$

Thus, we get the identity (32). □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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