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Fixed-point theorems for nonlinear operators with singular perturbations and applications

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Abstract

In this paper, using fixed-point index theory and approximation techniques, we consider the existence and multiplicity of fixed points of some nonlinear operators with singular perturbation. As an application we consider the existence and multiplicity of positive solutions of singular systems of multi-point boundary value problems, which improve the results in the literature.

Keywords: boundary value problems; singularity; fixed-point index

1 Introduction

In this paper we consider the problem

$$x = Ax + \lambda Bx,$$

where A is continuous and compact and B is a singular continuous and compact operator (defined in Section 2).

In the study of nonlinear phenomena many models give rise to singular boundary value problems (singular in the dependent variable) (see [1–3]). In [4], Taliaferro showed that the singular boundary value problem

$$\begin{cases} y'' + q(t)y^{-\alpha} = 0, & 0 < t < 1, \\ y(0) = 0 = y(1), \end{cases}$$

has a $C[0,1] \cap C^1(0,1)$ solution; here $\alpha > 0$, $q \in C(0,1)$ with $q > 0$ on $(0,1)$ and $\int_0^1 t(1-t)q(t)\,dt < \infty$. For more recent work we refer the reader to [5–14] and the references therein.

In this paper we consider abstract singular operators (defined in Section 2) and we consider the existence and multiplicity of fixed points of some nonlinear operators with singular perturbations. As an application we discuss the existence and multiplicity of positive solutions of singular systems of multi-point boundary value problems.

2 Fixed-point theorems

Let E be a Banach space, P a cone of E, $\Omega \subseteq E$ bounded and open. The following theorems are needed in our paper.

Theorem 2.1 ([8]) *Suppose $\theta \in \Omega$, $A : P \cap \overline{\Omega} \to P$ is continuous and compact and*

$$Ax \neq \mu x, \quad \forall x \in P \cap \partial\Omega, \mu \geq 1.$$

Then

$$i(A, P \cap \Omega, P) = 1.$$

Theorem 2.2 ([8]) *Assume that $A : P \cap \overline{\Omega} \to P$ is continuous and compact. If there exists a compact and continuous operator $K : P \cap \partial\Omega \to P$ such that*

(1) $\inf_{x \in P \cap \partial\Omega} \|Kx\| > 0$;
(2) $x - Ax \neq \lambda Kx$, $\forall x \in P \cap \partial\Omega$, $\lambda \geq 0$,

then

$$i(A, P \cap \Omega, P) = 0.$$

Now we give a new definition.

Definition 2.1 If $B : P - \{\theta\} \to P$ is continuous with

$$\lim_{x \to \theta, x \in (P - \{\theta\})} \|Bx\| = +\infty$$

and $B(\{x \in P | r \leq \|x\| \leq R\})$ is relatively compact, for any $0 < r < R < +\infty$, then $B : P - \{\theta\} \to P$ is called a singular continuous and compact operator.

Remark Consider

$$x''(t) + a(t)x^{-\gamma}(t) = 0, \quad t \in (0, 1),$$
$$x(0) = 0, \qquad x(1) = 0,$$

where $1 > \gamma > 0$ and $a(t) \in C((0, 1), (0, +\infty)) \cap L^1(0, 1)$ or equivalently

$$x(t) = \int_0^1 G(t, s)a(s)x^{-\gamma}(s)\, ds, \quad t \in [0, 1],$$

where

$$G(t, s) = \begin{cases} s(1 - t), & 0 \leq s \leq t \leq 1, \\ t(1 - s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Set

$$P := \{x \in C[0, 1] : x(t) \geq t(1 - t)\|x\|\},$$

where $\|x\| = \max_{t \in [0,1]} |x(t)|$. For $x \in P - \{\theta\}$, let

$$(Bx)(t) := \int_0^1 G(t, s)a(s)x^{-\gamma}(s)\, ds, \quad t \in [0, 1].$$

It is easy to see that $B : P - \{\theta\} \to P$ is a singular continuous and compact operator (see [7, 14]).

Theorem 2.3 *Suppose that $\theta \in \Omega$, $A : P \cap \overline{\Omega} \to P$ is continuous and compact and $B : P - \{\theta\} \to P$ is singular continuous and compact. Assume that*

$$Ax \neq \mu x, \quad \forall x \in P \cap \partial\Omega, \mu \geq 1. \tag{2.1}$$

Then there exists a $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, there exist $x_\lambda \in P \cap \Omega - \{\theta\}$ with

$$x_\lambda = Ax_\lambda + \lambda Bx_\lambda.$$

Proof Choose $x_0 \in P - \{\theta\}$, and define

$$B_n x = B\left(x + \frac{1}{n}x_0\right), \quad \forall x \in P, n \in \mathbb{N}.$$

Set

$$\gamma := \inf_{(\mu,x)\in[1,+\infty)\times P\cap\partial\Omega} \|\mu x - Ax\|.$$

Now we claim that

$$\gamma > 0. \tag{2.2}$$

If $\gamma = 0$, there exists $\{(\mu_n, x_n)\} \subseteq [1, +\infty) \times P \cap \partial\Omega$ such that

$$\lim_{n\to+\infty} \|\mu_n x_n - Ax_n\| = 0. \tag{2.3}$$

First, we show $\{\mu_n\}$ is bounded.

To see this suppose $\{\mu_n\}$ is unbounded. Without loss of generality, we assume that $\lim_{n\to+\infty} \mu_n = +\infty$. Then

$$0 \leftarrow \|\mu_n x_n - Ax_n\| \geq |\mu_n|\|x_n\| - \|Ax_n\| \geq \mu_n \inf_{x\in P\cap\partial\Omega} \|x\| - \|Ax_n\| \to +\infty,$$

and this is a contradiction.

Next, we show that there exists a $(\mu_0, x_0) \in [1, +\infty) \times P \cap \partial\Omega$ such that

$$\|\mu_0 x_0 - Ax_0\| = 0.$$

The boundedness of $\{\mu_n\}$ means that $\{\mu_n\}$ has a convergent subsequence. Without loss of generality, we assume that $\mu_n \to \mu_0 \geq 1$. Since $\{x_n\}$ is bounded and A is continuous and compact, $\{Ax_n\}$ has a convergent subsequence $\{Ax_{n_i}\}$ with $\lim_{n_i\to+\infty} Ax_{n_i} \to y_0$. From (2.3), we have

$$\lim_{n_i\to+\infty} \|\mu_{n_i} x_{n_i} - Ax_{n_i}\| = 0,$$

which implies that

$$\mu_{n_i} x_{n_i} \to y_0, \quad n_i \to +\infty.$$

Then

$$x_{n_i} \to \frac{1}{\mu_0} y_0, \quad \text{as } n_i \to +\infty.$$

Let $x_0 = \frac{1}{\mu_0} y_0$. Clearly, $x_0 \in P \cap \partial\Omega$ and

$$\|\mu_0 x_0 - Ax_0\| = \lim_{n_i \to +\infty} \|\mu_{n_i} x_{n_i} - Ax_{n_i}\| = 0,$$

which contradicts (2.1).

Let

$$\beta_n = \sup_{x \in P \cap \partial\Omega} \|B_n x\|.$$

Now we claim that

$$\sup_{n \in \mathbb{N}} \beta_n < +\infty. \tag{2.4}$$

To see this suppose that

$$\sup_{n \in \mathbb{N}} \beta_n = +\infty.$$

Without loss of generality, assume that

$$\lim_{n \to +\infty} \beta_n = +\infty,$$

which implies that there exists a sequence $\{x_n\} \in P \cap \partial\Omega$ such that

$$\lim_{n \to +\infty} \|B_n x_n\| = \lim_{n \to +\infty} \left\| B\left(x_n + \frac{1}{n} x_0\right) \right\| = +\infty. \tag{2.5}$$

For all $x \in P \cap \partial\Omega$, we have

$$\left\| x + \frac{1}{n} x_0 \right\| \leq \|x\| + \|x_0\| \leq \sup_{x \in P \cap \Omega} \|x\| + \|x_0\| := R < +\infty.$$

Since, for any $r > 0$, $B : P \cap \{x : r \leq \|x\| \leq R\}$ is relatively compact, (2.5) guarantees that there exists a subsequence $\{x_{n_i}\} \subseteq \{x_n\}$ such that

$$\lim_{n_i \to +\infty} \left\| x_{n_i} + \frac{1}{n_i} x_0 \right\| = 0.$$

Thus

$$\lim_{n_i \to +\infty} \|x_{n_i}\| = 0,$$

which implies that $\theta \in P \cap \partial\Omega$. This contradicts $\theta \in \Omega$. Hence, (2.4) holds.

Set

$$\beta := \sup_{n\in\mathbb{N}} \beta_n < +\infty$$

and

$$\lambda_* := \frac{\gamma}{\beta} > 0.$$

For $0 < \lambda < \lambda_*$, $x \in P \cap \partial\Omega$, $\mu \geq 1$, we have

$$\|Ax + \lambda B_n x - \mu x\| \geq \|Ax - \mu x\| - \lambda\|B_n x\| \geq \gamma - \beta\lambda > 0.$$

Theorem 2.1 guarantees that

$$i(A + \lambda B_n, P \cap \Omega, P) = 1. \tag{2.6}$$

Note (2.6) guarantees that, for any $\lambda \in (0, \lambda_*)$, there exists a $\{x_n\} \subseteq P \cap \Omega$ such that

$$x_n = Ax_n + \lambda B_n x_n, \quad n \in \mathbb{N}. \tag{2.7}$$

Now we show that

$$\inf_{n\in\mathbb{N}} \|x_n\| > 0, \tag{2.8}$$

which implies that

$$\inf_{n\in\mathbb{N}} \left\| x_n + \frac{1}{n} x_0 \right\| > 0.$$

To see this suppose that

$$\inf_{n\in\mathbb{N}} \|x_n\| = 0.$$

Then there exists a $\{x_{n_i}\}$ such that

$$\lim_{n_i\to+\infty} \|x_{n_i}\| = 0, \tag{2.9}$$

and so

$$\lim_{n_i\to+\infty} \left\| x_{n_i} + \frac{1}{n_i} x_0 \right\| = 0.$$

Thus

$$\lim_{n_i\to+\infty} \|B_{n_i} x_{n_i}\| = \lim_{n_i\to+\infty} \left\| B\left(x_{n_i} + \frac{1}{n_i} x_0 \right) \right\| = +\infty. \tag{2.10}$$

The compactness of A guarantees that $\{Ax_{n_i}\}$ has a convergent subsequence. Without loss of generality, we assume that $\lim_{n_i\to+\infty} Ax_{n_i} = y_0$. From (2.10), we have

$$\lim_{n_i\to+\infty}\|x_{n_i}\| = \lim_{n_i\to+\infty}\|Ax_{n_i} + \lambda B_{n_i}x_{n_i}\| \geq \lim_{n_i\to+\infty}\lambda\|B_{n_i}x_{n_i}\| - \lim_{n_i\to+\infty}\|Ax_{n_i}\| = +\infty,$$

which contradicts (2.9).

Now (2.8) guarantees that

$$0 < \inf_{n\in\mathbb{N}}\left\|x_n + \frac{1}{n}x_0\right\| \leq \left\|x_n + \frac{1}{n}x_0\right\| \leq \sup_{x\in P\cap\Omega}\|x\| + \|x_0\| < +\infty, \quad n \in \mathbb{N}.$$

Then $\{Ax_n + \lambda B_n x_n\}$ has a convergent subsequence. Without loss of generality, we assume that

$$Ax_n + \lambda B_n x_n \to y_1, \quad \text{as } n \to +\infty.$$

Then

$$x_n \to y_1, \quad \text{as } n \to +\infty.$$

Now (2.8) guarantees that $y_1 \neq \theta$. Letting $n \to +\infty$ in (2.7), and we have

$$y_1 = Ay_1 + \lambda By_1 \tag{2.11}$$

and $y_1 \in P \cap \Omega - \{\theta\}$. The proof is complete. □

Corollary 2.1 *Suppose that $\theta \in \Omega$, $A : P \cap \overline{\Omega} \to P$ is continuous and compact and $B : P - \{\theta\} \to P$ is singular continuous and compact. Assume that*

$$\|Ax\| < \|x\|, \quad \forall x \in P \cap \partial\Omega \tag{2.12}$$

or

$$Ax \not\geq x, \quad \forall x \in P \cap \partial\Omega. \tag{2.13}$$

Then there exists a $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, there exist $x_\lambda \in P \cap \Omega$ with

$$x_\lambda = Ax_\lambda + \lambda Bx_\lambda.$$

It is easy to see that (2.12) or (2.13) guarantees that (2.1) holds (see [8]).

Theorem 2.4 *Suppose that Ω_1, Ω_2 are bounded open sets and $\theta \in \Omega_1 \subseteq \Omega_2$, $A, K : P \cap \overline{\Omega}_2 \to P$ are continuous and compact and $B : P - \{\theta\} \to P$ is singular continuous and compact. Assume that*

(C_1) $Ax \neq \mu x$, $\forall x \in P \cap \partial\Omega_1$, $\mu \geq 1$;
(C_2) $\inf_{x\in P\cap\partial\Omega_2}\|Kx\| > 0$;
(C_3) $x - Ax \neq \mu Kx$, $\forall x \in P \cap \partial\Omega_2$, $\mu \geq 0$.

Then there exists a $\lambda_* > 0$ *such that, for any* $\lambda \in (0, \lambda_*)$, *there exist* $x_{\lambda,1} \in (P \cap \Omega_1 - \{\theta\})$ *and* $x_{\lambda,2} \in P \cap (\Omega_2 - \overline{\Omega}_1)$ *with*

$$x_{\lambda,1} = Ax_{\lambda,1} + \lambda Bx_{\lambda,1}, \qquad x_{\lambda,2} = Ax_{\lambda,2} + \lambda Bx_{\lambda,2}.$$

Proof Choose $x_0 \in P - \{\theta\}$, and define

$$B_n x = B\left(x + \frac{1}{n}x_0\right), \quad \forall x \in P, n \in \mathbb{N}.$$

Set

$$\gamma_1 := \inf_{(\mu,x)\in[1,+\infty)\times P\cap\partial\Omega_1} \|\mu x - Ax\|$$

and

$$\gamma_2 := \inf_{(\mu,x)\in[0,+\infty)\times P\cap\partial\Omega_2} \|x - Ax - \mu Kx\|.$$

We claim that

$$\gamma_1 > 0, \qquad \gamma_2 > 0. \tag{2.14}$$

An argument similar to that in (2.2) shows that

$$\gamma_1 > 0. \tag{2.15}$$

Now we show that

$$\gamma_2 > 0. \tag{2.16}$$

To see this suppose that $\gamma_2 = 0$. Then there exists $\{(\mu_n, x_n)\} \subseteq [0, +\infty) \times (P \cap \partial\Omega_2)$ such that

$$\lim_{n\to+\infty} \|x_n - Ax_n - \mu_n Kx\| = 0. \tag{2.17}$$

Now since

$$\|x_n - Ax_n - \mu_n Kx\| \geq \mu_n \inf_{x\in\partial\cap\Omega_2} \|Kx\| - \|x_n - Ax_n\|,$$

we have $\{\mu_n\}$ is bounded, which means that $\{\mu_n\}$ has a convergent subsequence. Without loss of generality, we assume that

$$\lim_{n\to+\infty} \mu_n = \mu_0.$$

Since $\{x_n\}$ is bounded and A and K are compact, $\{Ax_n\}$ and $\{Kx_n\}$ have convergent subsequences $\{Ax_{n_i}\}$ and $\{Kx_{n_i}\}$ with $\lim_{n_i\to+\infty} Ax_{n_i} = y_0$ and $\lim_{n_i\to+\infty} Kx_{n_i} = z_0$. Now

$$\lim_{n\to+\infty} \|x_n - Ax_n - \mu_n Kx_n\| = 0,$$

which implies that

$$\lim_{n_i \to +\infty} \|x_{n_i} - y_0 - \mu_0 z_0\| = 0.$$

Let $x_0 = y_0 + \mu_0 z_0$. Clearly, $x_0 \in P \cap \partial\Omega_2$. Now

$$\|x_0 - Ax_0 - \mu_0 Kx_0\| = \lim_{n_i \to +\infty} \|x_{n_i} - Ax_{n_i} - \mu_{n_i} Kx_{n_i}\| = 0,$$

which contradicts condition (C_3).

Consequently, (2.16) is true, which together with (2.15) yields (2.14).

Let $\gamma = \min\{\gamma_1, \gamma_2\}$. Obviously, $\gamma > 0$.

Let

$$\beta_{n,1} = \sup_{x \in P \cap \partial\Omega_1} \|B_n x\|, \qquad \beta_{n,2} = \sup_{x \in P \cap \partial\Omega_2} \|B_n x\|.$$

An argument similar to that in (2.4) shows that

$$\sup_{n \in \mathbb{N}} \beta_{n,1} < +\infty, \qquad \sup_{n \in \mathbb{N}} \beta_{n,2} < +\infty.$$

Let

$$\beta := \max\left\{\sup_{n \in \mathbb{N}} \beta_{n,1}, \sup_{n \in \mathbb{N}} \beta_{n,1}\right\} < +\infty$$

and

$$\lambda_* := \frac{\gamma}{\beta} > 0.$$

For $0 < \lambda < \lambda_*$, $x \in P \cap \partial\Omega_1$, $\mu \geq 1$, we have

$$\|Ax + \lambda B_n x - \mu x\| \geq \|Ax - \mu x\| - \lambda \|B_n x\| \geq \gamma - \beta\lambda > 0,$$

which guarantees that

$$i(A + \lambda B_n, P \cap \Omega_1, P) = 1, \quad n \in \mathbb{N}, \tag{2.18}$$

and for $x \in P \cap \partial\Omega_2$, $\mu \geq 0$, we have

$$\left\|x - (Ax + \lambda B_n x) - \mu Kx\right\| \geq \|x - Ax - \mu Kx\| - \lambda \|B_n x\| \geq \gamma - \beta\lambda > 0,$$

which guarantees that

$$i(A + \lambda B_n, P \cap \Omega_2, P) = 0.$$

Thus

$$i\big(A + \lambda B_n, P \cap (\Omega_2 - \overline{\Omega}_1), P\big) = -1. \tag{2.19}$$

Now (2.18) and (2.19) guarantee that there exist $\{x_{n,1}\} \subseteq P \cap \Omega_1$ and $\{x_{n,2}\} \subseteq P \cap (\Omega_2 - \overline{\Omega}_1)$ such that

$$x_{n,1} = Ax_{n,1} + \lambda B_n x_{n,1}, \qquad x_{n,2} = Ax_{n,2} + \lambda B_n x_{n,2}, \quad \lambda \in (0, \lambda_*), n \in \mathbb{N}.$$

An argument similar to that in (2.11) shows that there exist $y_1 \in P \cap \Omega_1 - \{\theta\}$ and $y_2 \in P \cap (\Omega_2 - \overline{\Omega}_1)$ with

$$y_1 = Ay_1 + \lambda By_1, \qquad y_2 = Ay_2 + \lambda By_2, \quad \lambda \in (0, \lambda_*).$$

The proof is complete. □

Corollary 2.2 *Suppose that Ω_1, Ω_2 are bounded open sets and $\theta \in \Omega_1 \subseteq \Omega_2$, $A : P \cap \overline{\Omega}_2 \to P$ is continuous and compact and $B : P - \{\theta\} \to P$ is singular continuous and compact. Assume that*

(C_4)

$$\|Ax\| < \|x\|, \quad \forall x \in P \cap \partial\Omega_1$$

or

$$Ax \ngeq x, \quad \forall x \in P \cap \partial\Omega_1;$$

(C_5)

$$\|Ax\| > \|x\|, \quad \forall x \in P \cap \partial\Omega_2$$

or $\exists u_0 \in P - \{\theta\}$ *such that*

$$x - Ax \neq \mu u_0, \quad \forall x \in P \cap \partial\Omega_2, \mu \geq 0$$

or

$$Ax \nleq x, \quad \forall x \in P \cap \partial\Omega_2.$$

Then there exists a $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, there exist $x_{\lambda,1} \in P \cap \Omega_1 - \{\theta\}$ and $x_{\lambda,2} \in P \cap (\Omega_2 - \overline{\Omega}_1)$ with

$$x_{\lambda,1} = Ax_{\lambda,1} + \lambda Bx_{\lambda,1}$$

and

$$x_{\lambda,2} = Ax_{\lambda,2} + \lambda Bx_{\lambda,2}.$$

It is easy to see that (C_4) and (C_5) guarantee that (C_1)-(C_3) hold (see [8]).

3 Applications for singular systems of multi-point boundary value problems

In [9], Henderson and Luca considered the system of nonlinear second-order ordinary differential equations

$$\begin{cases} u''(t) + f(t, v(t)) = 0, & t \in (0, T), \\ v''(t) + g(t, u(t)) = 0, & t \in (0, T) \end{cases} \tag{3.1}$$

with multi-point boundary conditions

$$\begin{cases} u(0) = 0, & u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i), \quad m \geq 3, \\ v(0) = 0, & v(T) = \sum_{i=1}^{n-2} c_i v(\eta_i), \quad n \geq 3. \end{cases} \tag{3.2}$$

The following conditions come from [9]:

(H1) $0 < \xi_1 < \cdots < \xi_{m-2} < T$, $0 < \eta_1 < \cdots < \eta_{n-2} < T$, $b_i > 0$, $i = 1, 2, \ldots, m-2$, $c_i \geq 0$, $i = 1, 2, \ldots, n-2$, $d = T - \sum_{i=1}^{m-2} b_i \xi_i > 0$, $e = T - \sum_{i=1}^{n-2} c_i \eta_i > 0$, $\sum_{i=1}^{m-2} b_i \xi_i > 0$, $\sum_{i=1}^{n-2} c_i \eta_i > 0$,

(H2) we have the functions $f, g \in C([0, T] \times [0, +\infty), [0, +\infty))$ and $f(t, 0) = 0$, $g(t, 0) = 0$ for all $t \in [0, T]$,

(H3) there exists a positive constant $p \in (0, 1]$ such that
  (1) $f_\infty^i = \lim_{u \to +\infty} \inf \inf_{t \in [0,T]} \frac{f(t,u)}{u^p} \in (0, +\infty]$;
  (2) $g_\infty^i = \lim_{u \to +\infty} \inf \inf_{t \in [0,T]} \frac{g(t,u)}{u^{\frac{1}{p}}} = +\infty$,

(H4) there exists a $r \in (0, +\infty)$ such that
  (1) $f_\infty^s = \lim_{u \to +\infty} \sup \sup_{t \in [0,T]} \frac{f(t,u)}{u^r} \in (0, +\infty]$;
  (2) $g_\infty^s = \lim_{u \to +\infty} \sup \sup_{t \in [0,T]} \frac{g(t,u)}{u^{\frac{1}{r}}} = 0$,

(H5) (1) $f_0^i = \lim_{u \to 0+} \inf \inf_{t \in [0,T]} \frac{f(t,u)}{u} \in (0, +\infty]$;
  (2) $g_0^i = \lim_{u \to 0+} \inf \inf_{t \in [0,T]} \frac{g(t,u)}{u} = +\infty$,

(H6) for each $t \in [0, T]$, $f(t, u)$ and $g(t, u)$ are nondecreasing with respect to u, and there exists a constant $N > 0$ such that

$$f\left(t, m_0 \int_0^T g(s, N)\, ds\right) < \frac{N}{m_0}, \quad \forall t \in [0, T],$$

where $m_0 = \frac{T^2}{4} \max\{a_1 T, \tilde{a}_1\}$ and a_1, $\tilde{a}_1$ are defined in [9].

Theorem 3.1 ([9]) *Assume that* (H1)-(H2) *and* (H4)-(H5) *hold. Then Problem* (3.1), (3.2) *has at least one positive solution* $(u(t), v(t))$, $t \in [0, T]$.

Theorem 3.2 ([9]) *Assume that* (H1)-(H3) *and* (H5)-(H6) *hold. Then Problem* (3.1), (3.2) *has at least two positive solutions* $(u_1(t), v_1(t))$, $(u_2(t), v_2(t))$, $t \in [0, T]$.

Here we consider

$$\begin{cases} u''(t) + f(t, v(t)) + \lambda u^{-\gamma} = 0, & t \in (0, T), \\ v''(t) + g(t, u(t)) = 0, & t \in (0, T) \end{cases} \tag{3.3}$$

with multi-point boundary conditions (3.2), where $1 > \gamma > 0$.

Let $C[0,T] := \{x : [0,T] \to R : x(t)$ is continuous on $[0,T]\}$ with norm

$$\|x\| = \max_{t\in[0,T]} \left|x(t)\right|.$$

Obviously, $C[0,T]$ is a Banach space. Let

$$P := \left\{x \in C[0,T] : x(t) \geq 0 \text{ is concave and } \inf_{t\in[\theta_0,T]} x(t) \geq \gamma \|x\|\right\},$$

where θ_0 and γ are defined in Section 2 in [9].

For $u \in P$, define an operator

$$(Au)(t) = \int_0^T G_1(t,s) f\left(s, \int_0^T G_2(s,\tau) g\big(\tau, u(\tau)\big)\, d\tau\right) ds, \quad t \in [0,T]$$

and for $u \in P - \{\theta\}$, define an operator

$$(Bu)(t) = \int_0^T G_1(t,s) u^{-\gamma}(s)\, ds, \quad t \in [0,T],$$

where $G_1(t,s)$ and $G_2(t,s)$ are defined in [9].

It is easy to see that $B : P - \{\theta\} \to P$ is a singular continuous and compact operator (see [9, 11]).

Theorem 3.3 *Assume that* (H1)-(H2) *and* (H4) *hold. Then there exists a* $\lambda^* > 0$ *such that Problem* (3.3), (3.2) *has at least one positive solution* $(u(t), v(t))$, $t \in [0,T]$ *for all* $\lambda \in (0, \lambda^*)$.

Proof Let B_R be defined as that in Theorem 3.2 of [9]. From the proof in [9], it is easy to see that

$$\|Ax\| < \|x\|, \quad \forall x \in P \cap \partial B_R.$$

Now Corollary 2.1 guarantees that there exists a $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, there exist $u \in (P \cap B_R - \{\theta\})$ such that

$$u = Au + \lambda Bu.$$

Let

$$v(t) = \int_0^T G_2(t,s) g\big(s, u(s)\big)\, ds, \quad t \in [0,T].$$

Then $(u(t), v(t))$ is a positive solution for (3.3), (3.2). The proof is complete. □

Theorem 3.4 *Assume that* (H1)-(H3) *and* (H6) *hold. Then Problem* (3.3), (3.2) *has at least two positive solutions* $(u_1(t), v_1(t))$, $(u_2(t), v_2(t))$, $t \in [0,T]$.

Proof Let B_N and B_L be defined as in Theorem 3.3 of [9]. From the proof in [9], it is easy to see that

$$\|Ax\| < \|x\|, \quad \forall x \in P \cap \partial B_N,$$

$$x \neq Ax + \lambda u_0, \quad \forall x \in P \cap \partial B_L, \lambda \geq 0, L > N.$$

Now Corollary 2.2 guarantees that there exists a $\lambda^* > 0$, for any $\lambda \in (0, \lambda^*)$, such that there exist $u_1 \in (P \cap B_N - \{\theta\})$ and $u_2 \in P \cap (B_L - \overline{B}_N)$ with

$$u_1 = Au_1 + \lambda Bu_1$$

and

$$u_2 = Au_2 + \lambda Bu_2.$$

Let

$$v_1(t) = \int_0^T G_2(t,s)g\bigl(s, u_1(s)\bigr)\,ds, \quad t \in [0, T]$$

and

$$v_2(t) = \int_0^T G_2(t,s)g\bigl(s, u_2(s)\bigr)\,ds, \quad t \in [0, T].$$

Then $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ are two positive solutions for (3.3), (3.2). The proof is complete. □

Remark Note that f and g have no singularity at $u = 0$ and $v = 0$ in [9, 10], so Theorems 3.3 and 3.4 improve the results in [9, 10].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the paper. All authors read and approved the final manuscript.

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