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Generalized α-ψ contractive mappings in quasi-metric spaces and related fixed-point theorems

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Abstract
In this paper, we characterize α-ψ contractive mappings in the setting of quasi-metric spaces and investigate the existence and uniqueness of a fixed point of such mappings. We notice that by using our result some fixed-point theorems in the context of G-metric space can be deduced.

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1 Introduction and preliminaries

Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ψ_1) ψ is nondecreasing;
(ψ_2) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

These functions are known in the literature as (c)-comparison functions. One can easily deduce that if ψ is a (c)-comparison function, then $\psi(t) < t$ for any $t > 0$.

Definition 1 Let X be a non-empty and let $d : X \times X \to [0,\infty)$ be a function which satisfies:

(d1) $d(x,y) = 0$ if and only if $x = y$,
(d2) $d(x,y) \le d(x,z) + d(z,y)$. Then d called a quasi-metric and the pair (X,d) is called a quasi-metric space.

Remark 2 Any metric space is a quasi-metric space, but the converse is not true in general.

Now, we give convergence and completeness on quasi-metric spaces.

Definition 3 Let (X,d) be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if

$$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0. \tag{1.1}$$

Definition 4 Let (X,d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is left-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geq m > N$.

Definition 5 Let (X,d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is right-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n > N$.

Definition 6 Let (X,d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.

Remark 7 A sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.

Definition 8 Let (X,d) be a quasi-metric space. We say that

(1) (X,d) is left-complete if and only if each left-Cauchy sequence in X is convergent.
(2) (X,d) is right-complete if and only if each right-Cauchy sequence in X is convergent.
(3) (X,d) is complete if and only if each Cauchy sequence in X is convergent.

Definition 9 Let (X,d) be a quasi-metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr), \quad \forall x,y \in X. \tag{1.2}$$

Remark 10 We easily see that any contractive mapping, that is, a mapping satisfying the Banach contraction, is an α-ψ contractive mapping with $\alpha(x,y) = 1$ for all $x,y \in X$ and $\psi(t) = kt$, $k \in (0,1)$.

Definition 11 Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. We say that T is α admissible if for all $x,y \in X$ we have

$$\alpha(x,y) \geq 1 \quad \Rightarrow \quad \alpha(Tx,Ty) \geq 1. \tag{1.3}$$

2 Main results

We start this section by the following definition, which is a characterization of α-ψ contractive mappings [1] in the context of a quasi-metric space.

Definition 12 (*cf.* [2]) Let (X,d) be a quasi-metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that for all $x,y \in X$, we have

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr). \tag{2.1}$$

Theorem 13 *Let (X,d) be a complete quasi-metric space. Suppose that $T : X \to X$ is a α-ψ contractive mapping which satisfies*

(i) *T is α admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *T is continuous.*

Then T has a fixed point.

Proof By (ii), there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$. Let us define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then it is evident that x_{n_0} is a fixed point of T. Consequently, throughout the proof, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Regarding the assumption (i), we derive

$$\alpha(x_1, x_0) = \alpha(Tx_0, x_0) \geq 1 \quad \Rightarrow \quad \alpha(Tx_1, Tx_0) = \alpha(x_2, x_1) \geq 1. \tag{2.2}$$

Recursively, we get

$$\alpha(x_{n+1}, x_n) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

Taking (2.1) and (2.3) into account, we find that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \alpha(x_n, x_{n-1}) d(Tx_n, Tx_{n-1}) \leq \psi\big(d(x_n, x_{n-1})\big), \tag{2.4}$$

for all $n \geq 1$. Inductively, we obtain

$$d(x_{n+1}, x_n) \leq \psi^n\big(d(x_1, x_0)\big), \quad \forall n \geq 1. \tag{2.5}$$

By using the triangular inequality and (2.5), for all $k \geq 1$, we get

$$\begin{aligned} d(x_{n+k}, x_n) &\leq d(x_{n+k}, x_{n+k-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \sum_{p=n}^{n+k-1} \psi^n\big(d(x_1, x_0)\big) \\ &\leq \sum_{p=n}^{\infty} \psi^n\big(d(x_1, x_0)\big). \end{aligned} \tag{2.6}$$

Letting $n \to \infty$ in the above inequality, we derive $\sum_{p=n}^{\infty} \psi^n(d(x_1, x_0)) \to 0$. Hence, $d(x_{n+k}, x_n) \to 0$ as $n \to \infty$. Therefore, $\{x_n\}$ is a left-Cauchy sequence in (X, d).

Analogously, we deduce that $\{x_n\}$ is a right-Cauchy sequence in (X, d). Indeed, by assumption (i), we obtain

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \quad \Rightarrow \quad \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1. \tag{2.7}$$

Recursively, we find that

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{2.8}$$

By combining (2.1) with (2.8), we find

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \leq \psi\big(d(x_{n-1}, x_n)\big), \tag{2.9}$$

for all $n \geq 1$. By iteration, we have

$$d(x_n, x_{n+1}) \leq \psi^n\big(d(x_0, x_1)\big), \quad \forall n \geq 1. \tag{2.10}$$

Due to the triangular inequality, together with (2.10), for all $k \geq 1$, we get

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+k-1}, x_{n+k}) \\ &\leq \sum_{p=n}^{n+k-1} \psi^n\big(d(x_0, x_1)\big) \\ &\leq \sum_{p=n}^{\infty} \psi^n\big(d(x_0, x_1)\big) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.11}$$

Consequently, $\{x_n\}$ is a right-Cauchy sequence in (X, d). By Remark 7, we deduce that $\{x_n\}$ is a Cauchy sequence in complete quasi-metric space (X, d). It implies that there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = \lim_{n\to\infty} d(u, x_n) = 0. \tag{2.12}$$

Then, by using the property (d1) together with the continuity of T, we obtain

$$\lim_{n\to\infty} d(x_n, Tu) = \lim_{n\to\infty} d(Tx_{n-1}, Tu) = 0 \tag{2.13}$$

and

$$\lim_{n\to\infty} d(Tu, x_n) = \lim_{n\to\infty} d(Tx_{n-1}, Tu) = 0. \tag{2.14}$$

Thus, we have

$$\lim_{n\to\infty} d(x_n, Tu) = \lim_{n\to\infty} d(Tu, x_n) = 0. \tag{2.15}$$

Keeping (2.12) and (2.15) in mind together with the uniqueness of the limit, we conclude that $u = Tu$, that is, u is a fixed point of T. □

Theorem 14 *Let (X, d) be a complete quasi-metric space. Suppose that $T : X \to X$ is an α-ψ contractive mapping which satisfies*:

(i) *T is α admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$*;
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_{n+1}, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x, x_{n(k)}) \geq 1$ for all k.*

Then T has a fixed point.

Proof Following the lines of the proof of Theorem 13, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, for all $n \geq 0$, converges for some $u \in X$. From (2.3) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(u, x_{n(k)}) \geq 1$ for all k. Applying (2.1), for all k, we get

$$d(Tu, x_{n(k)+1}) = d(Tu, Tx_{n(k)}) \leq \alpha(u, x_{n(k)}) d(Tu, Tx_{n(k)}) \leq \psi\bigl(d(u, x_{n(k)})\bigr). \tag{2.16}$$

Letting $k \to \infty$ in the above equality, we obtain

$$d(Tu, u) \leq 0. \tag{2.17}$$

Thus, we have $d(Tu, u) = 0$, that is, $Tu = u$. □

Definition 15 (*cf.* [3]) Let (X, d) be a quasi-metric space and $T : X \to X$ be a given mapping. We say that T is a generalized α-ψ contractive mapping of type A if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$ and we have

$$\alpha(x, y) d(Tx, Ty) \leq \psi\bigl(M(x, y)\bigr), \tag{2.18}$$

where

$$M(x, y) = \max\left\{d(x, y), d(Tx, x), d(Ty, y), \frac{1}{2}\bigl[d(Tx, y) + d(Ty, x)\bigr]\right\}. \tag{2.19}$$

Definition 16 Let (X, d) be a quasi-metric space and $T : X \to X$ be a given mapping. We say that T is a generalized α-ψ contractive mapping of type B if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$ and we have

$$\alpha(x, y) d(Tx, Ty) \leq \psi\bigl(N(x, y)\bigr), \tag{2.20}$$

where

$$N(x, y) = \max\left\{d(x, y), \frac{1}{2}\bigl[d(Tx, x) + d(Ty, y)\bigr], \frac{1}{2}\bigl[d(Tx, y) + d(Ty, x)\bigr]\right\}. \tag{2.21}$$

Theorem 17 *Let (X, d) be a complete quasi-metric space. Suppose that $T : X \to X$ is a generalized α-ψ contractive mapping of type A and satisfies*

(i) *T is α admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T is continuous.*

Then T has a fixed point.

Proof By assumption (ii), there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$. We construct a sequence $\{x_n\}$ in X in the following way:

$$x_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N}.$$

If $x_{n_0} = x_{n_0+1}$ for some n_0, then it is clear that x_{n_0} is a fixed point of T. Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Due to assumption (i), we have

$$\alpha(x_1, x_0) = \alpha(Tx_0, x_0) \geq 1 \quad \Rightarrow \quad \alpha(Tx_1, Tx_0) = \alpha(x_2, x_1) \geq 1. \tag{2.22}$$

If we continue in this way, we obtain

$$\alpha(x_{n+1}, x_n) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{2.23}$$

From (2.18) and (2.23), for all $n \geq 1$, we derive

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \alpha(x_n, x_{n-1})d(Tx_n, Tx_{n-1}) \leq \psi\bigl(M(x_n, x_{n-1})\bigr), \tag{2.24}$$

where

$$\begin{aligned} M(x_n, x_{n-1}) = {} & \max\biggl\{d(x_n, x_{n-1}), d(Tx_n, x_n), d(Tx_{n-1}, x_{n-1}), \\ & \frac{1}{2}\bigl[d(Tx_n, x_{n-1}) + d(Tx_{n-1}, x_n)\bigr]\biggr\} \\ = {} & \max\biggl\{d(x_n, x_{n-1}), d(x_{n+1}, x_n), d(x_n, x_{n-1}), \frac{1}{2}\bigl[d(x_{n+1}, x_{n-1}) + d(x_n, x_n)\bigr]\biggr\} \\ \leq {} & \max\bigl\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\bigr\}. \end{aligned} \tag{2.25}$$

Since ψ is a nondecreasing function, (2.24) implies that

$$d(x_{n+1}, x_n) \leq \psi\bigl(\max\bigl\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\bigr\}\bigr), \tag{2.26}$$

for all $n \geq 1$. We shall examine two cases. Suppose that $d(x_{n+1}, x_n) > d(x_n, x_{n-1})$. Since $d(x_{n+1}, x_n) > 0$, we obtain

$$d(x_{n+1}, x_n) \leq \psi\bigl(d(x_{n+1}, x_n)\bigr) < d(x_{n+1}, x_n), \tag{2.27}$$

a contradiction. Therefore, we find that $\max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\} = d(x_n, x_{n-1})$. Since $\psi \in \Psi$, (2.26) yields

$$d(x_{n+1}, x_n) \leq \psi\bigl(d(x_n, x_{n-1})\bigr) < d(x_n, x_{n-1}) \tag{2.28}$$

for all $n \geq 1$. Recursively, we derive

$$d(x_{n+1}, x_n) \leq \psi^n\bigl(d(x_1, x_0)\bigr), \quad \forall n \geq 1. \tag{2.29}$$

Together with (2.29) and the triangular inequality, for all $k \geq 1$, we get

$$\begin{aligned} d(x_{n+k}, x_n) & \leq d(x_{n+k}, x_{n+k-1}) + \cdots + d(x_{n+1}, x_n) \\ & \leq \sum_{p=n}^{n+k-1} \psi^n\bigl(d(x_1, x_0)\bigr) \\ & \leq \sum_{p=n}^{\infty} \psi^n\bigl(d(x_1, x_0)\bigr) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.30}$$

Therefore, $\{x_n\}$ is a left-Cauchy sequence in (X, d).

Analogously, we shall prove that $\{x_n\}$ is a right-Cauchy sequence in (X,d). Again by the assumption (i), we find that

$$\alpha(x_0,x_1) = \alpha(x_0,Tx_0) \geq 1 \quad \Rightarrow \quad \alpha(Tx_0,Tx_1) = \alpha(x_1,x_2) \geq 1. \tag{2.31}$$

Recursively, we obtain

$$\alpha(x_n,x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{2.32}$$

From (2.18) and (2.32), for all $n \geq 1$, we deduce that

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \leq \alpha(x_{n-1},x_n)d(Tx_{n-1},Tx_n) \leq \psi\bigl(M(x_{n-1},x_n)\bigr), \tag{2.33}$$

where

$$\begin{aligned} M(x_{n-1},x_n) = {} & \max\biggl\{d(x_{n-1},x_n), d(Tx_{n-1},x_{n-1}), d(Tx_n,x_n), \\ & \frac{1}{2}\bigl[d(Tx_{n-1},x_n) + d(Tx_n,x_{n-1})\bigr]\biggr\} \\ = {} & \max\biggl\{d(x_{n-1},x_n), d(x_n,x_{n-1}), d(x_{n+1},x_n), \frac{1}{2}\bigl[d(Tx_{n-1},x_n) + d(Tx_n,x_{n-1})\bigr]\biggr\} \\ \leq {} & \max\bigl\{d(x_{n-1},x_n), d(x_n,x_{n-1}), d(x_{n+1},x_n)\bigr\}. \end{aligned} \tag{2.34}$$

Since ψ is nondecreasing function, the inequality (2.33) turns into

$$d(x_n,x_{n+1}) \leq \psi\bigl(\max\bigl\{d(x_{n-1},x_n), d(x_n,x_{n-1}), d(x_{n+1},x_n)\bigr\}\bigr), \tag{2.35}$$

for all $n \geq 1$. We shall examine three cases.

Case 1. Assume that $\max\{d(x_{n-1},x_n), d(x_n,x_{n-1}), d(x_{n+1},x_n)\} = d(x_{n+1},x_n)$. Since $d(x_{n+1}, x_n) > 0$ we get

$$d(x_{n+1},x_n) \leq \psi\bigl(d(x_{n+1},x_n)\bigr) < d(x_{n+1},x_n), \tag{2.36}$$

a contradiction.

Case 2. Suppose that $\max\{d(x_{n-1},x_n), d(x_n,x_{n-1}), d(x_{n+1},x_n)\} = d(x_{n-1},x_n)$. Since $\psi \in \Psi$, from (2.34) we find that

$$d(x_n,x_{n+1}) \leq \psi\bigl(d(x_{n-1},x_n)\bigr) < d(x_{n-1},x_n) \tag{2.37}$$

for all $n \geq 1$. Inductively, we get

$$d(x_n,x_{n+1}) \leq \psi^n\bigl(d(x_0,x_1)\bigr), \quad \forall n \geq 1. \tag{2.38}$$

By using the triangular inequality and taking (2.38) into consideration, for all $k \geq 1$, we get

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+k-1}, x_{n+k}) \\ &\leq \sum_{p=n}^{n+k-1} \psi^n\bigl(d(x_0, x_1)\bigr) \\ &\leq \sum_{p=n}^{\infty} \psi^n\bigl(d(x_0, x_1)\bigr) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.39}$$

Case 3. Assume that $\max\{d(x_{n-1}, x_n), d(x_n, x_{n-1}), d(x_{n+1}, x_n)\} = d(x_n, x_{n-1})$. Regarding $\psi \in \Psi$ and (2.35), we obtain

$$d(x_n, x_{n+1}) \leq \psi\bigl(d(x_n, x_{n-1})\bigr) < d(x_n, x_{n-1}) \tag{2.40}$$

for all $n \geq 1$. From (2.18) and (2.23), for all $n \geq 1$, we derive

$$\begin{aligned} d(x_n, x_{n-1}) &= d(Tx_{n-1}, Tx_{n-2}) \\ &\leq \alpha(x_{n-1}, x_{n-2}) d(Tx_{n-1}, Tx_{n-2}) \\ &\leq \psi\bigl(M(x_{n-1}, x_{n-2})\bigr), \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} M(x_{n-1}, x_{n-2}) = {} & \max\biggl\{d(x_{n-1}, x_{n-2}), d(Tx_{n-1}, x_{n-1}), d(Tx_{n-2}, x_{n-2}), \\ & \frac{1}{2}\bigl[d(Tx_{n-1}, x_{n-2}) + d(Tx_{n-2}, x_{n-1})\bigr]\biggr\} \\ = {} & \max\biggl\{d(x_{n-1}, x_{n-2}), d(x_n, x_{n-1}), d(x_{n-1}, x_{n-2}), \\ & \frac{1}{2}\bigl[d(x_n, x_{n-2}) + d(x_{n-1}, x_{n-1})\bigr]\biggr\} \\ \leq {} & \max\bigl\{d(x_{n-1}, x_{n-2}), d(x_n, x_{n-1})\bigr\}. \end{aligned} \tag{2.42}$$

Since ψ is a nondecreasing function, (2.24) implies that

$$d(x_n, x_{n-1}) \leq \psi\bigl(\max\bigl\{d(x_{n-1}, x_{n-2}), d(x_n, x_{n-1})\bigr\}\bigr), \tag{2.43}$$

for all $n \geq 1$.

We shall examine two cases. Suppose that $d(x_n, x_{n-1}) > d(x_{n-1}, x_{n-2})$. Since $d(x_n, x_{n-1}) > 0$, we obtain

$$d(x_n, x_{n-1}) \leq \psi\bigl(d(x_n, x_{n-1})\bigr) < d(x_n, x_{n-1}), \tag{2.44}$$

a contradiction. Therefore, we find that $\max\{d(x_{n-1}, x_{n-2}), d(x_n, x_{n-1})\} = d(x_{n-1}, x_{n-2})$. Since $\psi \in \Psi$, (2.43) yields

$$d(x_n, x_{n-1}) \leq \psi\bigl(d(x_{n-1}, x_{n-2})\bigr) < d(x_{n-1}, x_{n-2}) \tag{2.45}$$

for all $n \geq 1$. Recursively, we derive

$$d(x_n, x_{n-1}) \leq \psi^{n-1}\big(d(x_1, x_0)\big), \quad \forall n \geq 1. \tag{2.46}$$

If we combine the inequalities (2.40) with (2.46), we derive

$$d(x_n, x_{n+1}) \leq \psi\big(d(x_n, x_{n-1})\big) < d(x_n, x_{n-1}) \leq \psi^{n-1}\big(d(x_1, x_0)\big), \quad \forall n \geq 1. \tag{2.47}$$

Together with (2.47) and the triangular inequality, for all $k \geq 1$, we get

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+k-1}, x_{n+k}) \\ &< d(x_n, x_{n-1}) + \cdots + d(x_{n+k-1}, x_{n+k-2}) \\ &\leq \sum_{p=n}^{n+k-1} \psi^{n-1}\big(d(x_1, x_0)\big) \\ &\leq \sum_{p=n}^{\infty} \psi^{n-1}\big(d(x_1, x_0)\big) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.48}$$

Therefore, by (2.39) and (2.48), we conclude that $\{x_n\}$ is a right-Cauchy sequence in (X, d).

From Remark 7, $\{x_n\}$ is a Cauchy sequence in complete quasi-metric space (X, d). This implies that there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = \lim_{n\to\infty} d(u, x_n) = 0. \tag{2.49}$$

Then, using property (d1) and the continuity of T, we obtain

$$\lim_{n\to\infty} d(x_n, Tu) = \lim_{n\to\infty} d(Tx_{n-1}, Tu) = 0 \tag{2.50}$$

and

$$\lim_{n\to\infty} d(Tu, x_n) = \lim_{n\to\infty} d(Tx_{n-1}, Tu) = 0. \tag{2.51}$$

Thus, we have

$$\lim_{n\to\infty} d(x_n, Tu) = \lim_{n\to\infty} d(Tu, x_n) = 0. \tag{2.52}$$

It follows from (2.49) and (2.52) that $u = Tu$, that is, u is a fixed point of T. □

Corollary 18 *Let (X, d) be a complete quasi-metric space. Suppose that $T : X \to X$ is a generalized α-ψ contractive mapping of type A and satisfies*:

(i) *T is α admissible*;

(ii) *there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$*;

(iii) *T is continuous.*

Then T has a fixed point.

The proof is evident due to Theorem 17. Indeed, ψ is nondecreasing and, hence,

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(N(x,y)\bigr) \leq \psi\bigl(M(x,y)\bigr),$$

where $M(x,y)$ and $N(x,y)$ are defined as in Definition 15 and Definition 16. The rest follows from Theorem 17.

In the following theorem we are able to remove the continuity condition for the α-ψ contractive mappings of type B.

Theorem 19 *Let (X,d) be a complete quasi-metric space. Suppose that $T: X \to X$ is a generalized α-ψ contractive mapping of type B which satisfies*:

(i) *T is α admissible*;

(ii) *there exists $x_0 \in X$ such that $\alpha(Tx_0,x_0) \geq 1$*;

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_{n+1},x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x,x_{n(k)}) \geq 1$ for all k.*

Then T has a fixed point.

Proof Following the lines in the proof of Theorem 17, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges for some $u \in X$. From (2.23) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(u,x_{n(k)}) \geq 1$ for all k. Applying (2.20), for all k, we get

$$d(Tu,x_{n(k)+1}) = d(Tu,Tx_{n(k)}) \leq \alpha(u,x_{n(k)})d(Tu,Tx_{n(k)}) \leq \psi\bigl(N(u,x_{n(k)})\bigr). \tag{2.53}$$

Also, using (2.21) we find

$$\begin{aligned} N(u,x_{n(k)}) = \max\biggl\{ & d(u,x_{n(k)}), \frac{1}{2}\bigl[d(Tu,u) + d(Tx_{n(k)},x_{n(k)})\bigr], \\ & \frac{1}{2}\bigl[d(Tu,x_{n(k)}) + d(Tx_{n(k)},u)\bigr]\biggr\}. \end{aligned} \tag{2.54}$$

Taking the limit as $k \to \infty$ in the above equality, we obtain

$$\lim_{k\to\infty} N(u,x_{n(k)}) = \frac{d(Tu,u)}{2}. \tag{2.55}$$

Assume that $d(Tu,u) > 0$. From (2.55), for k large enough, we have $N(u,x_{n(k)}) > 0$, which implies that $\psi(N(u,x_{n(k)})) < N(u,x_{n(k)})$. Then, from (2.53), we have

$$d(Tu,x_{n(k)+1}) < N(u,x_{n(k)}). \tag{2.56}$$

Taking the limit as $k \to \infty$ in the above equality, we get

$$d(Tu,u) \leq \frac{d(Tu,u)}{2}, \tag{2.57}$$

which is a contradiction. Therefore, we find $d(Tu,u) = 0$, that is, $Tu = u$. □

3 Consequences: fixed-point result on *G*-metric spaces

In this section, we note that some existing fixed-point results in the context of G-metric spaces are consequences of our main theorems. For the sake of completeness, we recollect some basic definitions and crucial results on the topic in the literature. For more details, see *e.g.* [4–6].

Definition 20 Let X be a non-empty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (rectangle inequality) for all $x, y, z, a \in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Note that every G-metric on X induces a metric d_G on X defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \text{for all } x, y \in X. \tag{3.1}$$

For a better understanding of the subject we give the following examples of G-metrics.

Example 21 Let (X, d) be a metric space. The function $G : X \times X \times X \to [0, +\infty)$, defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},$$

for all $x, y, z \in X$, is a G-metric on X.

Example 22 Let $X = [0, \infty)$. The function $G : X \times X \times X \to [0, +\infty)$, defined by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|,$$

for all $x, y, z \in X$, is a G-metric on X.

Definition 23 Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to $x \in X$ if

$$\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0,$$

that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to+\infty} x_n = x$.

Proposition 24 *Let (X, G) be a G-metric space. The following are equivalent*:

(1) $\{x_n\}$ *is G-convergent to x*,
(2) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$,
(3) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$,
(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to +\infty$.

Definition 25 Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $m,n,l \geq N$, that is, $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to +\infty$.

Proposition 26 *Let (X,G) be a G-metric space. Then the following are equivalent*:

(1) *the sequence $\{x_n\}$ is G-Cauchy,*

(2) *for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$, for all $m,n \geq N$.*

Definition 27 A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).

For more details of G-metric space, we refer *e.g.* to [7–9].

Theorem 28 *Let (X,G) be a G-metric space. Let $d : X \times X \to [0,\infty)$ be the function defined by $d(x,y) = G(x,y,y)$. Then*

(1) *(X,d) is a quasi-metric space*;

(2) *$\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X,d)*;

(3) *$\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X,d)*;

(4) *(X,G) is G-complete if and only if (X,d) is complete.*

Every quasi-metric induces a metric, that is, if (X,d) is a quasi-metric space, then the function $\delta : X \times X \to [0,\infty)$ defined by

$$\delta(x,y) = \max\bigl\{d(x,y), d(y,x)\bigr\}$$

is a metric on X.

As an immediate consequence of the definition above and Theorem 28, the following theorem is obtained.

Theorem 29 *Let (X,G) be a G-metric space. Let $d : X \times X \to [0,\infty)$ be the function defined by $d(x,y) = G(x,y,y)$. Then*

(1) *(X,d) is a quasi-metric space*;

(2) *$\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X,δ)*;

(3) *$\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X,δ)*;

(4) *(X,G) is G-complete if and only if (X,δ) is complete.*

Now, we state the characterization of Definition 9 and Definition 11 in the context of G-metric space.

Definition 30 (See *e.g.* [10, 11]) Let (X,G) be a G-metric space and $T : X \to X$ be a given mapping. We say that T is a β-ψ contractive mapping of type I if there exist two functions $\beta : X \times X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that for all $x,y \in X$, we have

$$\beta(x,y,y)G(Tx,Ty,Tz) \leq \psi\bigl(G(x,y,y)\bigr). \tag{3.2}$$

Definition 31 (See *e.g.* [10, 11]) Let $T : X \to X$ and $\beta : X \times X \times X \to [0,\infty)$. We say that T is β admissible if for all $x,y \in X$ we have

$$\beta(x,y,y) \geq 1 \quad \Rightarrow \quad \beta(Tx,Ty,Ty) \geq 1. \tag{3.3}$$

Lemma 32 *Let $T : X \to X$ where X is non-empty set. It is clear that the self-mapping T is β admissible if and only if T is α admissible.*

Proof It is sufficient to let $\alpha(x,y) = \beta(x,y,y)$. □

Theorem 33 *Let (X,G) be a complete G-metric space. Suppose that $T : X \to X$ is a β-ψ contractive mapping which satisfies:*

(i) *T is β admissible;*
(ii) *there exists $x_0 \in X$ such that $\beta(Tx_0, x_0, x_0) \geq 1$ and $\beta(x_0, Tx_0, Tx_0) \geq 1$;*
(iii) *T is continuous.*

Then T has a fixed point.

Proof Consider the quasi-metric $d(x,y) = G(x,y,y)$ for all $x, y \in X$. Due to Lemma 32 and (3.2), we have

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr), \quad \forall x,y \in X. \tag{3.4}$$

Then the result follows from Theorem 13. □

Definition 34 Let (X,G) be a G-metric space and $T : X \to X$ be a given mapping. We say that T is a generalized β-ψ contractive mapping of type A if there exist two functions $\beta : X \times X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$ we have

$$\beta(x,y,y)G(Tx,Ty,Ty) \leq \psi\bigl(M_1(x,y,y)\bigr), \tag{3.5}$$

where

$$M_1(x,y,y) = \max\left\{G(x,y,y), G(Tx,x,x), G(Ty,y,y), \frac{1}{2}\bigl[G(Tx,y,y) + G(Ty,x,x)\bigr]\right\}. \tag{3.6}$$

Definition 35 Let (X,G) be a G-metric space and $T : X \to X$ be a given mapping. We say that T is a generalized β-ψ contractive mapping of type B if there exist two functions $\beta : X \times X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$ we have

$$\beta(x,y,y)G(Tx,Ty,Ty) \leq \psi\bigl(N_1(x,y,y)\bigr), \tag{3.7}$$

where

$$\begin{aligned} N_1(x,y,y) = \max\Bigl\{&G(x,y,y), \frac{1}{2}\bigl[G(Tx,x,x) + G(Ty,y,y)\bigr], \\ &\frac{1}{2}\bigl[G(Tx,y,y) + d(Ty,x,x)\bigr]\Bigr\}. \end{aligned} \tag{3.8}$$

Remark 36 It is simple to see that every β-ψ contractive mapping is a generalized β-ψ contractive mapping of type A.

Similarly, every β-ψ contractive mapping is a generalized β-ψ contractive mapping of type B.

Theorem 37 *Let (X,G) be a complete G-metric space. Suppose that $T : X \to X$ is a generalized β-ψ contractive mapping of type A and satisfies*:

(i) *T is β admissible*;
(ii) *there exists $x_0 \in X$ such that $\beta(Tx_0, x_0, x_0) \geq 1$ and $\beta(x_0, Tx_0, Tx_0) \geq 1$*;
(iii) *T is continuous.*

Then T has a fixed point.

Proof Consider the quasi-metric $d(x,y) = G(x,y,y)$ for all $x,y \in X$. From Lemma 32 together with (3.5) and (3.6), we deduce that

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(M(x,y)\bigr), \quad \forall x,y \in X. \tag{3.9}$$

Then the result follows from Theorem 17. □

Theorem 38 *Let (X,G) be a complete G-metric space. Suppose that $T : X \to X$ is a β-ψ contractive mapping which satisfies*:

(i) *T is β admissible*;
(ii) *there exists $x_0 \in X$ such that $\beta(Tx_0, x_0, x_0) \geq 1$*;
(iii) *if $\{x_n\}$ is a sequence in X such that $\beta(x_{n+1}, x_n, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\beta(x, x_{n(k)}, x_{n(k)}) \geq 1$ for all k.*

Then T has a fixed point.

Proof Consider the quasi-metric $d(x,y) = G(x,y,y)$ for all $x,y \in X$. By Lemma 32 and (3.2), we find that

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr), \quad \forall x,y \in X. \tag{3.10}$$

Then the result follows from Theorem 14. □

Theorem 39 *Let (X,G) be a complete G-metric space. Suppose that $T : X \to X$ is a generalized β-ψ contractive mapping of type B which satisfies*:

(i) *T is β admissible*;
(ii) *there exists $x_0 \in X$ such that $\beta(Tx_0, x_0, x_0) \geq 1$*;
(iii) *if $\{x_n\}$ is a sequence in X such that $\beta(x_{n+1}, x_n, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\beta(x, x_{n(k)}, x_{n(k)}) \geq 1$ for all k.*

Then T has a fixed point.

Proof Consider the quasi-metric $d(x,y) = G(x,y,y)$ for all $x,y \in X$. Regarding Lemma 32, (3.7), and (3.8), we derive

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(N(x,y)\bigr), \quad \forall x,y \in X. \tag{3.11}$$

Then the result follows from Theorem 19. □

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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