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# On the multivalency of certain analytic functions

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**Abstract**

We prove several relations of the type $|\arg\{zf''(z)/f'(z)\}| \le |\arg\{f'(z)\}|$ for functions satisfying some geometric conditions.

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**Keywords:** analytic; univalent; multivalent; convex; starlike; strongly starlike; differential subordination

## 1 Introduction

Let $p$ be positive integer and let $\mathcal{A}(p)$ be the class of functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

which are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and denote $\mathcal{A} = \mathcal{A}(1)$.

The subclass of $\mathcal{A}(p)$ consisting of $p$-valently starlike functions is denoted by $\mathcal{S}^*(p)$. An analytic description of $\mathcal{S}^*(p)$ is given by

$$\mathcal{S}^*(p) = \left\{f \in \mathcal{A}(p) : \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}, z \in \mathbb{D}\right\}.$$

The subclass of $\mathcal{A}(p)$ consisting of $p$-valently and strongly starlike functions of order $\alpha$, $0 < \alpha \le 1$ is denoted by $\mathcal{S}^*_\alpha(p)$. An analytic description of $\mathcal{S}^*_\alpha(p)$ is given by

$$\mathcal{S}^*_\alpha(p) = \left\{f \in \mathcal{A}(p) : \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, z \in \mathbb{D}\right\}.$$

The subclass of $\mathcal{A}(p)$ consisting of $p$-valently convex functions and $p$-valently strongly convex functions of order $\alpha$, $0 < \alpha \le 1$, are denoted by $\mathcal{C}(p)$ and $\mathcal{C}_\alpha(p)$, respectively. The analytic descriptions of $\mathcal{C}(p)$ and $\mathcal{C}_\alpha(p)$ are given by

$$\mathcal{C}(p) = \left\{f \in \mathcal{A}(p) : \left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi}{2}, z \in \mathbb{D}\right\}$$

and

$$\mathcal{C}_\alpha(p) = \left\{f \in \mathcal{A}(p) : \left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\alpha\pi}{2}, z \in \mathbb{D}\right\}.$$

For $p = 1$ the classes $\mathcal{S}^*_\alpha(p)$ and $\mathcal{C}_\alpha(p)$ become the well known classes $\mathcal{S}^*_\alpha$ and $\mathcal{C}^*_\alpha$ of strongly starlike and strongly convex functions of order $\alpha$, respectively. The concept of strongly starlike and strongly convex functions of order $\alpha$ was introduced in [1] and [2] with their geometric interpretation. For $\alpha = 1$ the classes $\mathcal{S}^*_\alpha$ and $\mathcal{C}^*_\alpha$ become the classes $\mathcal{S}^*$ and $\mathcal{C}^*$ of starlike and convex functions; see for example [3]. In this paper, we need the following lemma.

**Lemma 1.1** *Assume that $f \in \mathcal{A}$ with $f(z)/z \neq 0$ in $\mathbb{D}$. Assume also that for all $\theta$, $0 \le \theta < 2\pi$, the function $f$ satisfies the following condition*:

$$\begin{aligned}\left(\mathfrak{Im}\frac{zf''(z)}{f'(z)}\right)\sin\theta &= \rho\left(\frac{\mathrm{d}\arg\{f'(\rho e^{i\theta})\}}{\mathrm{d}\rho}\right)\sin\theta \\ &= \frac{1}{2}\mathfrak{Re}\left\{\frac{zf''(z)}{f'(z)}(\overline{z}-z)\right\} \\ &= \frac{1}{2}\mathfrak{Re}\left\{\frac{f''(z)}{f'(z)}\left(|z|^2-z^2\right)\right\} \ge 0, \end{aligned} \tag{1.1}$$

*where $z = \rho e^{i\theta}$, $0 < \rho \le r < 1$. Then we have*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| \le \left|\arg\{f'(z)\}\right|, \quad z \in \mathbb{D}.$$

*Proof* First we note that from

$$0 \le \arg\{z_1\} \le \arg\{z_2\} \le \pi \quad \Rightarrow \quad \arg\{z_1\} \le \arg\{z_1+z_2\} \le \arg\{z_2\},$$

the implication

$$0 \le \arg\{z_1\} \le \cdots \le \arg\{z_n\} \le \pi \quad \Rightarrow \quad \arg\{z_1\} \le \arg\left\{\sum_{k=1}^n z_k\right\} \le \arg\{z_n\} \tag{1.2}$$

follows by mathematical induction.

For the case $0 \le \theta < \pi$, $z = re^{i\theta} \in \mathbb{D}$, we have

$$\begin{aligned}\arg\left\{\frac{f(z)}{z}\right\} &= \arg\left\{\frac{1}{re^{i\theta}}\int_0^r f'\left(\rho e^{i\theta}\right)e^{i\theta}\,\mathrm{d}\rho\right\} \\ &= \arg\left\{\int_0^r f'\left(\rho e^{i\theta}\right)\mathrm{d}\rho\right\}.\end{aligned} \tag{1.3}$$

Let $0 = \rho_0 < \rho_1 < \cdots < \rho_{n-1} < \rho_n = r$, $\Delta\rho_k = \rho_k - \rho_{k-1}$, $k = 1, \ldots, n$. By (1.1) $\arg\{f'(\rho e^{i\theta})\}$ is an increasing function with respect to $\rho$, thus

$$0 = \arg\left\{f'\left(\rho_0 e^{i\theta}\right)\right\} \le \arg\left\{f'\left(\rho_1 e^{i\theta}\right)\right\} \le \cdots \le \arg\left\{f'\left(\rho_n e^{i\theta}\right)\right\} = \arg\left\{f'\left(re^{i\theta}\right)\right\}. \tag{1.4}$$

Therefore, by (1.2) and by (1.4), we have

$$\arg\left\{\sum_{k=1}^n f'\left(\rho_k e^{i\theta}\right)\right\} \le \arg\left\{f'\left(re^{i\theta}\right)\right\}. \tag{1.5}$$

Using (1.5) in (1.3), we obtain

$$\begin{aligned}\arg\left\{\frac{f(z)}{z}\right\} &= \arg\left\{\int_0^r f'(\rho e^{i\theta})\,\mathrm{d}\rho\right\}\\ &= \arg\left\{\lim_{n\to\infty}\sum_{k=1}^{n} f'(\rho_k e^{i\theta})\,\Delta\rho_k\right\}\\ &= \lim_{n\to\infty}\arg\left\{\sum_{k=1}^{n} f'(\rho_k e^{i\theta})\,\Delta\rho_k\right\}\\ &\le \arg\{f'(re^{i\theta})\} \qquad (1.6)\end{aligned}$$

or we have

$$0 \le \arg\left\{\frac{f(z)}{z}\right\} \le \arg\{f'(z)\} \qquad (1.7)$$

for $z = re^{i\theta}$ and $0 \le \theta \le \pi$.

For the case $\pi \le \theta < 2\pi$, from the hypothesis (1.1), we find that $\arg\{f'(\rho e^{i\theta})\}$ is an decreasing function with respect to $\rho$, thus

$$0 = \arg\{f'(\rho_0 e^{i\theta})\} \ge \arg\{f'(\rho_1 e^{i\theta})\} \ge \cdots \ge \arg\{f'(\rho_n e^{i\theta})\} = \arg\{f'(re^{i\theta})\}$$

and

$$\arg\left\{\sum_{k=1}^{n} f'(\rho_k e^{i\theta})\right\} \ge \arg\{f'(re^{i\theta})\}.$$

Therefore, in a similar way to above, we obtain

$$\begin{aligned}\arg\left\{\frac{f(z)}{z}\right\} &= \arg\left\{\int_0^r f'(\rho e^{i\theta})\,\mathrm{d}\rho\right\}\\ &\ge \arg\{f'(re^{i\theta})\}\end{aligned}$$

and we also have

$$0 \ge \arg\left\{\frac{f(z)}{z}\right\} \ge \arg\{f'(z)\} \qquad (1.8)$$

for $z = re^{i\theta}$ and $\pi \le \theta \le 2\pi$. From (1.7) and (1.8), we have

$$\begin{aligned}\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| &= \left|\arg\{f'(z)\} - \arg\left\{\frac{f(z)}{z}\right\}\right|\\ &\le \left|\arg\{f'(z)\}\right|.\end{aligned}$$

It completes the proof of Lemma 1.1. □

**Corollary 1.2** *Assume that $f \in \mathcal{A}$ with $f(z)/z \neq 0$ in $\mathbb{D}$. Assume also that $f(z)$ satisfies the following condition*:

$$\left(\mathfrak{Im}\frac{zf''(z)}{f'(z)}\right)\mathfrak{Im}\{z\} \geq 0, \quad z \in \mathbb{D}, \tag{1.9}$$

*then we have*

$$\arg\left\{\frac{zf'(z)}{f(z)}\right\}\arg\{z\} \geq 0, \quad z \in \mathbb{D}. \tag{1.10}$$

*Proof* The conditions (1.1) and (1.9) are equivalent. If $\arg\{z\} \geq 0$, then $z = re^{i\theta} \in \mathbb{D}$ with $0 \leq \theta \leq \pi$. By (1.7), we have also

$$\arg\left\{\frac{zf'(z)}{f(z)}\right\} \geq 0, \quad z \in \mathbb{D}.$$

If $\arg\{z\} \leq 0$, then $z = re^{i\theta} \in \mathbb{D}$ with $\pi < \theta \leq 2\pi$. By (1.8), we have also

$$\arg\left\{\frac{zf'(z)}{f(z)}\right\} \leq 0, \quad z \in \mathbb{D}.$$

In both cases, we have (1.10). □

The inequality (1.10) can be written in the equivalent form

$$\left(\mathfrak{Im}\frac{zf'(z)}{f(z)}\right)\mathfrak{Im}\{z\} \geq 0, \quad z \in \mathbb{D}. \tag{1.11}$$

Recall that if $f(z)$ is analytic in $\mathbb{D}$ and $(\mathfrak{Im}\{f(z)\})(\mathfrak{Im}\{z\}) \geq 0$ in $\mathbb{D}$, then $f$ is called typically real function; see [4, Chapter 10]. Therefore, Corollary 1.2 says that if $zf''(z)/f'(z)$ is a typically real function, then $zf'(z)/f(z)$ is a typically real function, too.

## 2 Main result

**Theorem 2.1** *Let $f(z) \in \mathcal{A}$. Assume that for all $\theta$, $0 \leq \theta < 2\pi$, $f(z)$ satisfies the following condition*:

$$\begin{aligned}\left(\mathfrak{Im}\frac{zf''(z)}{f'(z)}\right)\sin\theta &= \rho\left(\frac{\mathrm{d}\arg\{f'(\rho e^{i\theta})\}}{\mathrm{d}\rho}\right)\sin\theta \\ &\geq 0,\end{aligned} \tag{2.1}$$

*where $z = \rho e^{i\theta}$, $0 \leq \rho \leq r < 1$, moves on the segment from $z = 0$ to $z = re^{i\theta}$ and*

$$\left|\arg\{f'(z)\}\right| \leq \frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{2.2}$$

*Then $f(z)$ is starlike in $\mathbb{D}$ or $f(z) \in \mathcal{S}^*$.*

*Proof* From the hypothesis (2.1) and the hypothesis (2.2) and applying Lemma 1.1, we have

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| \leq \left|\arg\{f'(z)\}\right| \leq \frac{\pi}{2}, \quad z \in \mathbb{D}.$$

This shows that $f(z)$ is starlike in $\mathbb{D}$. □

Applying the same method as in the proof of Lemma 1.1, we have the following lemma.

**Lemma 2.2** *Let $f(z) \in \mathcal{A}(2)$. Assume that for all $\theta$, $0 \le \theta < 2\pi$, $f(z)$ satisfies the following condition*:

$$\begin{aligned}\left(\mathfrak{Im}\frac{zf'''(z)}{f''(z)}\right)\sin\theta &= \rho\left(\frac{\mathrm{d}\arg\{f''(\rho e^{i\theta})\}}{\mathrm{d}\rho}\right)\sin\theta \\ &\ge 0,\end{aligned}$$

*where $z = \rho e^{i\theta}$, $0 \le \rho \le r < 1$ moves on the segment from $z = 0$ to $z = re^{i\theta}$. Then we have*

$$\left|\arg\left\{\frac{zf''(z)}{f'(z)}\right\}\right| \le \left|\arg\{f''(z)\}\right|, \quad z \in \mathbb{D}.$$

Applying Lemma 2.2, we have the following theorem.

**Theorem 2.3** *Let $f(z) \in \mathcal{A}(2)$. Suppose that for all $\theta$, $0 \le \theta < 2\pi$, $f(z)$ satisfies the following condition*:

$$\begin{aligned}\left(\mathfrak{Im}\frac{zf'''(z)}{f''(z)}\right)\sin\theta &= \rho\left(\frac{\mathrm{d}\arg\{f''(\rho e^{i\theta})\}}{\mathrm{d}\rho}\right)\sin\theta \\ &\ge 0,\end{aligned} \tag{2.3}$$

*where $z = \rho e^{i\theta}$, $0 \le \rho \le r < 1$, moves on the segment from $z = 0$ to $z = re^{i\theta}$ and*

$$\left|\arg\{f''(z)\}\right| < \frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{2.4}$$

*Then we have $f(z) \in \mathcal{C}(2) = \mathcal{C}_1(2)$ or $f(z)$ is 2-valently convex in $\mathbb{D}$.*

*Proof* From the hypothesis (2.3) and (2.4) and applying Lemma 2.2, we have

$$\left|\arg\left\{\frac{zf''(z)}{f'(z)}\right\}\right| \le \left|\arg\{f''(z)\}\right| < \frac{\pi}{2}, \quad z \in \mathbb{D}.$$

Therefore, we have

$$1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)} > 0, \quad z \in \mathbb{D}.$$

It completes the proof. □

Applying the same method as in the proof of Lemma 1.1 and Lemma 2.2, we can generalize Theorem 2.1 and Theorem 2.3 as follows.

**Lemma 2.4** *Let $f(z) \in \mathcal{A}(p)$. Suppose that for all $\theta$, $0 \le \theta < 2\pi$, $f(z)$ satisfies the following condition*:

$$\begin{aligned}\left(\mathfrak{Im}\frac{zf''(z)}{f'(z)}\right)\sin\theta &= \rho\left(\frac{\mathrm{d}(\arg\{f'(\rho e^{i\theta})\} - (p-1)\theta)}{\mathrm{d}\rho}\right)\sin\theta \\ &= \rho\left(\frac{\mathrm{d}\arg\{f'(z)/z^{p-1}\}}{\mathrm{d}\rho}\right)\sin\theta \\ &\ge 0,\end{aligned} \tag{2.5}$$

*where $z = \rho e^{i\theta}$, $0 \le \rho \le r < 1$ moves on the segment from $z = 0$ to $z = re^{i\theta}$. Then we have*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| \le \left|\arg\left\{\frac{f'(z)}{z^{p-1}}\right\}\right|, \quad z \in \mathbb{D}.$$

*Proof* For the case $0 \le \theta \le \pi$, from the hypothesis (2.5), we have

$$\begin{aligned}
\arg\left\{\frac{f(z)}{z^p}\right\} &= \arg\left\{\frac{1}{r^p e^{ip\theta}}\int_0^r f'(\rho e^{i\theta})e^{i\theta}\,\mathrm{d}\rho\right\} \\
&= \arg\left\{\int_0^r \left|f'(\rho e^{i\theta})\right| e^{i(\arg\{f'(\rho e^{i\theta})\}-(p-1)\theta)}\,\mathrm{d}\rho\right\} \\
&\le \arg\left\{f'(re^{i\theta})\right\} - (p-1)\theta \\
&= \arg\left\{\frac{f'(z)}{z^{p-1}}\right\}
\end{aligned}$$

and therefore we have

$$0 \le \arg\left\{\frac{f(z)}{z^p}\right\} \le \arg\left\{\frac{f'(z)}{z^{p-1}}\right\} \tag{2.6}$$

for $z = re^{i\theta}$ and $0 \le \theta \le \pi$.

For the case $\pi < \theta < 2\pi$, applying the same method as above and in the proof of Lemma 1.1 and Lemma 2.2, we have

$$0 \ge \arg\left\{\frac{f(z)}{z}\right\} \ge \arg\left\{\frac{f'(z)}{z^{p-1}}\right\} \tag{2.7}$$

for $z = re^{i\theta}$ and $\pi < \theta < 2\pi$. From (2.6) and (2.7), we have

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| = \left|\arg\left\{\frac{f'(z)}{z^p}\right\} - \arg\left\{\frac{f(z)}{z^p}\right\}\right| \le \left|\arg\left\{\frac{f'(z)}{z^{p-1}}\right\}\right|.$$

It completes the proof of Lemma 2.4. □

Thus, we have the following theorems.

**Theorem 2.5** *Let $f(z) \in \mathcal{A}(p)$. Assume that for all $\theta$, $0 \le \theta < 2\pi$, $f(z)$ satisfies the following condition*:

$$\begin{aligned}
\left(\frac{\mathrm{d}(\arg\{f'(\rho e^{i\theta})\} - (p-1)\theta)}{\mathrm{d}\rho}\right)\sin\theta &= \left(\frac{\mathrm{d}\arg\{f'(z)/z^{p-1}\}}{\mathrm{d}\rho}\right)\sin\theta \\
&\ge 0,
\end{aligned}$$

*where $z = \rho e^{i\theta}$, $0 \le \theta < 2\pi$, $0 \le \rho \le r < 1$, moves on the segment from $z = 0$ to $z = re^{i\theta}$ and suppose that*

$$\left|\arg\left\{\frac{f'(z)}{z^{p-1}}\right\}\right| \le \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},$$

*where $0 < \alpha \le 1$. Then we have $f(z) \in \mathcal{S}^*_\alpha(p)$ or $f(z)$ is $p$-valently and strongly starlike of order $\alpha$ in $\mathbb{D}$.*

**Theorem 2.6** *Let $f(z) \in \mathcal{A}(p)$, $p \geq 2$. Assume that for all $\theta$, $0 \leq \theta < 2\pi$, $f(z)$ satisfies the following condition*:

$$\begin{aligned}\left(\mathfrak{Im}\frac{zf'''z)}{f''(z)}\right)\sin\theta &= \rho\left(\frac{\mathrm{d}(\arg\{f''(\rho e^{i\theta})\} - (p-2)\theta)}{\mathrm{d}\rho}\right)\sin\theta \\ &= \rho\left(\frac{\mathrm{d}\arg\{f''(z)/z^{p-2}\}}{\mathrm{d}\rho}\right)\sin\theta \\ &\geq 0,\end{aligned}$$

*where $z = \rho e^{i\theta}$, $0 \leq \rho \leq r < 1$ moves on the segment from $z = 0$ to $z = re^{i\theta}$ and suppose that*

$$\left|\arg\left\{\frac{f'(z)}{z^{p-2}}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},$$

*where $0 < \alpha \leq 1$. Then we have $f(z) \in \mathcal{C}_\alpha(p)$ or $f(z)$ is $p$-valently and strongly convex of order $\alpha$.*

**Lemma 2.7** *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| \leq 1$ and suppose that it satisfies the following condition*:

$$\mathfrak{Re}\left\{\frac{zf''(z)}{f'(z)}\left(\overline{ze^{-i\alpha}} - ze^{-i\alpha}\right)\right\} \geq 0 \quad \text{in } |z| \leq 0, \tag{2.8}$$

*where $0 \leq \alpha \leq \pi$. Then for $\alpha \leq \theta \leq \alpha + \pi$ we have*

$$\begin{aligned}\rho\left(\frac{\mathrm{d}(\arg\{f'(\rho e^{i\theta})\})}{\mathrm{d}\rho}\right) &= \mathfrak{Im}\left\{\frac{zf'z)}{f(z)}\right\} \\ &\geq 0,\end{aligned} \tag{2.9}$$

*while for $\alpha + \pi \leq \theta \leq \alpha + 2\pi$ we have*

$$\begin{aligned}\rho\left(\frac{\mathrm{d}(\arg\{f'(\rho e^{i\theta})\})}{\mathrm{d}\rho}\right) &= \mathfrak{Im}\left\{\frac{zf'z)}{f(z)}\right\} \\ &\leq 0,\end{aligned} \tag{2.10}$$

*where $z = \rho e^{i\theta}$, $0 \leq \rho \leq |z| \leq 1$.*

*Proof* Let $z = \rho e^{i\theta}$, $0 \leq \rho \leq |z| \leq 1$. Then it follows that

$$\begin{aligned}&\mathfrak{Re}\left\{\frac{zf''(z)}{f'(z)}\left(\overline{ze^{-i\alpha}} - ze^{-i\alpha}\right)\right\} \\ &\quad = \mathfrak{Re}\left\{\frac{\mathrm{d}\log f'(z)}{\mathrm{d}z}\left(\rho e^{-i(\theta-\alpha)} - \rho e^{i(\theta-\alpha)}\right)\right\} \\ &\quad = \mathfrak{Re}\left\{\rho\left(\frac{\mathrm{d}\log|f'(\rho e^{i\theta})|}{\mathrm{d}\rho} + i\frac{\mathrm{d}\arg f'(\rho e^{i\theta})}{\mathrm{d}\rho}\right)(-2i)\right\}\sin(\theta-\alpha) \\ &\quad = 2\rho\frac{\mathrm{d}\arg f'(\rho e^{i\theta})}{\mathrm{d}\rho}\sin(\theta-\alpha) \\ &\quad \geq 0.\end{aligned}$$

This proves (2.9) and (2.10) and it shows that the function $\arg f'(\rho e^{i\theta})$ is an increasing function with respect to $\rho$, $0 \le \rho \le 1$, and $\alpha \le \theta \le \alpha + \pi$, and that the function $\arg f'(\rho e^{i\theta})$ is a decreasing function with respect to $\rho$, $0 \le \rho \le 1$, and $\alpha + \pi \le \theta \le \alpha + 2\pi$. □

**Theorem 2.8** *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ and suppose that it satisfies the following condition*:

$$\mathfrak{Re}\left\{\frac{zf''(z)}{f'(z)}\left(\overline{ze^{-i\alpha}} - ze^{-i\alpha}\right)\right\} \ge 0, \quad z \in \mathbb{D}, \tag{2.11}$$

*where* $0 \le \alpha \le \pi$ *and*

$$\left|\arg\left\{f'(z)\right\}\right| \le \frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{2.12}$$

*Then $f(z)$ is starlike in $\mathbb{D}$.*

*Proof* From Lemma 2.7 and (2.12), for the case $0 \le \theta \le \pi$, we have

$$\begin{aligned} 0 &= \left(\arg\frac{f(z)}{z}\right)_{z=0} \\ &\le \arg\left\{\frac{1}{\rho e^{i\theta}}\int_0^r f'\left(\rho e^{i\theta}\right)e^{i\theta}\,\mathrm{d}\rho\right\} \\ &= \arg\frac{f(z)}{z} \\ &= \arg\int_0^r f'\left(\rho e^{i\theta}\right)\mathrm{d}\rho \\ &= \arg\int_0^r \left|f'\left(\rho e^{i\theta}\right)\right| e^{i\arg\{f'(\rho e^{i\theta})\}}\,\mathrm{d}\rho \\ &\le \arg\left\{f'(z)\right\}. \end{aligned}$$

This shows that

$$0 \le \arg\left\{\frac{f(z)}{z}\right\} \le \arg\left\{f'(z)\right\}, \tag{2.13}$$

where $z = re^{i\theta}$, $0 \le r < 1$, and $\alpha \le \theta \le \alpha + \pi$.

For the case $\pi \le \theta \le 2\pi$, applying the same method as above, we have

$$0 \ge \arg\left\{\frac{f(z)}{z}\right\} \ge \arg\left\{f'(z)\right\}, \tag{2.14}$$

where $z = re^{i\theta}$, $0 \le r < 1$, and $\pi + \alpha \le \theta \le 2\pi + \alpha$. Applying (2.12), (2.13), and (2.14), we have

$$\begin{aligned} \left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| &= \left|\arg\left\{f'(z)\right\} - \arg\left\{\frac{f(z)}{z}\right\}\right| \le \left|\arg\left\{f'(z)\right\}\right| \\ &< \frac{\pi}{2}, \quad z \in \mathbb{D}. \end{aligned}$$

This completes the proof. □

**Remark 2.9** The functions $f(z) = z + \alpha z^2/2$ satisfy the conditions of Theorem 2.8 whenever $|\alpha| \leq 1/2$.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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**References**
1. Stankiewicz, J: Quelques problèmes extrêmaux dans les classes des fonctions $\alpha$-angulairement étoilées. Ann. Univ. Mariae Curie-Skłodowska, Sect. A **20**, 59-75 (1966)
2. Brannan, DA, Kirwan, WE: On some classes of bounded univalent functions. J. Lond. Math. Soc. **1**(2), 431-443 (1969)
3. Duren, PL: Univalent Functions. Springer, New York (1983)
4. Goodman, AW: Univalent Functions. Vols. I and II. Mariner Publishing Co., Tampa (1983)

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