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Inner functions as improving multipliers and zero sets of Besov-type spaces

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Abstract

Assume X and Y are two spaces of analytic functions in the unit disk $\mathbb{D}$ with $X \subseteq Y$. Let θ be an inner function. If every function $f \in X$ satisfying $f\theta \in Y$ must actually satisfy $f\theta \in X$, then θ is said to be (X, Y)-improving. In this paper, we characterize the inner functions in the Möbius invariant Besov-type spaces $F(p, p-2, s)$ as improving multipliers for $p > 1$ and $0 < s < 1$. Our result generalizes Peláez's result on Q_s spaces ($0 < s < 1$) (Peláez in J. Funct. Anal. 255:1403-1418, 2008).

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1 Introduction

We denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ by $\mathbb{D}$ and its boundary $\{z \in \mathbb{C} : |z| = 1\}$ by $\partial\mathbb{D}$. Let $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$. An analytic function in the unit disc $\mathbb{D}$ is called an inner function if it is bounded and the modulus equals 1 almost everywhere on the boundary $\partial\mathbb{D}$.

It is well known that every inner function has a factorization $e^{i\gamma}B(z)S(z)$, where $\gamma \in \mathbb{R}$, $B(z)$ is a Blaschke product and $S(z)$ is a singular inner function, that is,

$$B(z) = \prod_{k=1}^{\infty} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \bar{a}_k z}$$

and

$$S(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t)\right\},$$

where $\{a_k\}_{k=1}^{\infty}$ is a sequence of points in $\mathbb{D}$ which satisfies the Blaschke condition

$$\sum_{k=1}^{\infty} \bigl(1 - |a_k|\bigr) < \infty,$$

and μ is a finite positive Borel measure in $[0, 2\pi)$, which is singular with respect to Lebesgue measure. Let $\sigma(\theta)$ denote the singular set or boundary spectrum of inner function θ. From [1] we know that $\sigma(\theta) \subseteq \partial\mathbb{D}$ is the smallest closed set such that θ is analytic across $\partial\mathbb{D}\backslash\sigma(\theta)$, and $\sigma(\theta)$ consists of the accumulation points of zeros of θ and the closed support of the associated singular measure. See [2–6] for more information on inner functions.

Since a nontrivial inner function θ is extremely oscillatory near $\sigma(\theta)$, the same should happen to the product $f\theta$, where $f \in H(\mathbb{D})$ is smooth in some sense on $\partial\mathbb{D}$. But sometimes the product $f\theta$ inherits the nice properties of f, and it is possible that $f\theta$ has an added smoothness. In order to analyze this phenomenon, Dyakonov introduced the following notion in [7].

Suppose X and Y are two classes of analytic functions on $\mathbb{D}$, and $X \subseteq Y$. Let θ be an inner function, θ is said to be (X,Y)-improving, if every function $f \in X$ satisfying $f\theta \in Y$ must actually satisfy $f\theta \in X$.

In this paper, we study the inner functions in the Möbius invariant Besov-type spaces $F(p,p-2,s)$ as improving multipliers.

Let $0 < p < \infty$, $-2 < q < \infty$, $s \geq 0$, the $F(p,q,s)$ space is the set of $f \in H(\mathbb{D})$ [8] such that

$$\|f\|_{F(p,q,s)} = \left|f(0)\right| + \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^q g(z,a)^s \, dA(z)\right)^{\frac{1}{p}} < \infty,$$

where g denotes the Green function given by

$$g(z,a) = \log \frac{1}{|\varphi_a(z)|}, \quad z,a \in \mathbb{D}, z \neq a,$$

$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, $dA(z) = \frac{1}{\pi}\,dx\,dy$. It is easy to check that $F(p,p-2,s)$ is a Möbius invariant Besov-type space. From [8], when $0 < s < 1$, $F(2,0,s) = \mathcal{Q}_s$, for more information on $\mathcal{Q}_s$ spaces, we refer to [9] and [10]. When $s = 1$, $F(2,0,s) = BMOA$, the space of analytic functions in the Hardy space $H^1(\mathbb{D})$ whose boundary functions have bounded mean oscillation. When $s > 1$, $F(2,0,s) = \mathcal{B}$, the Bloch space (see [11]).

The following result was proved by Peláez in [12, Theorem 1].

Theorem A *Suppose that* $0 < s < 1$ *and* θ *is an inner function. Then the following conditions are equivalent*:

(1) $\theta \in \mathcal{Q}_s$;
(2) θ *is* $(\mathcal{Q}_s, BMOA)$*-improving*;
(3) θ *is* $(\mathcal{Q}_s, \mathcal{B})$*-improving.*

In this paper, we extend Theorem A from $\mathcal{Q}_s$ spaces to a more general space $F(p,p-2,s)$, $0 < s < 1$.

Theorem 1 *Let* $1 < p < \infty$, $0 < s < 1$. *Suppose that* θ *is an inner function. Then the following conditions are equivalent*:

(1) $\theta \in F(p,p-2,s)$;
(2) θ *is* $(F(p,p-2,s), BMOA)$*-improving*;
(3) θ *is* $(F(p,p-2,s), \mathcal{B})$*-improving.*

The proof of Theorem 1 is gave in Section 3. Theorem 1 can be farther generalized to a more general space $AB_p^t \cap F(p,p-2,s)$, where AB_p^t space is the set of functions $f \in H(\mathbb{D})$

such that (see [13])

$$\|f\|_{AB_p^t} = |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{(1-t)p-1}\, dA(z)\right)^{\frac{1}{p}} < \infty.$$

Corollary 1 *Let* $1 < p < \infty$, $0 < tp < 1$ *and* $0 < s < 1$. *Suppose* θ *is an inner function. Then the following conditions are equivalent*:

(1) $\theta \in AB_p^t \cap F(p,p-2,s)$;

(2) θ *is* $(AB_p^t \cap F(p,p-2,s), BMOA)$*-improving*;

(3) θ *is* $(AB_p^t \cap F(p,p-2,s), \mathcal{B})$*-improving*.

Notice the fact that $F(p,p-2,s) \subset AB_p^{\frac{1-s}{p}}$. The proof of Corollary 1 is similar to that of Theorem 1 and thus is omitted.

Remark 1 From Theorem 1, we know that any inner function $\theta \in F(p,p-2,s)$ is $(F(p,p-2,s), F(p,p-2,t))$-improving, when $p > 1$ and $0 < s < t < 1$. Conversely, by using the following results (Theorem 2 and Proposition 1) on zero sets of Besov-type spaces, we will prove that there exists an inner function θ which is $(F(p,p-2,s), F(p,p-2,t))$-improving, but θ does not belong to $F(p,p-2,s)$.

In order to state Theorem 2, we need a few notions.

A Blaschke product B with sequence of zeros $\{a_k\}_{k=1}^{\infty}$ is called interpolating if there exists a positive constant δ such that

$$\prod_{j\neq k} \varrho(a_j, a_k) \geq \delta, \quad k = 1, 2, \ldots.$$

Here $\varrho(a_j, a_k) = |\varphi_{a_j}(a_k)|$ denotes the pseudo-hyperbolic metric in $\mathbb{D}$. We also say that $\{a_k\}_{k=1}^{\infty}$ is an interpolating sequence or an uniformly separated sequence. A finite union of interpolating sequences is usually called a Carleson-Newman sequence. Similarly, a Carleson-Newman Blaschke product is a finite product of interpolating Blaschke products.

We recall that a function $g \in H(\mathbb{D})$ is called an outer function if $\log|g| \in L^1(\partial\mathbb{D})$ and

$$g(z) = \eta \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|g(e^{it})| \frac{e^{it}+z}{e^{it}-z}\, dt\right),$$

where $\eta \in \partial\mathbb{D}$.

We say that $Z = \{z_n\} \subset \mathbb{D}$ is a zero set of an analytic function space X defined on $\mathbb{D}$ if there is a $f \in X$ that vanishes on Z and nowhere else. Although the study of zero sets for analytic function spaces is a difficult problem, there are some excellent papers related to this question. We may refer to Carleson [14, 15], Caughran [16], Shapiro and Shields [17], Taylor and Williams [18]. Recently, Pau and Peláez obtained a theorem on zero sets of Dirichlet spaces $\mathcal{D}_s = AB_2^{\frac{1-s}{2}}$ in [19, Theorem 1]. In the next theorem, we characterize the zero sets of AB_p^t spaces which generalizes the Pau and Peláez's result in [19]. We characterize the Carleson-Newman sequences that are zero sets in AB_p^t spaces.

Theorem 2 *Suppose $p > 1$, $0 < t < 1$, $0 < pt < 1$ and $\{a_k\}_{k=1}^{\infty}$ is a Carleson-Newman sequence. Then the following conditions are equivalent*:

(1) *$\{a_k\}_{k=1}^{\infty}$ is an AB_p^t-zero set*;

(2) *there exists an outer function $g \in AB_p^t$ such that*

$$\sum_{k=1}^{\infty} \left|g(a_k)\right|^p \left(1 - |a_k|^2\right)^{1-tp} < \infty;$$

(3) *there exists an outer function $g \in AB_p^t$ such that*

$$\sum_{k=1}^{\infty} \left(1 - |a_k|^2\right)^{2-tp} \int_{\partial\mathbb{D}} \frac{|g(e^{it})|^p}{|e^{it} - a_k|^2}\, dt < \infty;$$

(4) *there exists an outer function $g \in AB_p^t$ such that*

$$\sum_{k=1}^{\infty} \left(1 - |a_k|^2\right)^{1-tp+p} \left(\int_{\partial\mathbb{D}} \frac{|g(e^{it})|}{|e^{it} - a_k|^2}\, dt\right)^p < \infty.$$

Using Theorem 2, we can deduce the following result.

Proposition 1 *Suppose $p > 1$, $0 < t < 1$, $0 < pt < 1$. There exists a Carleson-Newman sequence $\{a_k\}_{k=1}^{\infty}$ which is not an AB_p^t-zero set and with 1 as unique accumulation point.*

The proof of Proposition 1 is similar to that of Theorem 2 of [19] and is omitted here.

Applying Proposition 1, we can prove the following result whose proof (as well as the proof of Theorem 2) is given in Section 4.

Corollary 2 *Let $p > 1$ and $0 < s < t < 1$. Then there exists an inner function θ which is $(F(p,p-2,s), F(p,p-2,t))$-improving, but θ does not belong to $F(p,p-2,s)$.*

Throughout this paper, for two functions f and g, $f \asymp g$ means that $g \lesssim f \lesssim g$, that is, there are positive constants C_1 and C_2 depending only on the index $p, s, t, \ldots$, such that $C_1 g \le f \le C_2 g$.

2 Preliminaries

To prove Theorem 1 we need some auxiliary results. Lemmas 2.1 and 2.2 should be known to some experts, but we cannot find a reference. For the completeness of the paper, we give proofs below.

Lemma 2.1 *Let $0 < p < \infty$ and $s \ge 0$. Then $f \in F(p,p-2,s)$ if and only if*

$$\sup_{a\in\mathbb{D}} \left\|f \circ \varphi_a - f(a)\right\|_{AB_p^{\frac{1-s}{p}}} < \infty.$$

Proof From Theorem 2.4 of [8] and making the change of variables $w = \varphi_a(z)$, we have

$$\begin{aligned}\|f - f(0)\|^p_{F(p,p-2,s)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-2} g^s(w,a)\, dA(w)\\ &\asymp \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-2} (1-|\varphi_a(w)|^2)^s\, dA(w)\\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f\circ\varphi_a)'(z)|^p (1-|z|^2)^{s+p-2}\, dA(z)\\ &\asymp \sup_{a\in\mathbb{D}} \|f\circ\varphi_a - f(a)\|^p_{AB_p^{\frac{1-s}{p}}}.\end{aligned}$$

□

For $0 < p < \infty$, H^p denotes the Hardy space of $f \in H(\mathbb{D})$ with

$$\|f\|^p_{H^p} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta < \infty.$$

From [20], we know that

$$\|f\|_{BMOA} = |f(0)| + \sup_{a\in\mathbb{D}} \|f\circ\varphi_a - f(a)\|_{H^p} < \infty$$

can be defined as a norm of $BMOA$ space.

Lemma 2.2 *Let* $1 < p < \infty$, $0 < s < 1$. *Then* $F(p,p-2,s) \subseteq BMOA$.

Proof From Lemma 2.4 of [21], we know $AB_p^{\frac{1-s}{p}} \subset H^p$ and

$$\|f\|_{H^p} \le \|f\|_{AB_p^{\frac{1-s}{p}}}.$$

Let $f \in F(p,p-2,s)$. By Lemma 2.1, we get

$$\begin{aligned}\|f - f(0)\|_{BMOA} &= \sup_{a\in\mathbb{D}} \|f\circ\varphi_a - f(a)\|_{H^p}\\ &\le \sup_{a\in\mathbb{D}} \|f\circ\varphi_a - f(a)\|_{AB_p^{\frac{1-s}{p}}}\\ &\asymp \|f - f(0)\|_{F(p,p-2,s)} < \infty.\end{aligned}$$

That is, $f \in BMOA$. □

The following three lemmas will be used in the proof of Lemma 2.6. Their proofs can be found in [22, Corollary 2.4], [23, Lemma 2.5] or [24, Lemma 1] and [25, Lemma 2.1], respectively.

Lemma 2.3 *Let* θ *be an inner function and let* $1 \le p < \infty$, $-2 < q < \infty$ *and* $0 < s < \infty$ *such that* $0 < q + s + 1 < p$. *Then, for any* $f \in H(\mathbb{D})$ *and* $a \in \mathbb{D}$,

$$\begin{aligned}&\int_{\mathbb{D}} |f(z)|^p (1-|\theta(z)|^2)^p (1-|z|^2)^{q-p} (1-|\varphi_a(z)|^2)^s\, dA(z)\\ &\quad\asymp \int_{\mathbb{D}} |f(z)|^p |\theta'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s\, dA(z).\end{aligned}$$

Lemma 2.4 *Let $s > -1$, $r, t > 0$, and $t < s + 2 < r$. Then*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^s}{|1-\overline{w}z|^r |1-\overline{w}\zeta|^t}\, dA(w) \lesssim \frac{(1-|z|^2)^{2+s-r}}{|1-\overline{\zeta}z|^t}, \quad z, \zeta \in \mathbb{D}.$$

Lemma 2.5 *Let $1 < p < \infty$, and $a > -1$, $b \geq 0$ with $b < 2 + a$, $w \in \mathbb{D}$. If $f \in H(\mathbb{D})$, then*

$$\int_{\mathbb{D}} \left|f(z) - f(0)\right|^p \frac{(1-|z|^2)^a}{|1-\overline{w}z|^b}\, dA(z) \lesssim \int_{\mathbb{D}} \left(1-|z|^2\right)^p \left|f'(z)\right|^p \frac{(1-|z|^2)^a}{|1-\overline{w}z|^b}\, dA(z).$$

Lemma 2.6 plays an important role in the proof of Theorem 1, which generalizes Theorem 6 of [12], with a different proof motivated by [21] and [19].

Lemma 2.6 *Let $1 < p < \infty$, $0 < s < 1$, $f \in F(p, p-2, s)$ and B be a Carleson-Newman Blaschke product with a sequence of zeros $\{a_k\}_{k=1}^{\infty}$. Then $fB \in F(p, p-2, s)$ if and only if*

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} \left|f(a_k)\right|^p \left(1 - |\varphi_a(a_k)|^2\right)^s < \infty.$$

Proof Necessity. If $fB \in F(p, p-2, s)$, then it is easy to deduce that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f(z)\right|^p \left|B'(z)\right|^p \left(1-|z|^2\right)^{p-2} \left(1 - \left|\varphi_a(z)\right|^2\right)^s dA(z) < \infty.$$

We will prove that

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} \left|f(a_k)\right|^p \left(1 - \left|\varphi_a(a_k)\right|^2\right)^s \\
&\quad \lesssim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f(z)\right|^p \left|B'(z)\right|^p \left(1-|z|^2\right)^{p-2} \left(1 - \left|\varphi_a(z)\right|^2\right)^s dA(z).
\end{aligned}$$

From the sub-mean-value property of $|f|^p$ and the estimate

$$1 - |a_k|^2 \asymp 1 - |z|^2 \asymp |1 - \overline{a}_k z|, \quad z \in E(a_k, r),$$

for $r > 0$ (see [26, p.69]). Here and afterwards

$$E(a_k, r) = \left\{z : \left|\varphi_{a_k}(z)\right| < r\right\}.$$

We have

$$\begin{aligned}
&\sum_{k=1}^{\infty} \left|f(a_k)\right|^p \left(1 - \left|\varphi_a(a_k)\right|^2\right)^s \\
&\quad \lesssim \sum_{k=1}^{\infty} \left(1 - \left|\varphi_a(a_k)\right|^2\right)^s \int_{E(a_k,r)} \frac{|f(z)|^p}{|1-\overline{a}_k z|^2}\, dA(z) \\
&\quad \asymp \sum_{k=1}^{\infty} \int_{E(a_k,r)} \left|f(z)\right|^p \frac{(1-|a_k|^2)^p (1-|z|^2)^p}{|1-\overline{a}_k z|^{2p+2}} \left(1 - \left|\varphi_a(z)\right|^2\right)^s dA(z)
\end{aligned}$$

$$\begin{aligned}
&\asymp \sum_{k=1}^{\infty}\left(1-|a_k|^2\right)^p \int_{E(a_k,r)} \left|f(z)\right|^p \frac{(1-|z|^2)^{p-2}}{|1-\overline{a}_k z|^{2p}}\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z)\\
&\lesssim \sum_{k=1}^{\infty}\left(1-|a_k|^2\right)^p \int_{\mathbb{D}} \left|f(z)\right|^p \frac{(1-|z|^2)^{p-2}}{|1-\overline{a}_k z|^{2p}}\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z)\\
&= \int_{\mathbb{D}} \left|f(z)\right|^p \sum_{k=1}^{\infty}\left(1-\left|\varphi_{a_k}(z)\right|^2\right)^p \left(1-|z|^2\right)^{-2}\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z)\\
&\lesssim \int_{\mathbb{D}} \left|f(z)\right|^p \left(\sum_{k=1}^{\infty}\left(1-\left|\varphi_{a_k}(z)\right|^2\right)\right)^p \left(1-|z|^2\right)^{-2}\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z). \qquad (1)
\end{aligned}$$

Since B is a Carleson-Newman Blaschke product with a sequence of zeros $\{a_k\}_{k=1}^{\infty}$, we have

$$\sup_{z\in\mathbb{D}} \sum_{k=1}^{\infty}\left(1-\left|\varphi_z(a_k)\right|^2\right) \lesssim 1.$$

Note that

$$1-r^2 \le -2\log r, \quad 0<r\le 1,$$

we have

$$2\log\left|B(z)\right| = \sum_{n=1}^{\infty}\log\left|\varphi_{a_k}(z)\right|^2 \le -\sum_{k=1}^{\infty}\left(1-\left|\varphi_{a_k}(z)\right|^2\right).$$

Since the function $\frac{(1-e^{-x})}{x}$ is decreasing in $(0,\infty)$, we get

$$\begin{aligned}
1-\left|B(z)\right|^2 &\ge \sum_{k=1}^{\infty}\left(1-\left|\varphi_{a_k}(z)\right|^2\right)\frac{1-\exp(-\sup_{z\in\mathbb{D}}\sum_{k=1}^{\infty}(1-|\varphi_{a_k}(z)|^2))}{\sup_{z\in\mathbb{D}}\sum_{k=1}^{\infty}(1-|\varphi_{a_k}(z)|^2)}\\
&\asymp \sum_{k=1}^{\infty}\left(1-\left|\varphi_{a_k}(z)\right|^2\right).
\end{aligned}$$

Combining this with (1) and using Lemma 2.3 yield

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left|f(a_k)\right|^p\left(1-\left|\varphi_a(a_k)\right|^2\right)^s\\
&\quad\lesssim \int_{\mathbb{D}}\left|f(z)\right|^p\left|B'(z)\right|^p\left(1-|z|^2\right)^{p-2}\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z)<\infty.
\end{aligned}$$

Sufficiency. Applying the p-triangle inequality gives

$$\begin{aligned}
&\int_{\mathbb{D}}\left|\left(f(z)B(z)\right)'\right|^p\left(1-|z|^2\right)^{p-2}\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z)\\
&\quad\lesssim \int_{\mathbb{D}}\left|f'(z)\right|^p\left|B(z)\right|^p\left(1-|z|^2\right)^{p-2}\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z)\\
&\qquad+\int_{\mathbb{D}}\left|f(z)\right|^p\left|B'(z)\right|^p\left(1-|z|^2\right)^{p-2}\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z)\\
&\quad=: I_1+I_2.
\end{aligned}$$

Since $f \in F(p,p-2,s)$, $B \in H^\infty$, by Theorem 2.4 of [8], we have

$$\begin{aligned} I_1 &\lesssim \|B\|_{H^\infty}^p \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^{p-2} \left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) \\ &\lesssim \|f\|_{F(p,p-2,s)}^p. \end{aligned}$$

We now estimate I_2. Using the fact that $(1-|z|^2)|B'(z)| \le 1$ and

$$\left|B'(z)\right| \le \sum_{k=1}^\infty \frac{1-|a_k|^2}{|1-\overline{a}_k z|^2},$$

and employing the p-triangle inequality again, we have

$$\begin{aligned} I_2 &= \int_{\mathbb{D}} \left|f(z)\right|^p \left|B'(z)\right|^p \left(1-|z|^2\right)^{p-2} \left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) \\ &\lesssim \int_{\mathbb{D}} \left|f(z)\right|^p \left|B'(z)\right| \left(1-|z|^2\right)^{-1} \left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) \\ &\lesssim \int_{\mathbb{D}} \left|f(z)\right|^p \left(\sum_{k=1}^\infty \frac{1-|a_k|^2}{|1-\overline{a}_k z|^2}\right) \left(1-|z|^2\right)^{-1} \left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) \\ &= \sum_{k=1}^\infty \left(1-|a_k|^2\right) \int_{\mathbb{D}} \left|f(z)\right|^p \frac{(1-|z|^2)^{-1}}{|1-\overline{a}_k z|^2} \left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) \\ &\lesssim \sum_{k=1}^\infty \left(1-|a_k|^2\right) \int_{\mathbb{D}} \left|f(a_k)\right|^p \frac{(1-|z|^2)^{-1}}{|1-\overline{a}_k z|^2} \left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) \\ &\quad + \sum_{k=1}^\infty \left(1-|a_k|^2\right) \int_{\mathbb{D}} \left|f(z)-f(a_k)\right|^p \frac{(1-|z|^2)^{-1}}{|1-\overline{a}_k z|^2} \left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) \\ &=: I_3 + I_4. \end{aligned}$$

Applying Lemma 2.4 yields

$$\begin{aligned} I_3 &= \sum_{k=1}^\infty \left(1-|a_k|^2\right) \left|f(a_k)\right|^p \int_{\mathbb{D}} \frac{(1-|z|^2)^{-1}}{|1-\overline{a}_k z|^2} \left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) \\ &\lesssim \sup_{a\in\mathbb{D}} \sum_{k=1}^\infty \left|f(a_k)\right|^p \left(1-\left|\varphi_a(a_k)\right|^2\right)^s < \infty. \end{aligned}$$

Let $\varphi_\lambda(w) = e^{i\tau}\varphi_a(\varphi_{a_k}(w))$, $\tau \in [0,2\pi]$. Making the change of variables $z = \varphi_{a_k}(w)$ and using Lemma 2.5, we get

$$\begin{aligned} I_4 &\lesssim \sum_{k=1}^\infty \int_{\mathbb{D}} \left|(f\circ\varphi_{a_k})(w) - (f\circ\varphi_{a_k})(0)\right|^p \frac{(1-|\varphi_\lambda(w)|^2)^s}{(1-|w|^2)} dA(w) \\ &= \sum_{k=1}^\infty \left(1-|\lambda|^2\right)^s \int_{\mathbb{D}} \left|(f\circ\varphi_{a_k})(w) - (f\circ\varphi_{a_k})(0)\right|^p \frac{(1-|w|^2)^{s-1}}{|1-\overline{\lambda}w|^{2s}} dA(w) \\ &\lesssim \sum_{k=1}^\infty \int_{\mathbb{D}} \left|(f\circ\varphi_{a_k})'(w)\right|^p \left(1-|w|^2\right)^{p-1} \left(1-\left|\varphi_\lambda(w)\right|^2\right)^s dA(w) \end{aligned}$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1 - |z|^2\right)^{p-2} \left(1 - \left|\varphi_{a_k}(z)\right|^2\right) \left(1 - \left|\varphi_a(z)\right|^2\right)^s dA(z)$$

$$\lesssim \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1 - |z|^2\right)^{p-2} \left(1 - \left|\varphi_a(z)\right|^2\right)^s dA(z) \lesssim \|f\|^p_{F(p,p-2,s)}.$$

Hence, we have $f\theta \in F(p, p-2, s)$. □

The following well-known results will also be used in the proof of Theorem 1.

Lemma 2.7 ([22, Theorem 1.4]) *Let* $0 < s < 1$. *Then an inner function belongs to the Möbius invariant Besov-type space* $F(p, p-2, s)$ *for all* $p > \max\{s, 1-s\}$ *if and only if it is the Blaschke product associated with a sequence* $\{a_k\}_{k=1}^{\infty}$ *which satisfies*

$$\sup_{a \in \mathbb{D}} \sum_{k=1}^{\infty} \left(1 - \left|\varphi_a(a_k)\right|^2\right)^s < \infty.$$

Lemma 2.8 ([27, Lemma 21]) *Let* $\{a_k\}_{k=1}^{\infty}$ *be a sequence in* $\mathbb{D}$. *Then the measure* $d\mu_{a_k} = \sum_{k=1}^{\infty}(1 - |a_k|^2)\delta_{a_k}$ *is a Carleson measure, i.e.*

$$\sup_{a \in \mathbb{D}} \sum_{k=1}^{\infty} \left(1 - \left|\varphi_a(a_k)\right|^2\right) < \infty,$$

if and only if $\{a_k\}_{k=1}^{\infty}$ *is a finite union of uniformly separated sequences.*

If θ is an inner function, for $0 < \epsilon < 1$, define the level set of order ϵ of θ as

$$\Omega(\theta, \epsilon) = \left\{z \in \mathbb{D} : \left|\theta(z)\right| < \epsilon\right\}.$$

Lemma 2.9 ([28, Theorem 1]) *If* $f \in BMOA$ *and* θ *is an inner function, then the following conditions are equivalent*:

(1) $f\theta \in BMOA$;
(2) $\sup_{z \in \mathbb{D}} |f(z)|^2(1 - |\theta(z)|^2) < \infty$;
(3) $\sup_{z \in \Omega(\theta,\epsilon)} |f(z)| < \infty$, *for every* ϵ, $0 < \epsilon < 1$;
(4) $\sup_{z \in \Omega(\theta,\epsilon)} |f(z)| < \infty$, *for some* ϵ, $0 < \epsilon < 1$.

3 Proof of Theorem 1

(1) $\Rightarrow$ (2). For inner functions $\theta \in F(p, p-2, s)$, by Lemma 2.7, θ is a Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$, and

$$\sup_{a \in \mathbb{D}} \sum_{k=1}^{\infty} \left(1 - \left|\varphi_a(a_k)\right|^2\right)^s < \infty,$$

which implies that

$$\sup_{a \in \mathbb{D}} \sum_{k=1}^{\infty} \left(1 - \left|\varphi_a(a_k)\right|^2\right) < \infty.$$

From Lemma 2.8, θ is a Carleson-Nemwman Blaschke product. Suppose that $f \in F(p,p-2,s)$ and $f\theta \in BMOA$. Lemma 2.9 gives

$$\sup_{z\in\Omega(\theta,\epsilon)} |f(z)| < \infty, \quad 0 < \epsilon < 1.$$

Thus,

$$\begin{aligned}&\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} |f(a_k)|^p \left(1-|\varphi_a(a_k)|^2\right)^s \\ &\quad \le \Bigl(\sup_{z\in\Omega(\theta,\epsilon)} |f(z)|\Bigr)^p \sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} \left(1-|\varphi_a(a_k)|^2\right)^s < \infty.\end{aligned}$$

Applying Lemma 2.6 implies that $f\theta \in F(p,p-2,s)$. Hence, θ is $(F(p,p-2,s),BMOA)$-improving.

(2) $\Rightarrow$ (1), (3) $\Rightarrow$ (2). Their proofs are obvious.

(2) $\Rightarrow$ (3). Let θ be $(F(p,p-2,s),BMOA)$-improving. If the inner function $\theta \in F(p,p-2,s)$, then θ is a Carleson-Newman Blaschke product. For $f \in F(p,p-2,s)$, if $f\theta \in \mathcal{B}$, then any Carleson-Newman Blaschke product is $(BMOA,\mathcal{B})$-improving by Corollary 1 of [12]. Therefore, $f\theta \in BMOA$. Notice that θ is $(F(p,p-2,s),BMOA)$-improving, we have $f\theta \in F(p,p-2,s)$. Thus, θ is $(F(p,p-2,s),\mathcal{B})$-improving. The proof of Theorem 1 is completed.

4 Proofs of Theorem 2 and Corollary 2

In this section, we borrow the idea in [19] to study the zero set of AB_p^t spaces. Following the proof of Lemma 2.6, we have the next result.

Lemma 4.1 *Let $1 < p < \infty$, $0 < t < 1$, $0 < tp < 1$. Suppose $f \in AB_p^t$, B is a Carleson-Newman Blaschke product with sequence of zeros $\{a_k\}_{k=1}^{\infty}$. Then $fB \in AB_p^t$ if and only if*

$$\sum_{k=1}^{\infty} |f(a_k)|^p \left(1-|a_k|^2\right)^{1-tp} < \infty.$$

When $p = q$, $\sigma = 1$ or $\sigma = \frac{1}{p}$ in Theorem 1 of [13], we get the following lemma.

Lemma 4.2 *Let $1 \le p < \infty$, $0 < t < \frac{1}{p}$, $f \in H^p$. Then the following conditions are equivalent.*

(1) $f \in AB_p^t$;

(2) $\int_{\mathbb{D}}(\int_{\partial\mathbb{D}} |f(e^{i\eta}) - f(z)| \frac{1-|z|^2}{|e^{i\eta}-z|} \, d\eta)^p (1-|z|^2)^{-tp-1} \, dA(z) < \infty$;

(3) $\int_{\mathbb{D}}\int_{\partial\mathbb{D}} |f(e^{i\eta}) - f(z)|^p \frac{1-|z|^2}{|e^{i\eta}-z|} \, d\eta (1-|z|^2)^{-tp-1} \, dA(z) < \infty$.

Proof of Theorem 2 (1) $\Rightarrow$ (2). Since $\{a_k\}_{k=1}^{\infty}$ is an AB_p^t-zero set, there exists $f \in AB_p^t$ such that $f(a_k) = 0$ nowhere else. By $AB_p^t \subset H^p$, there exists a Blaschke product B, a singular inner function S and an outer function g, such that $f = BSg$. Since $S \neq 0$, $g \neq 0$, B is a Carleson-Newman Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$, from Corollary 3.1 of [29], we know that AB_p^s has the f-property (see [30] for the definition of the f-property). Then

$$gB = \frac{f}{S} \in AB_p^t.$$

Applying Lemma 4.1, we have (2).

(2) ⇒ (1). It is obvious from Lemma 4.1.

(3) ⇒ (2). By the Poisson integral formula

$$g(a_k) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} g\big(e^{i\eta}\big) \frac{1 - |a_k|^2}{|e^{i\eta} - a_k|^2}\, d\eta.$$

Applying Hölder's inequality, we have

$$\big|g(a_k)\big|^p \lesssim \int_{\partial\mathbb{D}} \big|g\big(e^{i\eta}\big)\big|^p \frac{1 - |a_k|^2}{|e^{i\eta} - a_k|^2}\, d\eta.$$

The desired result follows.

(2) ⇒ (3). By the p-triangle inequality, we obtain

$$\begin{aligned} &\sum_{k=1}^{\infty} \big(1 - |a_k|^2\big)^{2-tp} \int_{\partial\mathbb{D}} \frac{|g(e^{i\eta})|^p}{|e^{i\eta} - a_k|^2}\, d\eta \\ &\quad \lesssim \sum_{k=1}^{\infty} \big(1 - |a_k|^2\big)^{2-tp} \int_{\partial\mathbb{D}} \frac{|g(a_k)|^p + |g(e^{i\eta}) - g(a_k)|^p}{|e^{i\eta} - a_k|^2}\, d\eta. \end{aligned}$$

Let

$$I_1 =: \sum_{k=1}^{\infty} \big(1 - |a_k|^2\big)^{2-tp} \int_{\partial\mathbb{D}} \frac{|g(a_k)|^p}{|e^{i\eta} - a_k|^2}\, d\eta$$

and

$$I_2 =: \sum_{k=1}^{\infty} \big(1 - |a_k|^2\big)^{2-tp} \int_{\partial\mathbb{D}} \frac{|g(e^{i\eta}) - g(a_k)|^p}{|e^{i\eta} - a_k|^2}\, d\eta.$$

It is obvious that

$$\begin{aligned} I_1 &= \sum_{k=1}^{\infty} \big(1 - |a_k|^2\big)^{1-tp} \big|g(a_k)\big|^p \int_{\partial\mathbb{D}} \frac{(1 - |a_k|^2)}{|e^{i\eta} - a_k|^2}\, d\eta \\ &\asymp \sum_{k=1}^{\infty} \big(1 - |a_k|^2\big)^{1-tp} \big|g(a_k)\big|^p < \infty. \end{aligned}$$

We next estimate I_2. Since $\{a_k\}_{k=1}^{\infty}$ is a finite union of interpolating sequences, we can write

$$\{a_k\}_{k=1}^{\infty} =: \bigcup_{1\le i\le n} \bigcup_{1\le j\le\infty} \{a_{ij}\}$$

and there exist a positive integer n and $\delta_i > 0$, $i = 1, 2, \dots, n$, such that

$$\prod_{j \neq l} \varrho(a_{ij}, a_{il}) \geq \delta_i.$$

Then for fixed i, the pseudo-hyperbolic disks $\{E(a_{ij}, \frac{\delta_i}{4})\}_{j=1}^{\infty}$ are pairwise disjoint. Notice that

$$G(z) =: \int_0^{2\pi} \left|g\left(e^{i\eta}\right) - g(z)\right|^p \frac{1 - |z|^2}{|e^{i\eta} - z|^2}\, d\eta$$

has the generalized sub-mean-value property. Therefore,

$$\begin{aligned} I_2 &= \sum_{i=1}^{n} \sum_{j=1}^{\infty} \left(1 - |a_{ij}|^2\right)^{1-tp} \int_0^{2\pi} \left|g\left(e^{i\eta}\right) - g(a_{ij})\right|^p \frac{1 - |a_{ij}|^2}{|e^{i\eta} - a_{ij}|^2}\, d\eta \\ &\lesssim \sum_{i=1}^{n} \sum_{j=1}^{\infty} \left(1 - |a_{ij}|^2\right)^{-1-tp} \int_{E(a_{ij}, \frac{\delta_i}{4})} G(u)\, dA(u) \\ &\lesssim \sum_{i=1}^{n} \sum_{j=1}^{\infty} \int_{E(a_{ij}, \frac{\delta_i}{4})} \frac{G(u)}{(1 - |u|)^{1+tp}}\, dA(u) \\ &\lesssim \int_{\mathbb{D}} \frac{G(u)}{(1 - |u|)^{1+tp}}\, dA(u). \end{aligned}$$

Bearing in mind Lemma 4.2, we have (3).

Now, we prove that $G(z)$ has the generalized sub-mean-value property. Since $|g(e^{i\eta}) - g(z)|^p$ is subharmonic, using the sub-mean-value property, we have

$$\left|g\left(e^{i\eta}\right) - g(z)\right|^p \lesssim \frac{1}{|E(z, \delta)|} \int_{E(z,\delta)} \left|g\left(e^{i\eta}\right) - g(w)\right|^p dA(w), \quad 0 < \delta < 1.$$

From [26, p.69 and Lemma 4.30], we know

$$1 - |z| \asymp 1 - |w| \asymp |1 - \bar{z}w|, \qquad |1 - \bar{a}w| \asymp |1 - \bar{a}z|$$

for $w \in E(z, \delta)$, $0 < \delta < 1$, $a \in \overline{\mathbb{D}}$. Thus, it follows that

$$\begin{aligned} G(z) &\lesssim \frac{1}{|E(z, \delta)|} \int_{E(z,\delta)} \int_{\partial\mathbb{D}} \left|g\left(e^{i\eta}\right) - g(w)\right|^p \frac{1 - |w|^2}{|e^{i\eta} - w|^2}\, d\eta\, dA(w) \\ &= \frac{1}{|E(z, \delta)|} \int_{E(z,\delta)} G(w)\, dA(w). \end{aligned}$$

That is, $G(z)$ has the generalized sub-mean-value property.

(2) $\Leftrightarrow$ (4). The proof is similar to that of (2) $\Leftrightarrow$ (3) and thus is omitted. The theorem is proved. □

Lemma 4.3 *Let $1 \leq p < \infty$ and $0 < s < 1$. Suppose $f = I_f \mathcal{O}_f \in F(p, p-2, s)$, where I_f and $\mathcal{O}_f$ are inner-outer factors. Let I be an inner function dividing I_f. Then $f/I \in F(p, p-2, s)$.*

Proof Let $f \in F(p,p-2,s) \subseteq AB_p^{\frac{1-s}{p}}$. By the Möbius invariant property of $F(p,p-2,s)$, we have $f \circ \varphi_a \in F(p,p-2,s)$. From Corollary 3.1 of [29], we get

$$\begin{aligned}
&\int_{\mathbb{D}} \left| \left(\frac{f(z)}{I(z)} \right)' \right|^p \left(1-|z|^2\right)^{p-2} \left(1-\left|\varphi_a(z)\right|^2\right)^s \, dA(z) \\
&\quad = \int_{\mathbb{D}} \left| \left(\frac{f \circ \varphi_a(z)}{I \circ \varphi_a(z)} \right)' \right|^p \left(1-|z|^2\right)^{p-2+s} \, dA(z) \\
&\quad \lesssim \int_{\mathbb{D}} \left|(f \circ \varphi_a)'(z)\right|^p \left(1-|z|^2\right)^{p-2+s} \, dA(z) \\
&\quad = \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^{p-2} \left(1-\left|\varphi_a(z)\right|^2\right)^s \, dA(z).
\end{aligned}$$

The desired result follows. □

Proof of Corollary 2 Taking the same sequence $\{a_k\}_{k=1}^{\infty}$ as in Proposition 1. Let $B(z)$ be the associated Blaschke product. Notice the fact that $F(p,p-2,t) \subseteq AB_p^{\frac{1-t}{p}}$, it implies that $\{a_k\}_{k=1}^{\infty}$ is not a $F(p,p-2,t)$-zero set. We deduce that $B(z)$ does not belong to $F(p,p-2,s)$. Combining this with Lemma 4.3, the proof of the rest is similar to that of Theorem 2 of [12] and thus is omitted. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in the preparation of this article. All authors read and approved the final manuscript.

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