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Some limit theorems for the log-optimal portfolio

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Abstract
In this paper, we mainly study the strong limit theorems for the log-optimal portfolio of the $*$-mixing stock market. First, we establish a strong limit theorem for the log-optimal portfolio of any sequence investment, then we obtain the result that the average return of the long term behavior of a sequence investment converges to the average of the expectation return in every period with probability 1 under some conditions. We also obtain another strong limit theorem for the log-optimal portfolio for a class of sequence investments.
MSC: Primary 60F15; secondary 91B28

Keywords: log-optimal portfolio; $*$-mixing sequence; limit theorem

1 Introduction

This paper considers the continuous investment behaviors in the stock market under a discrete time framework. If there are m stocks in the market, the market can be denoted by $\{X_n = (X_{n1}, X_{n2}, \ldots, X_{nm})^T, X_{ni} \geq 0, n \geq 1, i = 1, 2, \ldots, m\}$ (the superscript T means the transpose of a vector). X_{ni} represents the relative price of the ith stock in the nth period (the ratio of the stock price at time $n+1$ to the stock price at time n). Let $\{b_n = (b_{n1}, b_{n2}, \ldots, b_{nm})^T, b_{ni} \geq 0, \sum_{i=1}^m b_{ni} = 1, n \geq 1\}$ be the investment strategies, where b_n represents the investment strategy in the nth period and b_{ni} $(i = 1, 2, \ldots, m)$ is the investment share of the ith stock in the nth period. In general, suppose that b_n is measurable with respect to $\sigma(X_1, X_2, \ldots, X_{n-1})$, *i.e.* b_n is required to be totally decided by the information of the $n-1$ periods before it. At the end of the nth period, the relative wealth of investor (the ratio of wealth at time $n+1$ to time n) is $S_n = \prod_{k=1}^n (b_k^T X_k)$. The purpose of the investor is to choose the investment strategy such that $E \log S_n = \sum_{k=1}^n E \log(b_k^T X_k)$ becomes maximum. This is the so-called log-optimal investment portfolio problem. Here, we suppose that $\{b_n, n \geq 1\}$ are the collection of all the optimized investment portfolio.

We mainly study the strong limit theorems for the log-optimal portfolio in the stock investment by using the limit properties for arbitrary random variables. Until now, there are some typical conclusions as regards the limit theorems of the random variables such as Chung's classical strong law of large numbers for the sequence of independent random variables in [1], Chow's convergence theorems for martingale difference sequence in [2], and Liu's limit theorems of the sequence of m-valued random variables in [3]. The application of the above typical conclusions in the financial market has excited the scholars' interests. Reference [4] considered the problem of the optimal portfolio and proved the limit theorems for independent and identically distributed (i.i.d.) and stationary markets. Reference [5] also discussed this problem. However, we know that the real stock market does not have these good conditions. Reference [6] gave the strong limit theorems for arbitrary random variables and the strong law of large numbers for the ∗-mixing sequence. Therefore, we suppose that the relative prices of the stocks are a ∗-mixing sequence which is gradually independent and the investment strategy in the nth period is correlated with the information of the stock prices of the previous $n-1$ periods. Then we apply the conclusions in [6] to the financial field and obtain some limit theorems of the stock log-optimal portfolio. Our results extend the results in [4] and [5].

In fact, the investment of the stock is generally the long term behavior and the stock market is ∗-mixing, *i.e.* the stock prices in two time periods which are sufficiently far away from each other can be approached as being independent. Therefore, the assumptions in this paper are reasonable and meaningful in the financial market.

2 Main results

Lemma 2.1 [6] *Let $\{X_n, n \geq 1\}$ be a sequence of arbitrary random variables. Let $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ and $\mathcal{F}_{-n} = \{\phi, \Omega\}$, $n \geq 1$. Let $\{\varphi_n, n \geq 1\}$ be a sequence of non-negative even functions of x,*

$$\frac{\varphi_n(x)}{|x|} \uparrow, \qquad \frac{\varphi_n(x)}{x^2} \downarrow, \tag{1}$$

and let $\{a_n, n \geq 0\}$ be an increasing sequence of positive numbers. If

$$\sum_{n=1}^{\infty} \frac{E[\varphi_n(X_n)]}{\varphi_n(a_n)} < +\infty, \tag{2}$$

and $a_n \uparrow \infty$, then $\forall m \geq 1$,

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} \bigl\{X_k - E[X_k|\mathcal{F}_{k-m}]\bigr\} = 0 \quad a.e. \tag{3}$$

Lemma 2.2 *Let $\{X_n, n \geq 1\}$ be the m-dimension stock relative price sequence as given above and $\{b_n, n \geq 1\}$ be the m-dimension investment strategy sequence as given above. Suppose that b_n is measurable with respect to $\mathcal{F}_{n-1} = \sigma(X_1, X_2, \dots, X_{n-1})$ and $\{\varphi_n, n \geq 1\}$ is defined as in Lemma 2.1. Let $Y_n = \log(b_n^T X_n)$, if*

$$\sum_{n=1}^{\infty} \frac{E[\varphi_n(Y_n)]}{\varphi_n(n)} < +\infty, \tag{4}$$

then $\forall m \geq 1$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \bigl\{\log\bigl(b_k^T X_k\bigr) - E\bigl[\log\bigl(b_k^t X_k\bigr)|\mathcal{F}_{k-m}\bigr]\bigr\} = 0 \quad a.e., \tag{5}$$

where $\mathcal{F}_{-n} = \{\phi, \Omega\}$, $n \geq 1$.

Proof Because $\{Y_n, \mathcal{F}_n, n \geq 1\}$ is a stochastic adapted sequence, from Lemma 2.1, this lemma holds. □

Definition 2.3 Suppose that $\{X_n, n \geq 1\}$ is m-dimension vector sequence and $\mathcal{F}_n^l = \sigma(X_n, \ldots, X_l)$. If there exists a sequence of numbers $\{\phi(n), n \geq N\}$ that satisfies $\lim_{n\to\infty} \phi(n) = 0$, such that when $n \geq N$, $\forall l \geq 1$, $\forall A \in \mathcal{F}_1^l$, $B \in \mathcal{F}_{n+l}^{+\infty} = \sigma(X_{n+l}, X_{n+l+1}, \ldots)$, we have

$$\bigl|P(AB) - P(A)P(B)\bigr| \leq \phi(n)P(A)P(B), \tag{6}$$

then $\{X_n, n \geq 1\}$ is called a $*$-*mixing sequence*.

We easily see that (3) is equivalent to the following equation: $\forall B \in \mathcal{F}_{n+l}^{+\infty}$,

$$\bigl|P\bigl(B|\mathcal{F}_1^l\bigr) - P(B)\bigr| \leq \phi(n)P(B) \quad \text{a.e.} \tag{7}$$

Lemma 2.4 *Let $\{X_n, n \geq 1\}$ be a $*$-mixing sequence defined by Definition* 2.3. *Suppose* 1-*dimension random variable $Y \in \mathcal{F}_{n+l}^{+\infty}$ and $E|Y| < +\infty$, then $\forall l \geq 1$,*

$$\bigl|E\bigl[Y|\mathcal{F}_1^l\bigr] - E[Y]\bigr| \leq \phi(n)E|Y| \quad \textit{a.e.}$$

Proof The proof procedure is almost similar to [7], p.139. □

Theorem 2.5 *Let $\{X_n, n \geq 1\}$ be an m stock relative price sequence and a $*$-mixing sequence. $\{b_n, n \geq 1\}$ is the optimized investment strategy sequence. Suppose that $N \geq 1$, $b_n \in \sigma(X_{n-N}, X_{n-N+1}, \ldots, X_{n-1})$ and $\forall n \geq 1$, $E|\log(b_n^T X_n)| \leq k < +\infty$. If* (4) *holds, then*

$$\lim_{n\to\infty} \frac{1}{n}\bigl(\log S_n - E[\log S_n]\bigr) = 0 \quad \textit{a.e.} \tag{8}$$

Proof From Lemma 2.2, we know that $\forall m \geq N + 1$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \bigl\{\log\bigl(b_k^T X_k\bigr) - E\bigl[\log\bigl(b_k^T X_k\bigr)|\mathcal{F}_{k-m}\bigr]\bigr\} = 0 \quad \text{a.e.}, \tag{9}$$

then from Lemma 2.4, we have

$$\begin{aligned} &\left|\frac{1}{n} \sum_{k=1}^{n} \bigl\{E\bigl[\log\bigl(b_k^T X_k\bigr)|\mathcal{F}_{k-m}\bigr] - E\bigl[\log\bigl(b_k^T X_k\bigr)\bigr]\bigr\}\right| \\ &\quad \leq \frac{1}{n} \sum_{k=1}^{n} \bigl|E\bigl[\log\bigl(b_k^T X_k\bigr)|\mathcal{F}_{k-m}\bigr] - E\bigl[\log\bigl(b_k^T X_k\bigr)\bigr]\bigr| \\ &\quad \leq \frac{1}{n} \sum_{k=1}^{n} E\bigl|\log\bigl(b_k^T X_k\bigr)\bigr|\phi(m-N) \\ &\quad \leq k\phi(m-N). \end{aligned} \tag{10}$$

Since when $\phi(m-N) \to 0$ $(m \to \infty)$, we can choose m such that $k\phi(m-N)$ is sufficiently small, from (9) and (10), we know that (8) holds. □

Remark 2.6 In fact, a $*$-mixing sequence means that the sequence is gradually independent. In this paper, we suppose the relative price sequence satisfying the $*$-mixing condition, which means that the relative stock prices in two periods gradually become independent when the interval between two periods is longer and longer. In the last section, we will give two numerical examples, which show that a market satisfying the $*$-mixing condition exists. Therefore, Theorem 2.5 is meaningful.

Remark 2.7 In Theorem 2.5, we suppose that b_n is measurable with respect to $\sigma(X_{n-N}, X_{n-N+1}, \ldots, X_{n-1})$. It means that the investment strategy in the nth period is totally decided by the information of the stock prices of the N periods before it. Because we consider the long term behavior of a sequence investment, this assumption is meaningful for the financial market.

Remark 2.8 The equality (8) shows that the average return of the long term behavior of a sequence investment converges to the average of the expectation return in every period in probability 1 under the conditions in Theorem 2.5.

Corollary 2.9 *Let* $\{X_n, n \geq 1\}$ *be the* m *stock relative price sequence and be the* $*$*-mixing sequence. Let* $\{b_n, n \geq 1\}$ *be the optimized investment strategy sequence. Suppose that* $N \geq 1$, $b_n \in \sigma(X_{n-N}, X_{n-N+1}, \ldots, X_{n-1})$ *and* $\forall n \geq 1$, $E|\log(b_n^T X_n)| \leq k$, *where* $k > 0$ *is a constant. If*

$$\sum_{n=1}^{\infty} \frac{E|Y_n|^p}{n^p} < +\infty, \quad 1 \leq p \leq 2, \tag{11}$$

then

$$\lim_{n\to\infty} \frac{1}{n}\big(\log S_n - E[\log S_n]\big) = 0 \quad a.e. \tag{12}$$

Proof Letting $\varphi_n(x) = |x|^p$ in Theorem 2.5, this corollary follows. □

Theorem 2.10 *Let* $\{X_n, n \geq 1\}$ *and* $\{b_n, n \geq 1\}$ *be defined as in Theorem* 2.5 *and let condition* (1) *in Lemma* 2.1 *be replaced by the following condition*: *as* $|x|$ *increases*,

$$\varphi_n(x) \uparrow, \qquad \frac{\varphi_n(x)}{|x|} \downarrow. \tag{13}$$

If (4) *holds, then*

$$\lim_{n\to\infty} \frac{1}{n} \log S_n = 0 \quad a.e. \tag{14}$$

Proof Let $Y_n = \log(b_n^T X_n)$ and $Z_n = Y_n I_{\{|Y_n| \leq n\}}$. By (13), we have

$$\begin{aligned} E\left(\sum_{n=1}^{\infty} \frac{|Z_n|}{n}\right) &= E\left(\sum_{n=1}^{\infty} \frac{|Y_n| I_{\{|Y_n| \leq n\}}}{n}\right) \\ &\leq E\left(\sum_{n=1}^{\infty} \frac{\varphi_n(Y_n) I_{\{|Y_n| \leq n\}}}{\varphi_n(n)}\right) \end{aligned}$$

$$\begin{aligned} &\le E\left(\sum_{n=1}^{\infty}\frac{\varphi_n(Y_n)}{\varphi_n(n)}\right) \\ &< \infty. \end{aligned}$$

So

$$\sum_{n=1}^{\infty}\frac{Z_n}{n} \quad \text{converges a.e.} \tag{15}$$

In addition, from (13),

$$\begin{aligned} \sum_{n=1}^{\infty}P(Z_n \neq Y_n) &= \sum_{n=1}^{\infty}P\bigl(|Y_n| > n\bigr) \\ &\le E\left(\sum_{n=1}^{\infty}\frac{\varphi_n(Y_n)I_{\{|Y_n|\le n\}}}{\varphi_n(n)}\right) \\ &\le E\left(\sum_{n=1}^{\infty}\frac{\varphi_n(Y_n)}{\varphi_n(n)}\right) \\ &< \infty, \end{aligned}$$

thus, according to the Borel-Cantelli lemma, we have

$$\sum_{n=1}^{\infty}\frac{Z_n - Y_n}{n} \quad \text{converges a.e.} \tag{16}$$

By (15) and (16), we find that the conclusion holds. □

3 Numerical examples

The key hypothesis of this paper is that the stock market is $*$-mixing, which means that the relative stock prices in two periods are independent when the two periods are sufficiently far away from each other. All the results are obtained based on this hypothesis. In the following, we give two numerical examples with the past 50 years' historical data of the daily closing prices of Dow Jones Industrial Average Index and Standard & Poors 500 Index to show that the stock markets in this paper exist. [a]

The verification process

The selection of the underlying stocks

The Dow Jones Industrial Average (DJIA) Index and Standard & Poors 500 (S&P 500) Index are both portfolios and universally representative, so the verifications with these two indices are reasonable.

Step 1: *generating the relative price sequence of the stock*

Suppose the sequence of the stock prices is $\{C_n, n \ge 1\}$, the data of the stock prices $\{c_n, n \ge 1\}$ are a realization of $\{C_n, n \ge 1\}$. The first step is to generate the relative price data $\{x_n = \frac{c_{n+1}}{c_n}, n \ge 1\}$, which are a realization of the relative price sequence $\{X_n = \frac{C_{n+1}}{C_n}, n \ge 1\}$.

Step 2: *the discretization of the relative price*
We divide the value range of X_n into 21 intervals:

$$I_{-10} \triangleq (-\infty, 0.905),$$
$$I_{-9} \triangleq [0.905, 0.915),$$
$$I_{-8} \triangleq [0.915, 0.925),$$
$$\ldots,$$
$$I_0 \triangleq [0.995, 1.05),$$
$$\ldots,$$
$$I_9 \triangleq [1.085, 1.095),$$
$$I_{10} \triangleq [1.095, +\infty).$$

These 21 intervals represent 21 cases of the return ratio which are less than –9%, greater than +9% and the others 19 cases varying from –9% to +9%. Now we established the sample space $\Omega = \{I_{-10}, I_{-9}, \ldots, I_9, I_{10}\}$ of the discrete random variable X_n.

Step 3: *evaluating the probability distribution of the relative prices*
For every random variable X_m, we evaluate its probability distribution by 1 year's data before it. Since 1 year contains 250 trading days around, we count 250 data of $\{x_n, n \geq 1\}$ before it to compute the frequencies of $I_{-10}, \ldots, I_{10}$, and from them we get the evaluation of the probability distribution. In Definition 2.3, there is a condition that $n \geq N$, $\forall l \geq 1$, $\forall A \in \mathcal{F}_1^l$, $B \in \mathcal{F}_{n+l}^{+\infty} = \sigma(X_{n+l}, X_{n+l+1}, \ldots)$. So we can regard B as the most recent event, and A as the event that happened n days ago.

Specifically, suppose the relative price sequence data $\{x_n\}$ contain M data, *i.e.*, our sequence data are $\{x_n, 1 \leq n \leq M\}$, x_M is the most recent datum. Now, let B be the elementary event of random variable X_M, A be the elementary event of random variable X_{M-n}. Then we use the data $\{x_{M-n-249}, \ldots, x_{M-n}\}$ and $\{x_{M-249}, \ldots, x_M\}$ to evaluate the probability distribution of X_M and X_{M-n}, respectively. As for the joint probability of AB, we use the 250 data pairs, $\{(x_{M-n-249}, x_{M-249}), \ldots, (x_{M-n}, x_M)\}$, to evaluate the joint probability distribution of (X_{M-n}, X_M). For the sake of the condition mentioned above, $n >= 250$ necessarily holds.

Step 4: *evaluating the number sequence* $\phi(n)$
According to Definition 2.3, if there exists a number sequence $\{\phi(n)\}$ that converges to zero when n goes to infinity, then the sequence is a $*$-mixing sequence. So, the next step is to evaluate the number sequence $\phi(n)$. From (6), for every possible value pair of (X_{M-n}, X_M), we compute the value

$$\varphi_{\{x,y\}} = \frac{|P(AB) - P(A)P(B)|}{P(A)P(B)}, \quad x, y \in \Omega,$$

here A, B are the elementary events of random variables (X_{M-n}, X_M), respectively. Let

$$\varphi(n) = \frac{\sum_{s,t=-10}^{10} \varphi_{\{I_s, I_t\}}}{21 \times 21}.$$

Now we can use the average value $\varphi(n)$ of $\varphi_{\{x,y\}}$ to evaluate $\phi(n)$.

By similar steps, we can compute the value of $\varphi(n)$ once every 250 days, *i.e.*, we just compute $\varphi(n)$, $n = 250d$, $d = 1, 2, \ldots$.

Example 1: figures by data of DJIA

Figure 1 is the daily closing price of DJIA.

Figure 2 is the logarithm of the return ratio (*i.e.*, $\log \frac{x_{n+1}}{x_n}$).

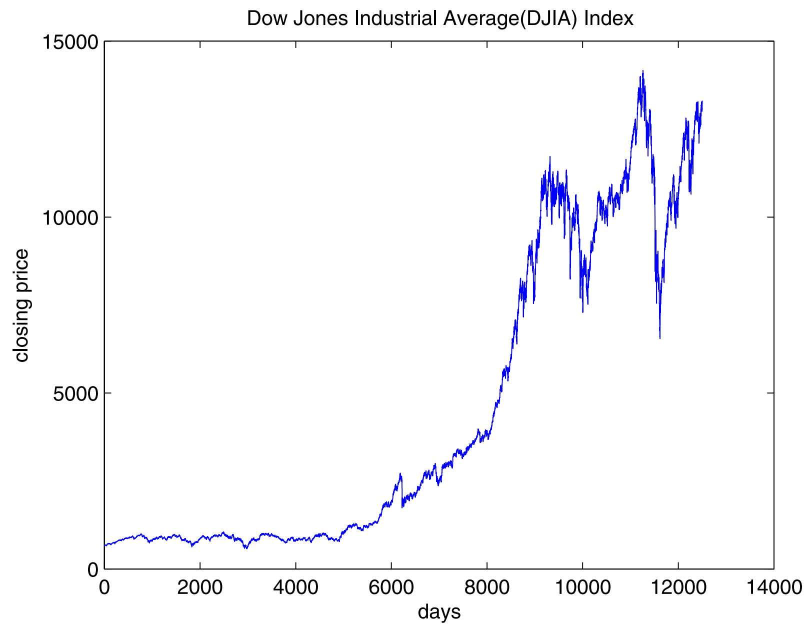

Figure 1 The daily closing price of DJIA.

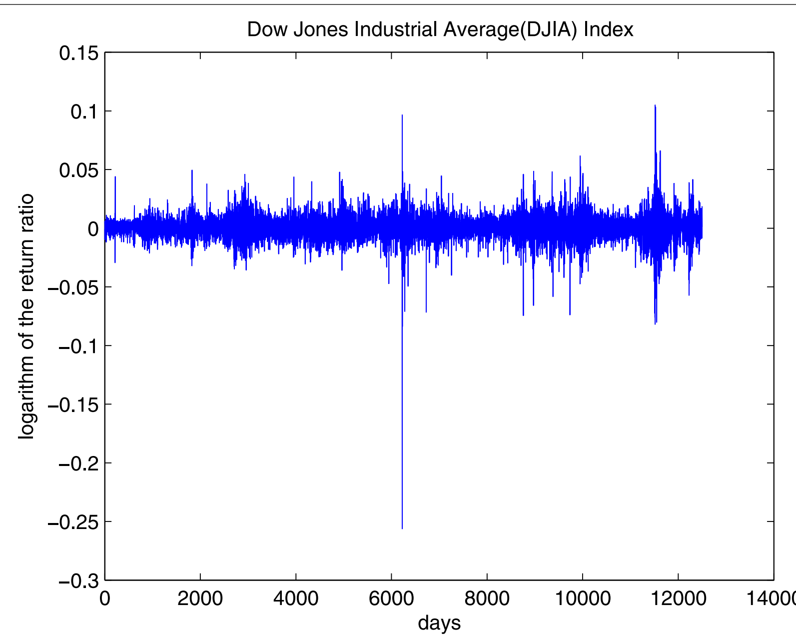

Figure 2 The logarithm of return ratio of DJIA. The logarithm of the return ratio, $\log \frac{x_{n+1}}{x_n}$, of DJIA.

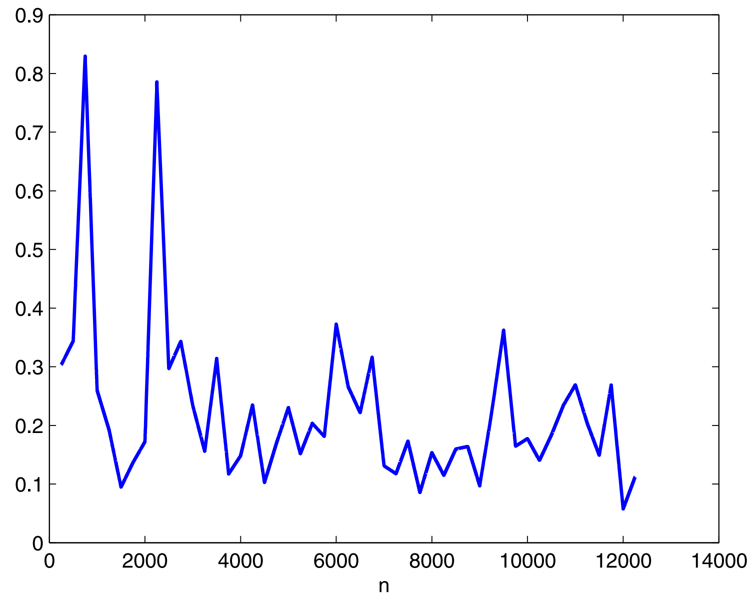

Figure 3 $\varphi(n)$ **of DJIA.** $\varphi(n)$ with the independent variable n varied 250 days a step.

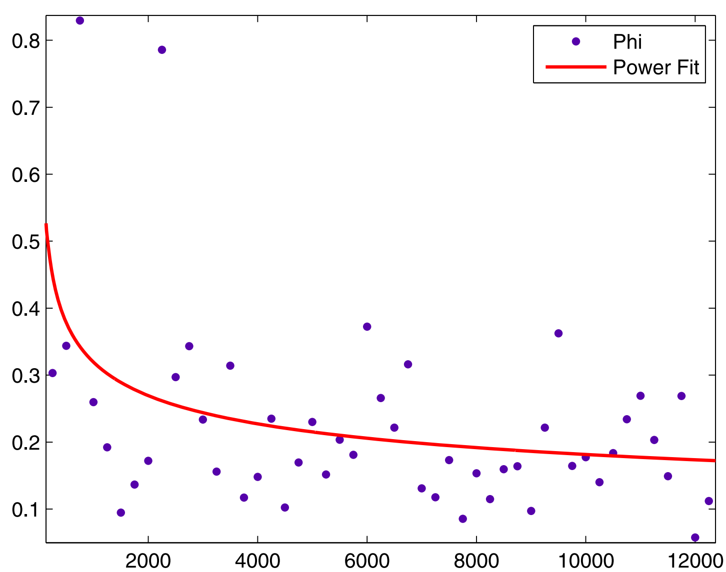

Figure 4 The fitting curve of $\varphi(n)$ **of DJIA.**

Figure 3 is $\varphi(n)$ where $n = 250d$, $d = 1, 2, \ldots$.

Figure 4 is the fitting curve to the sequence of the point $\{(\varphi(n), n), n = 250d, d = 1, 2, \ldots\}$. Our general fitting model is a power function: $f(x) = ax^b$. Through the least squares method, we get the value of the coefficients (with 95% confidence bounds):

$$a = 1.738(-0.3096, 3.785),$$

$$b = -0.2452(-0.3916, -0.09885).$$

From Figure 3 and Figure 4, we can see the trend of $\varphi(n)$ approaching zero when n is going to infinity, which shows that the market is $*$-mixing.

Example 2: figures by data of S&P 500

Figure 5 is the graph of $\varphi(n)$ computed by the data of S&P 500.

Figure 6 is the fitting curve to the sequence of points $\{(\varphi(n), n), n = 250d, d = 1, 2, \ldots\}$ generated by the data of S&P 500. The general fitting model is also by power functions: $f(x) = ax^b$. Through the least squares method, we get the value of the coefficients (with

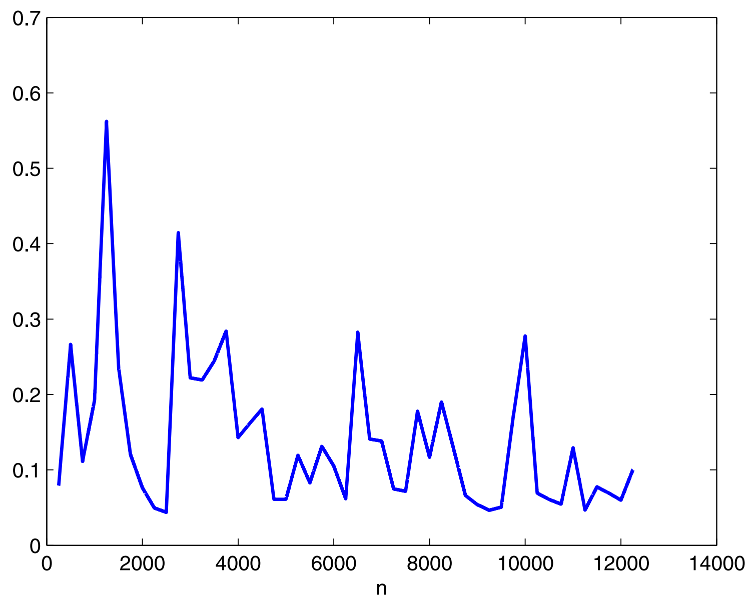

Figure 5 $\varphi(n)$ of S&P 500. $\varphi(n)$ with the independent variable n varied 250 days a step.

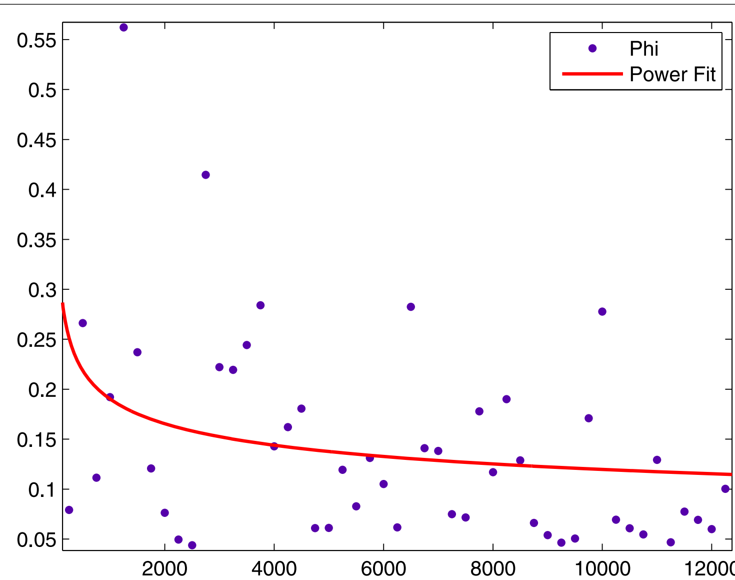

Figure 6 The fitting curve of $\varphi(n)$ of S&P 500.

95% confidence bounds):

$$a = 0.7636(-0.3526, 1.88),$$
$$b = -0.2012(-0.3805, -0.02189).$$

In Figure 5 and Figure 6, $\varphi(n)$ also approaches zero when n is going to infinity, which shows that this market is $*$-mixing too.

4 Conclusion

In the stock market, we generally observe the information of the stocks' price before the period when we decide the investment strategy of the stock in a period. In this paper, we suppose that the stock market is $*$-mixing and the investment strategy of the stock is correlated with the prices in the N periods before this period. Under the above conditions, we prove the limit properties of a log-optimal portfolio. In the end, we give the examples in a real market. In the following study, we will discuss a similar problem under the assumption of a nonhomogeneous Markov chain, and so on.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have equal contributions. All authors read and approved the final manuscript.

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Endnote

[a] Data source: http://www.forecasts.org/.

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