Lu and Zhou *Journal of Inequalities and Applications* 2014, **2014**:285
http://www.journalofinequalitiesandapplications.com/content/2014/1/285

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Estimates for fractional type Marcinkiewicz integrals with non-doubling measures

Guanghui Lu and Jiang Zhou*

*Correspondence:
zhoujiangshuxue@126.com
Department of Mathematics,
Xinjiang University, Urumqi, 830046,
People's Republic of China

Abstract

Under the assumption that μ is a non-doubling measure on $\mathbb{R}^d$ satisfying the growth condition, the authors prove that the fractional type Marcinkiewicz integral $\mathcal{M}$ is bounded from the Hardy space $H^{1,\infty,0}_{\mathrm{fin}}(\mu)$ to the Lebesgue space $L^q(\mu)$ for $\frac{1}{q} = 1 - \frac{\alpha}{n}$ with kernel satisfying a certain Hörmander-type condition. In addition, the authors show that for $p = \frac{n}{\alpha}$, $\mathcal{M}$ is bounded from the Morrey space $M^p_q(\mu)$ to the space RBMO(μ) and from the Lebesgue space $L^{\frac{n}{\alpha}}(\mu)$ to the space RBMO(μ).

MSC: Primary 46A20; secondary 42B25; 42B35

Keywords: non-doubling measure; fractional type Marcinkiewicz integral; Hardy space; RBMO(μ)

1 Introduction

Let μ be a nonnegative Radon measure on $\mathbb{R}^d$ which satisfies the following growth condition: for all $x \in \mathbb{R}^d$ and all $r > 0$,

$$\mu\bigl(B(x,r)\bigr) \le C_0 r^n, \tag{1.1}$$

where C_0 and n are positive constants and $n \in (0,d]$, $B(x,r)$ is the open ball centered at x and having radius r. So μ is claimed to be non-doubling measure. If there exists a positive constant C such that for any $x \in \operatorname{supp}(\mu)$ and $r > 0$, $\mu(B(x,2r)) \le C\mu(B(x,r))$, the μ is said to be doubling measure. It is well known that the doubling condition on underlying measures is a key assumption in the classical theory of harmonic analysis. Especially, in recent years, many classical results concerning the theory of Calderón-Zygmund operators and function spaces have been proved still valid if the underlying measure is a nonnegative Radon measure on $\mathbb{R}^d$ which only satisfies (1.1) (see [1–8]). The motivation for developing the analysis with non-doubling measures and some examples of non-doubling measures can be found in [9]. We only point out that the analysis with non-doubling measures played a striking role in solving the long-standing open Painlevé's problem by Tolsa in [10].

Let $K(x,y)$ be a μ-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$. Assume that there exists a positive constant C such that for any $x, y \in \mathbb{R}^d$ with $x \ne y$,

$$\bigl|K(x,y)\bigr| \le C|x-y|^{-(n-1)}, \tag{1.2}$$

and for any $x, y, y' \in \mathbb{R}^d$,

$$\int_{|x-y|\geq 2|y-y'|} \bigl[\bigl|K(x,y) - K\bigl(x,y'\bigr)\bigr| + \bigl|K(y,x) - K\bigl(y',x\bigr)\bigr|\bigr] \frac{1}{|x-y|}\,d\mu(x) \leq C. \tag{1.3}$$

The fractional type Marcinkiewicz integral $\mathcal{M}$ associated to the above kernel $K(x,y)$ and the measure μ as in (1.1) is defined by

$$\mathcal{M}(f)(x) = \biggl(\int_0^\infty \biggl|\int_{|x-y|\leq t} \frac{K(x,y)}{|x-y|^{-\alpha}} f(y)\,d\mu(y)\biggr|^2 \frac{dt}{t^3}\biggr)^{\frac{1}{2}}, \quad x \in \mathbb{R}^d, 0 < \alpha < n. \tag{1.4}$$

If μ is the d-dimensional Lebesgue measure in $\mathbb{R}^d$, and

$$K(x,y) = \frac{\Omega(x-y)}{|x-y|^{n-1}}, \tag{1.5}$$

with Ω homogeneous of degree zero and $\Omega \in \mathrm{Lip}_\gamma(S^{d-1})$ for some $\gamma \in (0,1]$, then K satisfies (1.2) and (1.3). Under these conditions, $\mathcal{M}$ in (1.4) is introduced by Si *et al.* in [11]. As a special case, by letting $\alpha = 0$, we recapture the classical Marcinkiewicz integral operators that Stein introduced in 1958 (see [12]). Since then, many works have appeared about Marcinkiewicz type integral operators. A nice survey has been given by Lu in [13].

In 2007, the Hörmander-type condition was introduced by Hu *et al.* in [14], which was slightly stronger than (1.3) and was defined as follows:

$$\begin{aligned} &\sup_{\substack{\ell>0, y, y' \in \mathbb{R}^d \\ |y-y'|\leq \ell}} \sum_{k=1}^\infty k \int_{2^k\ell<|x-y|\leq 2^{k+1}\ell} \bigl[\bigl|K(x,y) - K\bigl(x,y'\bigr)\bigr| \\ &\quad + \bigl|K(y,x) - K\bigl(y',x\bigr)\bigr|\bigr] \frac{1}{|x-y|}\,d\mu(x) \leq C. \end{aligned} \tag{1.6}$$

However, in this paper, we discover that the kernel should satisfy some other kind of smoothness condition to replace (1.6).

Definition 1.1 Let $1 \leq s < \infty$, $0 < \varepsilon < 1$. The kernel K is said to satisfy a Hörmander-type condition if there exist $c_s > 1$ and $C_s > 0$ such that for any $x \in \mathbb{R}^d$ and $\ell > c_s|x|$,

$$\begin{aligned} &\sup_{\substack{\ell>0, y, y' \in \mathbb{R}^d \\ |y-y'|\leq \ell}} \sum_{k=1}^\infty 2^{k\varepsilon}\bigl(2^k\ell\bigr)^n \biggl(\frac{1}{(2^k\ell)^n} \int_{2^k\ell<|x-y|\leq 2^{k+1}\ell} \biggl[\bigl(\bigl|K(x,y) - K\bigl(x,y'\bigr)\bigr| \\ &\quad + \bigl|K(y,x) - K\bigl(y',x\bigr)\bigr|\bigr) \frac{1}{|x-y|}\biggr]^s d\mu(x)\biggr)^{\frac{1}{s}} \leq C_s. \end{aligned} \tag{1.7}$$

We denote by $\mathcal{H}^s$ the class of kernels satisfying this condition. It is clear that these classes are nested,

$$\mathcal{H}^{s_2} \subset \mathcal{H}^{s_1} \subset \mathcal{H}^1, \quad 1 < s_1 < s_2 < \infty.$$

We should point out that $\mathcal{H}^1$ is not condition (1.6).

The purpose of this paper is to get some estimates for the fractional type Marcinkiewicz integral $\mathcal{M}$ with kernel K satisfying (1.2) and (1.7) on the Hardy-type space and the RBMO(μ) space. To be precise, we establish the boundedness of $\mathcal{M}$ in $H^{1,\infty,0}_{\text{fin}}(\mu)$ for $\frac{1}{q} = 1 - \frac{\alpha}{n}$ in Section 2. In Section 3, we prove that $\mathcal{M}$ is bounded from the space RBMO(μ) to the Morrey space $M^p_q(\mu)$, from the space RBMO(μ) to the Lebesgue space $L^{\frac{n}{\alpha}}(\mu)$ for $p = \frac{n}{\alpha}$.

Before stating our results, we need to recall some necessary notation and definitions. For a cube $Q \subset \mathbb{R}^d$, we mean a closed cube whose sides are parallel to the coordinate axes. We denote its center and its side length by x_Q and $\ell(Q)$, respectively. Let $\eta > 1$, ηQ denote the cube with the same center as Q and $\ell(\eta Q) = \eta\ell(Q)$. Given two cubes $Q \subset R$ in $\mathbb{R}^d$, set

$$S_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{[\ell(2^k Q)]^n},$$

where $N_{Q,R}$ is the smallest positive integer k such that $\ell(2^k Q) \geq \ell(R)$. The concept $S_{Q,R}$ was introduced in [15], where some useful properties of $S_{Q,R}$ can be found.

Lemma 1.2 *For a function* $b \in L^1_{\text{loc}}(\mu)$, $0 < \beta \leq 1$, *conditions* (i) *and* (ii) *below are equivalent.*

(i) *There exist some constant* C_2 *and a collection of numbers* b_Q *such that these two properties hold*: *for any cube* Q,

$$\frac{1}{\mu(2Q)} \int_Q \left|b(x) - b(y)\right| d\mu(x) \leq C_2 \ell(Q)^\beta, \tag{1.8}$$

and for any cube R *such that* $Q \subset R$ *and* $\ell(R) \leq 2\ell(Q)$,

$$|b_Q - b_R| \leq C_2 \ell(Q)^\beta. \tag{1.9}$$

(ii) *For any given* p, $1 \leq p \leq \infty$, *there is a constant* $C(p) \geq 0$ *such that for every cube* Q, *then*

$$\left[\frac{1}{\mu(Q)} \int_Q \left|b(x) - m_Q(b)\right|^p d\mu(x)\right]^{\frac{1}{p}} \leq C(p)\ell(Q)^\beta, \tag{1.10}$$

where

$$m_Q(b) = \frac{1}{\mu(Q)} \int_Q b(y)\, d\mu(y),$$

and also for any cube R *such that* $Q \subset R$ *and* $\ell(R) \leq 2\ell(Q)$,

$$\left|m_Q(b) - m_R(b)\right| \leq C(p)\ell(Q)^\beta.$$

Remark 1.3 Lemma 1.2 is a slight variant of Theorem 2.3 in [16]. To be precise, if we replace all balls in Theorem 2.3 of [16] by cubes, we then obtain Lemma 1.2.

Remark 1.4 For $0 < \beta \leq 1$, (1.9) is equivalent to

$$|b_Q - b_R| \leq CS_{Q,R}\ell(R)^\beta \tag{1.11}$$

for any two cubes $Q \subset R$ with $\ell(R) \leq 2\ell(Q)$ (see Remark 2.7 in [16]).

Lemma 1.5 *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $q \geq \frac{n}{n-\alpha}$. *Then the fractional integral operator* I_α *defined by*

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy$$

is bounded from $L^p(\mu)$ *to* $L^r(\mu)$ (*see* [17]).

Lemma 1.6 *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Suppose that* $K(x,y)$ *satisfies* (1.2) *and* (1.3) *and* $\mathcal{M}$ *is as in* (1.4). *Then there exists a positive constant* $C > 0$ *such that for all bounded functions* f *with compact support*,

$$\|\mathcal{M}(f)\|_{L^q(\mu)} \leq C\|f\|_{L^p(\mu)}.$$

Proof of Lemma 1.6 By Minkowski's inequality, we have

$$\begin{aligned}
\mathcal{M}(f)(x) &= \left(\int_0^\infty \left|\int_{|x-y|\leq t} \frac{K(x,y)}{|x-y|^{-\alpha}} f(y)\,d\mu(y)\right|^2 \frac{dt}{t^3}\right)^{1/2} \\
&\leq \int_{\mathbb{R}^d} \frac{|K(x,y)|}{|x-y|^{-\alpha}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3}\right)^{\frac{1}{2}} d\mu(y) \\
&\leq C \int_{\mathbb{R}^d} \frac{1}{|x-y|^{n-\alpha-1}} |f(y)| \frac{1}{|x-y|}\,d\mu(y) \\
&\leq C \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,d\mu(y) \\
&\leq C I_\alpha(|f|)(x).
\end{aligned}$$

By Lemma 1.5 then

$$\|\mathcal{M}(f)\|_{L^q(\mu)} \leq C\|f\|_{L^p(\mu)}. \qquad \square$$

Throughout this paper, we use the constant C with subscripts to indicate its dependence on the parameters. For a μ-measurable set E, χ_E denotes its characteristic function. For any $p \in [1,\infty]$, we denote by p' its conjugate index, namely $\frac{1}{p} + \frac{1}{p'} = 1$.

2 Boundedness of $\mathcal{M}$ in Hardy spaces

This section is devoted to the behavior of $\mathcal{M}$ in Hardy spaces. In order to define the Hardy space $H^1(\mu)$, Tolsa introduced the grand maximal operator M_ϕ in [18].

Definition 2.1 Given $f \in L^1_{\text{loc}}(\mu)$, $M_\phi f$ is defined as

$$M_\phi f(x) = \sup_{\varphi\sim x}\left|\int_{\mathbb{R}^d} f\varphi\,d\mu\right|,$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d)$ and satisfies

(1) $\|\varphi\|_{L^1(\mu)} \leq 1$,
(2) $0 \leq \varphi(y) \leq \frac{1}{|x-y|^n}$ for all $y \in \mathbb{R}^d$,
(3) $|\varphi'(y)| \leq \frac{1}{|x-y|^{n+1}}$ for all $y \in \mathbb{R}^d$.

Based on Theorem 1.2 in [18], we can define the Hardy space $H^1(\mu)$ as follows (see [15]).

Definition 2.2 The Hardy space $H^1(\mu)$ is the set of all functions $f \in L^1(\mu)$ satisfying that $\int_{\mathbb{R}^d} f\, d\mu = 0$ and $M_\phi f \in L^1(\mu)$. Moreover, the norm of $f \in H^1(\mu)$ is defined by

$$\|f\|_{H^1(\mu)} = \|f\|_{L^1(\mu)} + \|M_\phi f\|_{L^1(\mu)}.$$

We recall the atomic Hardy space $H^{1,\infty,0}_{\mathrm{atb}}(\mu)$ as follows.

Definition 2.3 Let $\rho > 1$. A function $h \in L^1_{\mathrm{loc}}(\mu)$ is called an atomic block if

(1) there exists some cube R such that $\operatorname{supp} h \subset R$,
(2) $\int_{\mathbb{R}^d} h(x)\, d\mu(x) = 0$,
(3) for $i = 1, 2$, there are functions a_i supported on cubes $Q_i \subset R$ and numbers $\lambda_i \in \mathbb{R}$ such that $h = \lambda_1 a_1 + \lambda_2 a_2$, and

$$\|a_i\|_{L^\infty(\mu)} \leq \left[\mu(\rho Q_i) S_{Q_i,R}\right]^{-1}.$$

Then define

$$|h|_{H^{1,\infty,0}_{\mathrm{atb}}(\mu)} = |\lambda_1| + |\lambda_2|.$$

Define $H^{1,\infty,0}_{\mathrm{atb}}(\mu)$ and $H^{1,\infty,0}_{\mathrm{fin}}(\mu)$ as follows:

$$\|f\|_{H^{1,\infty,0}_{\mathrm{atb}}(\mu)} = \inf\left\{\sum_{j}^{\infty} |h_j|_{H^{1,\infty,0}_{\mathrm{atb}}(\mu)} : f = \sum_{j=1}^{\infty} h_j, \{h_j\}_{j\in\mathbb{N}} \text{ are } (1,\infty,0)\text{-atoms}\right\}$$

and

$$\|f\|_{H^{1,\infty,0}_{\mathrm{fin}}(\mu)} = \inf\left\{\sum_{j}^{k} |h_j|_{H^{1,\infty,0}_{\mathrm{atb}}(\mu)} : f = \sum_{j=1}^{k} h_j, \{h_j\}_{j=1}^{k} \text{ are } (1,\infty,0)\text{-atoms}\right\},$$

where the infimum is taken over all possible decompositions of f in atomic blocks, $H^{1,\infty,0}_{\mathrm{fin}}(\mu)$ is the set of all finite linear combinations of $(1,\infty,0)$-atoms.

Remark 2.4 It was proved in [15] that for each $\rho > 1$, the atomic Hardy space $H^{1,\infty,0}_{\mathrm{atb}}(\mu)$ is independent of the choice of ρ.

Applying the theory of Meda *et al.* in [19], we easily get the result as follows.

Theorem 2.5 *Let* $0 < \alpha < n$, $\frac{1}{q} = 1 - \frac{\alpha}{n}$. *Suppose that* K *satisfies* (1.2) *and the* $\mathcal{H}^q$ *condition and* $f \in H^{1,\infty,0}_{\mathrm{fin}}(\mu)$. *Then* $\mathcal{M}$ *is bounded from the Hardy space into the Lebesgue space,* *namely there exists a positive constant C such that*

$$\|\mathcal{M}(f)\|_{L^q(\mu)} \le C\|f\|_{H^{1,\infty,0}_{\mathrm{fin}}(\mu)}.$$

Proof of Theorem 2.5 Without loss of generality, we may assume that $\rho = 4$ and $f = \sum h$ as a finite of atomic blocks defined in Definition 2.3. It is easy to see that we only need to prove the theorem for one atomic block h. Let R be a cube such that $\operatorname{supp} h \subset R$, $\int_{\mathbb{R}^d} h(x)\,d\mu(x) = 0$, and

$$h(x) = \lambda_1 a_1(x) + \lambda a_2(x), \tag{2.1}$$

where λ_i for $i = 1,2$ is a real number, $|h_i|_{H^{1,\infty,0}_{\mathrm{atb}}(\mu)} = \lambda_1 + \lambda_2$, a_i for $i = 1,2$ is a bounded function supported on some cubes $Q_i \subset R$ and it satisfies

$$\|a_i\|_{L^\infty(\mu)} \le \left[\mu(4Q_i)S_{Q_i,R}\right]^{-1}. \tag{2.2}$$

Write

$$\begin{aligned}
\|\mathcal{M}(h)\|_{L^q(\mu)} &\le \left(\int_{2R}\left|\mathcal{M}(h)(x)\right|^q d\mu(x)\right)^{\frac{1}{q}} + \left(\int_{\mathbb{R}^d\setminus 2R}\left|\mathcal{M}(h)(x)\right|^q d\mu(x)\right)^{\frac{1}{q}} \\
&\le \left(\int_{2R}\left|\mathcal{M}(h)(x)\right|^q d\mu(x)\right)^{\frac{1}{q}} \\
&\quad + \left\{\int_{\mathbb{R}^d\setminus 2R}\left(\int_0^{|x-x_R|+2\ell(R)}\left|\int_{|x-y|\le t}\frac{K(x,y)}{|x-y|^{-\alpha}}h(y)\,d\mu(y)\right|^2\frac{dt}{t^3}\right)^{\frac{q}{2}} d\mu(x)\right\}^{\frac{1}{q}} \\
&\quad + \left\{\int_{\mathbb{R}^d\setminus 2R}\left(\int_{|x-x_R|+2\ell(R)}^{\infty}\left|\int_{|x-y|\le t}\frac{K(x,y)}{|x-y|^{-\alpha}}h(y)\,d\mu(y)\right|^2\frac{dt}{t^3}\right)^{\frac{q}{2}} d\mu(x)\right\}^{\frac{1}{q}} \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

By (2.1), we have

$$\begin{aligned}
\mathrm{I} &= \left(\int_{2R}\left|\mathcal{M}(h)(x)\right|^q d\mu(x)\right)^{\frac{1}{q}} \\
&\le |\lambda_1|\left(\int_{2R}\left|\mathcal{M}(a_1)(x)\right|^q d\mu(x)\right)^{\frac{1}{q}} + |\lambda_2|\left(\int_{2R}\left|\mathcal{M}(a_2)(x)\right|^q d\mu(x)\right)^{\frac{1}{q}} \\
&= \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

To estimate I_1, we write

$$\begin{aligned}
\mathrm{I}_1 &\le |\lambda_1|\left(\int_{2Q_1}\left|\mathcal{M}(a_1)(x)\right|^q d\mu(x)\right)^{\frac{1}{q}} + |\lambda_1|\left(\int_{2R\setminus 2Q_1}\left|\mathcal{M}(a_1)(x)\right|^q d\mu(x)\right)^{\frac{1}{q}} \\
&= \mathrm{I}_{11} + \mathrm{I}_{12}.
\end{aligned}$$

Choose p_1 and q_1 such that $1 < p_1 < \frac{n}{\alpha}$, $1 < q < q_1$ and $\frac{1}{q_1} = \frac{1}{p_1} - \frac{n}{\alpha}$. By the Hölder inequality, the fact that $S_{Q_1,R} \ge 1$ and the $(L^{p_1}(\mu), L^{q_1}(\mu))$-boundedness of $\mathcal{M}$ (see Lemma 1.6), we have that

$$\begin{aligned}
\mathrm{I}_{11} &\leq |\lambda_1| \left[\int_{2Q_1} \left| \mathcal{M}(a_1)(x) \right|^{q_1} d\mu(x) \right]^{\frac{1}{q_1}} \mu(2Q_1)^{\frac{1}{q}-\frac{1}{q_1}} \\
&\leq C|\lambda_1| \|a_1\|_{L^{p_1}(\mu)} \mu(2Q_1)^{\frac{1}{q}-\frac{1}{q_1}} \\
&\leq C|\lambda_1| \|a_1\|_{L^{\infty}(\mu)} \mu(2Q_1)^{\frac{1}{p_1}+\frac{1}{q}-\frac{1}{q_1}} \\
&\leq C|\lambda_1|.
\end{aligned}$$

Denote $N_{2Q_1,2R}$ simply by N_1. Invoking the fact that $\|a_1\|_{L^\infty(\mu)} \leq [\mu(4Q_i) S_{Q_i,R}]^{-1}$, we thus get

$$\begin{aligned}
\mathrm{I}_{12} &\leq C|\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \int_{2^{k+1}Q_1 \setminus 2^k Q_1} \left[\int_0^\infty \left| \int_{|x-y|\leq t} \frac{a_1(y)}{|x-y|^{n-\alpha-1}} \, d\mu(y) \right|^2 \frac{dt}{t^3} \right]^{\frac{q}{2}} d\mu(x) \right\}^{\frac{1}{q}} \\
&\leq C|\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \ell\big(2^k Q_1\big)^{q(\alpha-n)} \right. \\
&\quad \left. \times \int_{2^{k+1}Q_1 \setminus 2^k Q_1} \left[\int_{Q_1} \frac{|a_1(y)|}{|x-y|^{n-1-\alpha}} \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{\frac{1}{2}} d\mu(y) \right]^q d\mu(x) \right\}^{\frac{1}{q}} \\
&\leq C|\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \ell\big(2^k Q_1\big)^{q(\alpha-n)} \int_{2^{k+1}Q_1 \setminus 2^k Q_1} \left[\int_{Q_1} \left| a_1(y) \right| d\mu(y) \right]^q d\mu(x) \right\}^{\frac{1}{q}} \\
&\leq C|\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \ell\big(2^k Q_1\big)^{q(\alpha-n)} \mu\big(2^{k+1}Q_1\big) \|a_1\|_{L^\infty(\mu)}^q \mu(Q_1)^q \right\}^{\frac{1}{q}} \\
&\leq C|\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \ell\big(2^k Q_1\big)^{q(\alpha-n)} \mu(4Q_1)^{-q} S_{Q_1,R}^{-q} \mu\big(2^{k+1}Q_1\big) \|a_1\|_{L^\infty(\mu)}^q \mu(Q_1)^q \right\}^{\frac{1}{q}} \\
&\leq C|\lambda_1| \left(S_{Q_1,R}^{-q} \sum_{k=2}^{N_1+1} \frac{\mu(2^k Q_1)}{\ell(2^k Q_1)^n} \right)^{\frac{1}{q}} \\
&\leq C|\lambda_1|.
\end{aligned}$$

Here we have used the fact that

$$\sum_{k=2}^{N_1+1} \frac{\mu(2^k Q)}{\ell(2^k Q)^n} \leq C S_{Q,R},$$

see [16] for details.

The estimates for I_{11} and I_{12} give the desired estimate for I_1. With a similar argument, we have

$$\mathrm{I}_2 \leq C|\lambda_2|.$$

Combining the estimates for I_1 and I_2 yields the estimate for I.

For $i = 1, 2$, $y \in Q_i \subset R$, $x \in \mathbb{R}^d \setminus (2R)$, we have $|x-y| \sim |x-x_R| \sim |x-x_R| + 2\ell(R)$, by Minkowski's inequality, we get

$$\begin{aligned}
\mathrm{II} &\le \left\{ \int_{\mathbb{R}^d\setminus(2R)} \left[\int_R \frac{h(y)}{|x-y|^{n-1-\alpha}} \left(\int_{|x-y|}^{|x-x_R|+2\ell(R)} \frac{dt}{t^3} \right)^{\frac{1}{2}} \right]^q d\mu(x) \right\}^{\frac{1}{q}} \\
&\le C \int_R \left\{ \int_{\mathbb{R}^d\setminus(2R)} \left[\left| \frac{1}{(|x-x_R| + 2\ell(R))^2} - \frac{1}{|x-y|^2} \right|^{\frac{1}{2}} \frac{|h(y)|}{|x-y|^{n-1-\alpha}} \right]^q d\mu(x) \right\}^{\frac{1}{q}} d\mu(y) \\
&\le C \int_R \left\{ \int_{\mathbb{R}^d\setminus(2R)} \left(\frac{\ell(R)^{\frac{1}{2}}}{|x-y|^{\frac{3}{2}}} \cdot \frac{|h(y)|}{|x-y|^{n-1-\alpha}} \right)^q d\mu(x) \right\}^{\frac{1}{q}} d\mu(y) \\
&\le C \int_R \left\{ \sum_{k=1}^{\infty} \int_{2^{k+1}R\setminus(2^kR)} \left(\frac{\ell(R)^{\frac{1}{2}}}{|x-y|^{n-\alpha+\frac{1}{2}}} \right)^q d\mu(x) \right\}^{\frac{1}{q}} \bigl|h(y)\bigr| \, d\mu(y) \\
&\le C \left(\sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^1(\mu)} \right) \left\{ \sum_{k=1}^{\infty} \ell(R)^{\frac{1}{2}} \ell\bigl(2^k R\bigr)^{-n+\alpha-\frac{1}{2}} \mu\bigl(2^{k+1}R\bigr)^{\frac{1}{q}} \right\} \\
&\le C \left(\sum_{j=1}^{2} |\lambda_j| \right).
\end{aligned}$$

For any $y \in R$, we have $|x-y| \le |x-x_R| + |y-x_R| \le |x-x_R| + 2\ell(R) \le t$. It follows that

$$\begin{aligned}
\mathrm{III} &\le \left\{ \int_{\mathbb{R}^d\setminus 2R} \left(\int_{|x-x_R|+2\ell(R)}^{\infty} \left| \int_{|x-y|\le t} \left[\frac{K(x,y)}{|x-y|^{-\alpha}} - \frac{K(x,x_R)}{|x-x_R|^{-\alpha}} \right] h(y) \, d\mu(y) \right|^2 \frac{dt}{t^3} \right)^{\frac{q}{2}} d\mu(x) \right\}^{\frac{1}{q}} \\
&\le \left\{ \int_{\mathbb{R}^d\setminus 2R} \left[\int_R \left| \frac{K(x,y)}{|x-y|^{-\alpha}} - \frac{K(x,x_R)}{|x-x_R|^{-\alpha}} \right| \left(\int_{|x-x_R|+2\ell(R)}^{\infty} \frac{dt}{t^3} \right)^{\frac{1}{2}} \bigl|h(y)\bigr| \, d\mu(y) \right]^q d\mu(x) \right\}^{\frac{1}{q}} \\
&\le C \int_R \sum_{k=1}^{\infty} \left\{ \int_{2^{k+1}R\setminus 2^kR} \left[\left| \frac{K(x,y)}{|x-y|^{-\alpha}} - \frac{K(x,x_R)}{|x-x_R|^{-\alpha}} \right| \cdot \frac{1}{|x-y|} \right]^q d\mu(x) \right\}^{\frac{1}{q}} \bigl|h(y)\bigr| \, d\mu(y) \\
&\le C \int_R \sum_{k=1}^{\infty} \left\{ \int_{2^{k+1}R\setminus 2^kR} \left[\left| \frac{K(x,y)}{|x-y|^{-\alpha}} - \frac{K(x,y)}{|x-x_R|^{-\alpha}} \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{K(x,y)}{|x-x_R|^{-\alpha}} - \frac{K(x,x_R)}{|x-x_R|^{-\alpha}} \right| \cdot \frac{1}{|x-y|} \right]^q d\mu(x) \right\}^{\frac{1}{q}} \bigl|h(y)\bigr| \, d\mu(y) \\
&\le C \int_R \sum_{k=1}^{\infty} \left\{ \int_{2^{k+1}R\setminus 2^kR} \left[\left| \frac{K(x,y)}{|x-y|^{-\alpha}} - \frac{K(x,y)}{|x-x_R|^{-\alpha}} \right| \cdot \frac{1}{|x-y|} \right]^q d\mu(x) \right\}^{\frac{1}{q}} \bigl|h(y)\bigr| \, d\mu(y) \\
&\quad + C \int_R \sum_{k=1}^{\infty} \left\{ \int_{2^{k+1}R\setminus 2^kR} \left[\left| \frac{K(x,y)}{|x-x_R|^{-\alpha}} - \frac{K(x,x_R)}{|x-x_R|^{-\alpha}} \right| \cdot \frac{1}{|x-y|} \right]^q d\mu(x) \right\}^{\frac{1}{q}} \bigl|h(y)\bigr| \, d\mu(y) \\
&\le C \int_R \sum_{k=1}^{\infty} \ell(R) \left\{ \int_{2^{k+1}R\setminus 2^kR} \frac{1}{|x-y|^{q(n-\alpha+1)}} \, d\mu(x) \right\}^{\frac{1}{q}} \bigl|h(y)\bigr| \, d\mu(y) \\
&\quad + \int_R \sum_{k=1}^{\infty} \left(\int_{2^{k+1}R\setminus 2^kR} \left[\ell\bigl(2^kR\bigr)^{\alpha} \frac{|K(x,y) - K(x,x_R)|}{|x-y|} \right]^q d\mu(x) \right)^{\frac{1}{q}} \bigl|h(y)\bigr| \, d\mu(y) \\
&\le C \left(\sum_{j=1}^{2} |\lambda_j| \right).
\end{aligned}$$

Here we have used the fact that $\frac{1}{q} = 1 - \frac{\alpha}{n}$.

Combining the estimates for I, II and III yields that

$$\|\mathcal{M}(h)\|_{L^q(\mu)} \leq C|h|_{H^{1,\infty,0}_{\mathrm{atb}}(\mu)},$$

and this is the result of Theorem 2.5. □

3 Boundedness of $\mathcal{M}$ in RBMO(μ) spaces

In this section, we discuss the boundedness for $\mathcal{M}$ as in (1.4) in the space RBMO(μ) for $f \in M^q_p(\mu)$ and $f \in L^{\frac{n}{\alpha}}(\mu)$, respectively.

Firstly, we need to recall the definition of Morrey space with non-doubling measure denoted by $M^p_q(\mu)$, which was introduced by Sawano and Tanaka in [20].

Definition 3.1 Let $\nu > 1$ and $1 \leq q \leq p < \infty$. The Morrey space $M^p_q(\mu)$ is defined by

$$M^p_q(\mu) = \{f \in L^q_{\mathrm{loc}}(\mu) : \|f\|_{M^p_q(\mu)} < \infty\},$$

where the norm $\|f\|_{M^p_q(\mu)}$ is given by

$$\|f\|_{M^p_q(\mu)} = \sup_Q \mu(\nu Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \, d\mu(x)\right)^{\frac{1}{q}}.$$

We should note that the parameter $\nu > 1$ appearing in the definition does not affect the definition of the space $M^p_q(\mu)$, and $M^p_q(\mu)$ is a Banach space with its norms (see [20]). By using the Hölder inequality to (1.4), it is easy to see that for all $1 \leq q_2 \leq q_1 \leq p$, then

$$L^p(\mu) = M^p_p(\mu) \subset M^p_{q_1}(\mu) \subset M^p_{q_2}(\mu).$$

Theorem 3.2 *Let $0 < \alpha < n$, $1 \leq q < p = \frac{n}{\alpha}$. Suppose that $K(x,y)$ satisfies* (1.2) *and the $\mathcal{H}^{p'}$ condition, $\mathcal{M}$ is defined as in* (1.4). *Then there exists a positive constant C such that for all $f \in M^p_q(\mu)$,*

$$\|\mathcal{M}(f)\|_{\mathrm{RBMO}(\mu)} \leq C\|f\|_{M^p_q(\mu)}.$$

Theorem 3.3 *Let $0 < \alpha < n$ and $p = \frac{n}{\alpha}$. Suppose that $K(x,y)$ satisfies* (1.2) *and the $\mathcal{H}^{\frac{n}{n-\alpha}}$ condition, $\mathcal{M}$ is defined as in* (1.4). *Then there exists a positive constant C such that for all bounded functions f with compact support,*

$$\|\mathcal{M}(f)\|_{\mathrm{RBMO}(\mu)} \leq C\|f\|_{L^{\frac{n}{\alpha}}(\mu)}.$$

Remark 3.4 As a special condition, we take $p = q = \frac{n}{\alpha}$, Theorem 3.3 can be deduced with a similar method of Theorem 3.2.

Proof of Theorem 3.2 For any cubes Q and R in $\mathbb{R}^d$ such that $Q \subset R$ satisfies $\ell(R) \leq 2\ell(Q)$, let

$$a_Q = m_Q[\mathcal{M}(f\chi_{\mathbb{R}^d \setminus \frac{3}{2}Q})]$$

and

$$a_R = m_R\big[\mathcal{M}(f\chi_{\mathbb{R}^d\setminus\frac{3}{2}R})\big].$$

It is easy to see that a_Q and a_R are real numbers. By Lemma 1.2, we need to show that for some fixed $r > q$, there exists a constant $C > 0$ such that

$$\left(\frac{1}{\mu(2Q)}\int_Q \big|\mathcal{M}(f)(x) - a_Q\big|^r\,d\mu(x)\right)^{\frac{1}{r}} \le C\|f\|_{M_q^p(\mu)} \tag{3.1}$$

and

$$|a_Q - a_R| \le C\|f\|_{M_q^p(\mu)}. \tag{3.2}$$

Let us first prove estimate (3.1). For a fixed cube Q and $x \in Q$, decompose $f = f_1 + f_2$, where $f_1 = f\chi_{\frac{3}{2}Q}$ and $f_2 = f - f_1$. Write that

$$\begin{aligned}
&\frac{1}{\mu(2Q)}\int_Q \big|\mathcal{M}(f)(x) - a_Q\big|^r\,d\mu(x)\\
&\quad= \frac{1}{\mu(2Q)}\int_Q \big|\mathcal{M}(f_1+f_2)(x) - a_Q\big|^r\,d\mu(x)\\
&\quad\le \frac{1}{\mu(2Q)}\int_Q \big|\mathcal{M}(f_1)(x)\big|^r\,d\mu(x) + \frac{1}{\mu(2Q)}\int_Q \big|\mathcal{M}(f_2)(x) - a_Q\big|^r\,d\mu(x)\\
&\quad= \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

For $\frac{1}{r} = \frac{1}{q} - \frac{\alpha}{n}$ and $p = \frac{\alpha}{n}$, it follows that

$$\begin{aligned}
\mathrm{I}_1 &= \frac{1}{\mu(2Q)}\int_Q \big|\mathcal{M}(f_1)(x)\big|^r\,d\mu(x)\\
&\le C\frac{1}{\mu(2Q)}\left(\int_{\frac{3}{2}Q}\big|f(x)\big|^q\,d\mu(x)\right)^{\frac{r}{q}}\\
&\le C\frac{1}{\mu(2Q)}\left(\mu(2Q)^{\frac{1}{p}-\frac{1}{q}}\int_{\frac{3}{2}Q}\big|f(x)\big|^q\,d\mu(x)\right)^{\frac{r}{q}}\mu(2Q)^{r(\frac{1}{q}-\frac{1}{p})}\\
&\le C\|f\|^r_{M_q^p(\mu)}\mu(2Q)^{r(\frac{1}{q}-\frac{1}{p})-1}\\
&\le C\|f\|^r_{M_q^p(\mu)}.
\end{aligned}$$

Now let us estimate the term I_2,

$$\begin{aligned}
\mathrm{I}_2 &= \frac{1}{\mu(2Q)}\int_Q \big|\mathcal{M}(f_2)(x) - a_Q\big|^r\,d\mu(x)\\
&= \frac{1}{\mu(2Q)}\int_Q \left|\mathcal{M}(f_2)(x) - \frac{1}{\mu(Q)}\int_Q \mathcal{M}(f\chi_{\mathbb{R}^d\setminus\frac{3}{2}Q})(y)\,d\mu(y)\right|^r d\mu(x)\\
&= \frac{1}{\mu(2Q)}\int_Q \left|\frac{1}{\mu(Q)}\int_Q \mathcal{M}(f_2)(x)\,d\mu(y) - \frac{1}{\mu(Q)}\int_Q \mathcal{M}(f\chi_{\mathbb{R}^d\setminus\frac{3}{2}Q})(y)\,d\mu(y)\right|^r d\mu(x)\\
&\le \frac{1}{\mu(2Q)}\frac{1}{\mu(Q)}\int_Q\int_Q \big|\mathcal{M}(f_2)(x) - \mathcal{M}(f_2)(y)\big|^r\,d\mu(x)\,d\mu(y).
\end{aligned}$$

In order to estimate $|\mathcal{M}(f_2)(x) - \mathcal{M}(f_2)(y)|$, we write

$$D_1(x,y) = \left(\int_0^\infty \left[\int_{|x-z|\le t<|y-z|} \frac{|K(x,z)|}{|x-z|^{-\alpha}} f_2(z)\,d\mu(z)\right]^2 \frac{dt}{t^3}\right)^{\frac{1}{2}},$$

$$D_2(x,y) = \left(\int_0^\infty \left[\int_{|y-z|\le t<|x-z|} \frac{|K(y,z)|}{|y-z|^{-\alpha}} f_2(z)\,d\mu(z)\right]^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}$$

and

$$D_3(x,y) = \left(\int_0^\infty \left[\int_{\substack{|x-z|\le t\\ |y-z|\le t}} \left|\frac{K(x,z)}{|x-z|^{-\alpha}} - \frac{K(y,z)}{|y-z|^{-\alpha}}\right| \left|f_2(z)\right| d\mu(z)\right]^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}.$$

It is easy to get that for any $x, y \in Q$,

$$\begin{aligned}
&\left|\mathcal{M}(f_2)(x) - \mathcal{M}(f_2)(y)\right| \\
&\quad = \left|\left(\int_0^\infty \left|\int_{|x-z|\le t} \frac{K(x,z)}{|x-z|^{\alpha}}\,d\mu(z)\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} - \left(\int_0^\infty \left|\int_{|y-z|\le t} \frac{K(y,z)}{|y-z|^{\alpha}}\,d\mu(z)\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}\right| \\
&\quad \le \left(\int_0^\infty \left|\int_{|x-z|\le t} \frac{K(x,z)}{|x-z|^{-\alpha}} f_2(z)\,d\mu(z) - \int_{|y-z|\le t} \frac{K(y,z)}{|y-z|^{-\alpha}} f_2(z)\,d\mu(z)\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\quad \le \left(\int_0^\infty \left|\int_{|x-z|\le t<|y-z|} \frac{K(x,z)}{|x-z|^{-\alpha}} f_2(z)\,d\mu(z) + \int_{|y-z|\le t} \frac{K(x,z)}{|x-z|^{-\alpha}} f_2(z)\,d\mu(z)\right.\right. \\
&\qquad \left.\left. - \int_{|y-z|\le t<|x-z|} \frac{K(y,z)}{|y-z|^{-\alpha}} f_2(z)\,d\mu(z) - \int_{|x-z|\le t} \frac{K(y,z)}{|y-z|^{-\alpha}} f_2(z)\,d\mu(z)\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\quad \le \left(\int_0^\infty \left|\int_{|x-z|\le t<|y-z|} \frac{K(x,z)}{|x-z|^{-\alpha}} f_2(z)\,d\mu(z)\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\qquad + \left(\int_0^\infty \left|\int_{|y-z|\le t<|x-z|} \frac{K(y,z)}{|y-z|^{-\alpha}} f_2(z)\,d\mu(z)\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\qquad + \left\{\int_0^\infty \left[\int_{\substack{|x-z|\le t\\ |y-z|\le t}} \left(\frac{K(x,z)}{|x-z|^{-\alpha}} - \frac{K(y,z)}{|y-z|^{-\alpha}}\right) f_2(z)\,d\mu(z)\right]^2 \frac{dt}{t^3}\right\}^{\frac{1}{2}} \\
&\quad \le \sum_{j=1}^{3} D_j(x,y).
\end{aligned}$$

For $D_1(x,y)$, since $x, y \in Q$, $z \in \frac{3}{2}Q$, thus we get

$$\begin{aligned}
D_1(x,y) &\le C\left(\int_0^\infty \left[\int_{|x-z|\le t<|y-z|} \frac{|f_2(z)|}{|x-z|^{n-\alpha-1}}\,d\mu(z)\right]^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\le C \int_{|x-z|<|y-z|} \frac{|f_2(z)|}{|x-z|^{n-\alpha-1}} \left(\int_{|x-z|}^{|y-z|} \frac{dt}{t^3}\right)^{\frac{1}{2}} d\mu(z) \\
&\le C\ell(Q)^{\frac{1}{2}} \int_{|x-z|<|y-z|} \frac{|f_2(z)|}{|x-z|^{n-\alpha+\frac{1}{2}}}\,d\mu(z)
\end{aligned}$$

$$\begin{aligned}
&\le C\ell(Q)^{\frac{1}{2}} \int_{\mathbb{R}^d\setminus\frac{3}{2}Q} \frac{|f_2(z)|}{|x-z|^{n-\alpha+\frac{1}{2}}}\,d\mu(z) \\
&\le C\ell(Q)^{\frac{1}{2}} \sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} \frac{|f_2(z)|}{|x-z|^{n-\alpha+\frac{1}{2}}}\,d\mu(z) \\
&\le C\ell(Q)^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{1}{\ell(\frac{3}{2}2^kQ)^{n-\alpha+\frac{1}{2}}} \int_{2^{k+1}Q} \bigl|f_2(z)\bigr|\,d\mu(z) \\
&\le C\sum_{k=1}^{\infty} 2^{-\frac{k}{2}} \frac{1}{\ell(\frac{3}{2}2^kQ)^{n-\alpha}} \biggl(\int_{2^{k+1}Q} \bigl|f_2(z)\bigr|^q\,d\mu(z)\biggr)^{\frac{1}{q}} \mu\biggl(\frac{3}{2}2^kQ\biggr)^{1-\frac{1}{q}} \\
&\le C\|f\|_{M_q^p(\mu)} \sum_{k=1}^{\infty} 2^{-\frac{k}{2}} \\
&\le C\|f\|_{M_q^p(\mu)}.
\end{aligned}$$

By a similar argument, it follows that

$$D_2(x,y) \le C\|f\|_{M_q^p(\mu)}.$$

Finally, by the condition $\mathcal{H}^{p'}$, which the kernel $K(x,y)$ conditions, applying Minkowski's inequality, and the fact that $\alpha = \frac{n}{p}$, we have

$$\begin{aligned}
D_3(x,y) &= \biggl(\int_0^{\infty}\biggl[\int_{\substack{|x-z|\le t\\|y-z|\le t}} \biggl|\frac{K(x,z)}{|x-z|^{-\alpha}} - \frac{K(y,z)}{|y-z|^{-\alpha}}\biggr| \bigl|f_2(z)\bigr|\,d\mu(z)\biggr]^2 \frac{dt}{t^3}\biggr)^{\frac{1}{2}} \\
&\le C\int_{\mathbb{R}^d\setminus\frac{3}{2}Q} \biggl|\frac{K(x,z)}{|x-z|^{-\alpha}} - \frac{K(y,z)}{|y-z|^{-\alpha}}\biggr| \bigl|f(z)\bigr| \biggl(\int_{\substack{|x-z|\le t\\|y-z|\le t}} \frac{dt}{t^3}\biggr)^{\frac{1}{2}}\,d\mu(z) \\
&\le C\sum_{k=1}^{\infty} \int_{\frac{3}{2}2^{k+1}Q\setminus\frac{3}{2}2^kQ} \biggl|\frac{K(x,z)}{|x-z|^{-\alpha}} - \frac{K(y,z)}{|y-z|^{-\alpha}}\biggr| \frac{|f(z)|}{|y-z|}\,d\mu(z) \\
&\le C\|f\|_{M_q^p(\mu)} \sum_{k=1}^{\infty} \mu\bigl(2^kQ\bigr)^{\frac{1}{q}-\frac{1}{p}} \\
&\quad\times \biggl\{\int_{\frac{3}{2}2^{k+1}Q\setminus\frac{3}{2}2^kQ} \biggl[\frac{1}{|y-z|}\biggl|\frac{K(x,z)}{|x-z|^{-\alpha}} - \frac{K(y,z)}{|y-z|^{-\alpha}}\biggr|\biggr]^{q'}\,d\mu(z)\biggr\}^{\frac{1}{q'}} \\
&\le C\|f\|_{M_q^p(\mu)} \sum_{k=1}^{\infty} \ell\biggl(\frac{3}{2}2^kQ\biggr)^{\frac{n}{q}-\frac{n}{p}} \\
&\quad\times \biggl\{\int_{\frac{3}{2}2^{k+1}Q\setminus\frac{3}{2}2^kQ} \biggl[\frac{1}{|y-z|}\biggl|\frac{K(x,z)}{|x-z|^{-\alpha}} - \frac{K(x,z)}{|y-z|^{-\alpha}} \\
&\quad + \frac{K(x,z)}{|y-z|^{-\alpha}} - \frac{K(y,z)}{|y-z|^{-\alpha}}\biggr|\biggr]^{q'}\,d\mu(z)\biggr\}^{\frac{1}{q'}} \\
&\le C\|f\|_{M_q^p(\mu)} \sum_{k=1}^{\infty} \ell\biggl(\frac{3}{2}2^kQ\biggr)^{\alpha-\frac{n}{p}} \ell\biggl(\frac{3}{2}2^kQ\biggr)^{n} \\
&\quad\times \biggl\{\frac{1}{\ell(\frac{3}{2}2^kQ)^n} \int_{\frac{3}{2}2^{k+1}Q\setminus\frac{3}{2}2^kQ} \biggl[\bigl|K(x,z)-K(y,z)\bigr|\frac{1}{|y-z|}\biggr]^{q'}\,d\mu(z)\biggr\}^{\frac{1}{q'}}
\end{aligned}$$

$$\begin{aligned}
&+ C\|f\|_{M^p_q(\mu)} \sum_{k=1}^{\infty} \ell\left(\frac{3}{2}2^k Q\right)^{\frac{n}{q}-\frac{n}{p}} \ell(Q)^{\alpha} \left(\int_{\frac{3}{2}2^{k+1}Q\setminus\frac{3}{2}2^k Q} \frac{1}{|y-z|^{nq'}}\, d\mu(z)\right)^{\frac{1}{q'}} \\
&\le C\|f\|_{M^p_q(\mu)}.
\end{aligned}$$

Combining these estimates, we conclude that

$$\mathrm{I}_2 \le C\|f\|_{M^p_q(\mu)},$$

and so estimate (3.1) is proved.

We proceed to show (3.2). For any cubes $Q \subset R$ with $x \in Q$, denote $N_{Q,R+1}$ simply by N. Write

$$\begin{aligned}
|a_Q - a_R| &\le \left|m_R\left[\mathcal{M}(f\chi_{\mathbb{R}^d\setminus 2^N Q})\right] - m_Q\left[\mathcal{M}(f\chi_{\mathbb{R}^d\setminus 2^N R})\right]\right| \\
&\quad + \left|m_Q\left[\mathcal{M}(f\chi_{2^N Q\setminus\frac{3}{2}Q})\right]\right| + \left|m_R\left[\mathcal{M}(f\chi_{2^N Q\setminus\frac{3}{2}R})\right]\right| \\
&= E_1 + E_2 + E_3.
\end{aligned}$$

As in the estimate for the term I_2, then

$$E_2 \le C\|f\|_{M^p_q(\mu)}.$$

We conclude from $y \in R$, $z \in 2^N Q \setminus \frac{3}{2}Q$ that

$$\begin{aligned}
\mathcal{M}(f\chi_{2^N Q\setminus\frac{3}{2}R})(y) &\le C\int_{2^N Q\setminus\frac{3}{2}R} \left|\frac{K(y,z)}{|y-z|^{-\alpha}}\right| \left(\int_{|y-z|}^{\infty} \frac{dt}{t^3}\right)^{\frac{1}{2}} d\mu(z) \\
&\le C\int_{2^N Q\setminus\frac{3}{2}R} \frac{|f(z)|}{|y-z|^{n-\alpha}}\, d\mu(z) \\
&\le C\ell(R)^{\alpha-n} \int_{2^N Q\setminus\frac{3}{2}R} \left|f(z)\right| d\mu(z) \\
&\le C\ell(R)^{\alpha-n} \left(\int_{2^N Q\setminus\frac{3}{2}R} \left|f(z)\right|^q d\mu(z)\right)^{\frac{1}{q}} \mu\left(2^N Q\right)^{1-\frac{1}{q}} \\
&\le C\ell(R)^{\alpha-n} \mu\left(2^N Q\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{2^N Q} \left|f(z)\right|^q d\mu(z)\right)^{\frac{1}{q}} \mu\left(2^N Q\right)^{1-\frac{1}{p}} \\
&\le C\|f\|_{M^p_q(\mu)} \ell\left(2^N Q\right)^{\alpha-\frac{n}{p}} \\
&\le C\|f\|_{M^p_q(\mu)}.
\end{aligned}$$

Taking mean over $y \in R$, we obtain

$$E_3 \le C\|f\|_{M^p_q(\mu)}.$$

Analysis similar to that in the estimates for E_3 shows that

$$E_2 \le C\|f\|_{M^p_q(\mu)}.$$

Finally, we get (3.2) and this is precisely the assertion of Theorem 3.2. □

Competing interests
The authors declare that they do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
Jiang Zhou is supported by the National Science Foundation of China (Grant No. 11261055) and the National Natural Science Foundation of Xinjiang (Grant Nos. 2011211A005, BS120104).

Received: 26 November 2013 Accepted: 11 July 2014 Published: 18 August 2014

References

1. Deng, D, Han, Y, Yang, D: Besov spaces with non-doubling measures. Trans. Am. Math. Soc. **358**(7), 2965-3001 (2006)
2. Han, Y, Yang, D: Triebel-Lizorkin spaces with non-doubling measures. Stud. Math. **162**(2), 105-140 (2004)
3. Hu, G, Meng, Y, Yang, D: New atomic characterization of H^1 space with non-doubling measures and its applications. Math. Proc. Camb. Philos. Soc. **138**(1), 151-171 (2005)
4. Nazarov, F, Treil, S, Volberg, A: Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on non-homogeneous spaces. Int. Math. Res. Not. **9**, 463-487 (1998)
5. Nazarov, F, Treil, S, Volberg, A: Accretive system Tb-theorems on non-homogeneous spaces. Duke Math. J. **113**(2), 259-312 (2002)
6. Nazarov, F, Treil, S, Volberg, A: The Tb-theorems on non-homogeneous spaces. Acta Math. **190**(2), 151-239 (2003)
7. Tolsa, X: Littlewood-Paley theory and the $T(1)$ theorem with non-doubling measures. Adv. Math. **164**(1), 57-116 (2001)
8. Yang, D, Yang, D: Uniform boundedness for approximations of the identity with non-doubling measures. J. Inequal. Appl. **2007**, Article ID 19574 (2007)
9. Verdera, J: The fall of doubling condition in Calderón-Zygmund theory. Publ. Math. **Extra**, 275-292 (2002)
10. Tolsa, X: Painlevé's problem and the semiadditivity of analytic capacity. Acta Math. **190**(1), 105-149 (2003)
11. Si, Z, Wang, L, Jiang, Y: Fractional type Marcinkiewicz integral on Hardy spaces. J. Math. Res. Expo. **31**(2), 233-241 (2011)
12. Stein, E: On the function of Littlewood-Paley, Lusin and Marcinkiewicz. Trans. Am. Math. Soc. **88**, 430-466 (1958)
13. Lu, S: Marcinkiewicz integrals with rough kernels. Front. Math. China **3**, 1-14 (2008)
14. Hu, G, Lin, H, Yang, D: Marcinkiewicz integrals with non-doubling measures. Integral Equ. Oper. Theory **58**, 205-238 (2007)
15. Tolsa, X: BMO, H^1 and Calderón-Zygmund operators for non-doubling measures. Math. Ann. **319**, 89-149 (2001)
16. García-Cuerva, J, Gatto, A: Lipschitz spaces and Calderón-Zygmund operators associated to non-doubling measures. Publ. Mat. **49**, 258-296 (2005)
17. Ding, Y, Yang, D: Weighted norm inequalities for fractional integral operators with rough kernel. Can. J. Math. **50**(1), 29-39 (1998)
18. Tolsa, X: The space H^1 for non-doubling measure in terms of a grand maximal operator. Trans. Am. Math. Soc. **355**, 315-348 (2003)
19. Meda, S, Sjögren, P, Vallarino, M: On the H^1-L^1 boundedness of operators. Proc. Am. Math. Soc. **136**, 2921-2931 (2008)
20. Sawano, Y, Tanaka, H: Morrey space for non-doubling measures. Acta Math. Sin. **21**(6), 1535-1544 (2005)

doi:10.1186/1029-242X-2014-285
Cite this article as: Lu and Zhou: **Estimates for fractional type Marcinkiewicz integrals with non-doubling measures.** *Journal of Inequalities and Applications* 2014 **2014**:285.