Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Viscosity approximation methods for two nonexpansive semigroups in CAT(0) spaces

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Abstract

The purpose of this paper is by using the viscosity approximation method to study the strong convergence problem for two *one-parameter continuous semigroups of nonexpansive mappings in CAT(0) spaces*. Under suitable conditions, some strong convergence theorems for the proposed implicit and explicit iterative schemes to converge to a common fixed point of two one-parameter continuous semigroups of nonexpansive mappings are proved, which is also a unique solution of some kind of variational inequalities. The results presented in this paper extend and improve the corresponding results of some others.

MSC: 47J05; 47H09; 49J25

Keywords: viscosity approximation method; nonexpansive semigroup; implicit and explicit iterative scheme; CAT(0) space; fixed point

1 Introduction

Throughout this paper, we assume that X is a CAT(0) space, $\mathbb{N}$ is the set of positive integers, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ is the set of nonnegative real numbers and C is a nonempty closed and convex subset of a complete CAT(0) space X.

A family of mappings $\mathcal{T} := \{T(t) : t \in \mathbb{R}^+\} : C \to C$ is called a *one-parameter continuous semigroup of nonexpansive mappings* if the following conditions are satisfied:

(i) for each $t \in \mathbb{R}^+$, $T(t)$ is a nonexpansive mapping on C, *i.e.*,

$$d\big(T(t)x, T(t)y\big) \le d(x,y), \quad \forall x,y \in C;$$

(ii) $T(s+t) = T(t) \circ T(s)$ for all $t,s \in \mathbb{R}^+$;

(iii) for each $x \in X$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into C is continuous.

A family of mappings $\mathcal{T} := \{T(t) : t \in \mathbb{R}^+\}$ is called a *one-parameter strongly continuous semigroup of nonexpansive mappings* if conditions (i), (ii), (iii) and the following condition are satisfied:

(iv) $T(0)x = x$ for all $x \in C$.

In the sequel, we shall denote by $\mathcal{F}$ the common fixed point set of $\mathcal{T}$, that is,

$$\mathcal{F} := F(\mathcal{T}) = \big\{x \in C : T(t)x = x, t \in \mathbb{R}^+\big\} = \bigcap_{t\in\mathbb{R}^+} F\big(T(t)\big).$$

It is well known that one classical way to study nonexpansive mappings is to use the contractions to approximate nonexpansive mappings. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t = tu + (1-t)Tx, \quad \forall x \in C, \tag{1.1}$$

where $u \in C$ is an arbitrary fixed element. In the case of T having a fixed point, Browder [1] proved that x_t converged strongly to a fixed point of T that is nearest to u in the framework of Hilbert spaces. Reich [2] extended Browder's result to the setting of a uniformly smooth Banach space and proved that x_t converged strongly to a fixed point of T.

Halpern [3] introduced the following explicit iterative scheme (1.2) for a nonexpansive mapping T on a subset C of a Hilbert space:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n. \tag{1.2}$$

He proved that the sequence $\{x_n\}$ converged to a fixed point of T. In [4], Shioji and Takahashi introduced the following implicit iteration in a Hilbert space:

$$x_n = \alpha_n u + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(t)x_n\,dt. \tag{1.3}$$

Under suitable conditions, they proved strong convergence of $\{x_n\}$ to a member of $\mathcal{F}$.

Later, Suzuki [5] introduced in a Hilbert space the following iteration process:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad \forall n \geq 1, \tag{1.4}$$

where $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on C such that $\mathcal{F} \neq \emptyset$. Under suitable conditions he proved that $\{x_n\}$ converged strongly to the element of $\mathcal{F}$ nearest to u. Using Moudafi's viscosity approximation methods, Song and Xu [6], Cho and Kang [7] introduced the following iteration process:

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad \forall n \geq 1, \tag{1.5}$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad \forall n \geq 1. \tag{1.6}$$

They proved that $\{x_n\}$ defined by (1.5) and (1.6) both converged to the same point of $\mathcal{F}$ in a reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm.

In a similar way, Dhompongsa *et al.* [8] extended Browder's implicit iteration to a strongly continuous semigroup of nonexpansive mappings $\{T(t) : t \geq 0\}$ in a complete CAT(0) space X. Under suitable conditions he proved that the sequence converged strongly to the element of $\mathcal{F}$ nearest to u. Using Moudafi's viscosity approximation methods, Shi and Chen [9] studied the convergence theorems of the following Moudafi's viscosity iterations for a nonexpansive mapping T:

$$x_t = tf(x_t) \oplus (1-t)Tx_t, \tag{1.7}$$

and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n. \tag{1.8}$$

They proved that $\{x_t\}$ defined by (1.7) and $\{x_n\}$ defined by (1.8) converged strongly to a fixed point of T in the framework of CAT(0) spaces.

Very recently, Wangkeeree and Preechasilp [10] extended the results of [9] to a one-parameter continuous semigroup of nonexpansive mappings $\mathcal{T} := \{T(t) : t \in \mathbb{R}^+\}$ in CAT(0) spaces. Under suitable conditions they proved that the iterative schemes $\{x_n\}$ both converged strongly to the same point $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$, which is the unique solution of the variational inequality

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{1.9}$$

Motivated and inspired by the research going on in this direction, especially inspired by Wangkeeree and Preechasilp [10], in this paper we study the strong convergence theorems of Moudafi's viscosity approximation methods for two one-parameter continuous semigroups of nonexpansive mappings in CAT(0) spaces. We prove that the implicit and explicit iteration algorithms both converge strongly to the same point $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$, which is the unique solution of the variational inequality (1.9) where $\mathcal{F}$ is the set of common fixed points of the two semigroups of nonexpansive mappings.

2 Preliminaries and lemmas

In this paper, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x,z) = td(x,y), \qquad d(y,z) = (1-t)d(x,y). \tag{2.1}$$

Lemma 2.1 [11] *A geodesic space X is a CAT(0) space if and only if the following inequality*

$$d^2\big((1-t)x \oplus ty, z\big) \leq (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y) \tag{2.2}$$

is satisfied for all $x,y,z \in X$ and $t \in [0,1]$. In particular, if x, y, z are points in a CAT(0) space and $t \in [0,1]$, then

$$d\big((1-t)x \oplus ty, z\big) \leq (1-t)d(x,z) + td(y,z). \tag{2.3}$$

Lemma 2.2 [12] *Let X be a CAT(0) space, $p,q,r,s \in X$ and $\lambda \in [0,1]$. Then*

$$d\big(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s\big) \leq \lambda d(p,r) + (1-\lambda)d(q,s).$$

By induction, we write

$$\bigoplus_{m=1}^{n} \lambda_m x_m := (1-\lambda_n)\left(\frac{\lambda_1}{1-\lambda_n}x_1 \oplus \frac{\lambda_2}{1-\lambda_n}x_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1-\lambda_n}x_{n-1}\right) \oplus \lambda_n x_n. \tag{2.4}$$

Lemma 2.3 *Let X be a CAT(0) space, then, for any sequence $\{\lambda_m\}_{m=1}^n$ in $[0,1]$ satisfying $\sum_{m=1}^n \lambda_m = 1$ and for any $\{x_m\}_{m=1}^n \subset X$, the following conclusions hold*:

$$d\left(\bigoplus_{m=1}^{n} \lambda_m x_m, x\right) \leq \sum_{m=1}^{n} \lambda_m d(x_m, x), \quad x \in X; \tag{2.5}$$

and

$$d^2\left(\bigoplus_{m=1}^{n} \lambda_m x_m, x\right) \leq \sum_{m=1}^{n} \lambda_m d^2(x_m, x) - \lambda_1\lambda_2 d^2(x_1, x_2), \quad x \in X. \tag{2.6}$$

Proof It is obvious that (2.5) holds for $n = 2$. Suppose that (2.5) holds for some $n \geq 2$. From (2.3) and (2.4) we have

$$\begin{aligned}
&d\left(\bigoplus_{m=1}^{n+1} \lambda_m x_m, x\right) \\
&\quad = d\left((1-\lambda_{n+1})\left(\frac{\lambda_1}{1-\lambda_{n+1}}x_1 \oplus \frac{\lambda_2}{1-\lambda_{n+1}}x_2 \oplus \cdots \oplus \frac{\lambda_n}{1-\lambda_{n+1}}x_n\right) \oplus \lambda_{n+1}x_{n+1}, x\right) \\
&\quad \leq (1-\lambda_{n+1})d\left(\frac{\lambda_1}{1-\lambda_{n+1}}x_1 \oplus \frac{\lambda_2}{1-\lambda_{n+1}}x_2 \oplus \cdots \oplus \frac{\lambda_n}{1-\lambda_{n+1}}x_n, x\right) + \lambda_{n+1}d(x_{n+1}, x) \\
&\quad \leq \lambda_1 d(x_1, x) + \lambda_2 d(x_2, x) + \cdots + \lambda_n d(x_n, x) + \lambda_{n+1} d(x_{n+1}, x) \\
&\quad = \sum_{m=1}^{n+1} \lambda_m d(x_m, x).
\end{aligned}$$

This implies that (2.5) holds.

Next, we prove that (2.6) holds.

Indeed, it is obvious that (2.6) holds for $n = 2$. Suppose that (2.6) holds for some $n \geq 2$. Next we prove that (2.6) is also true for $n + 1$.

In fact, we have

$$d^2\left(\bigoplus_{m=1}^{n+1} \lambda_m x_m, x\right) = d^2\left(\bigoplus_{m=1}^{n} \lambda_m x_m \oplus \lambda_{n+1}x_{n+1}, x\right).$$

From (2.2) and (2.4) and the assumption of induction, we have

$$\begin{aligned}
&d^2\left(\bigoplus_{m=1}^{n+1} \lambda_m x_m, x\right) \\
&\quad = d^2\left(\bigoplus_{m=1}^{n} \lambda_m x_m \oplus \lambda_{n+1}x_{n+1}, x\right) \\
&\quad = d^2\left((1-\lambda_{n+1})\bigoplus_{m=1}^{n} \frac{\lambda_m}{1-\lambda_{n+1}}x_m \oplus \lambda_{n+1}x_{n+1}, x\right) \\
&\quad \leq (1-\lambda_{n+1})d^2\left(\bigoplus_{m=1}^{n} \frac{\lambda_m}{1-\lambda_{n+1}}x_m, x\right) + \lambda_{n+1}d^2(x_{n+1}, x)
\end{aligned}$$

$$\leq (1-\lambda_{n+1})\sum_{m=1}^{n}\frac{\lambda_m}{1-\lambda_{n+1}}d^2(x_m,x) - \lambda_1\lambda_2 d^2(x_1,x_2) + \lambda_{n+1}d^2(x_{n+1},x)$$
$$= \sum_{m=1}^{n+1}\lambda_m d^2(x_m,x) - \lambda_1\lambda_2 d^2(x_1,x_2).$$

This completes the proof of (2.6). □

The concept of Δ-convergence introduced by Lim [13] in 1976 was shown by Kirk and Panyanak [14] in CAT(0) spaces to be very similar to the weak convergence in the Banach space setting (see also [15]). Now, we give the concept of Δ-convergence.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r\bigl(x,\{x_n\}\bigr) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\bigl(\{x_n\}\bigr) = \inf_{x\in X}\bigl\{r\bigl(x,\{x_n\}\bigr)\bigr\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\bigl(\{x_n\}\bigr) = \bigl\{x\in X : r\bigl(x,\{x_n\}\bigr) = r\bigl(\{x_n\}\bigr)\bigr\}.$$

It is known from Proposition 7 of [16] that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

The uniqueness of an asymptotic center implies that a CAT(0) space X satisfies *Opial's property*, *i.e.*, for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n,x) < \limsup_{n\to\infty} d(x_n,y).$$

Lemma 2.4 [14] *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Berg and Nikolaev [17] *introduced the concept of quasilinearization as follows. Let us denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is defined as a map $\langle\cdot,\cdot\rangle : (X\times X)\times(X\times X) \to \mathbb{R}$ defined by*

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = \frac{1}{2}\bigl(d^2(a,d)+d^2(b,c)-d^2(a,c)-d^2(b,d)\bigr) \quad (a,b,c,d\in X). \tag{2.7}$$

It is easily seen that $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = \langle\overrightarrow{cd},\overrightarrow{ab}\rangle$, $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = -\langle\overrightarrow{ba},\overrightarrow{cd}\rangle$ and $\langle\overrightarrow{ax},\overrightarrow{cd}\rangle + \langle\overrightarrow{xb},\overrightarrow{cd}\rangle = \langle\overrightarrow{ab},\overrightarrow{cd}\rangle$ for all $a,b,c,d \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle \leq d(a,b)d(c,d) \tag{2.8}$$

for all $a,b,c,d \in X$.

Recently, Dehghan and Rooin [18] presented a characterization of metric projection in CAT(0) spaces as follows.

Lemma 2.5 *Let C be a nonempty convex subset of a complete CAT(0) space X, $x \in X$ and $u \in C$. Then $u = P_C x$ if and only if*

$$\langle \overrightarrow{yu}, \overrightarrow{ux} \rangle \leq 0, \quad \forall y \in C. \tag{2.9}$$

Lemma 2.6 [19] *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0$ for all $y \in X$.*

Lemma 2.7 [20] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n$, $n \geq 0$, where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset \mathbb{R}$ such that*
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\alpha_n\beta_n| < \infty$.
Then $\{a_n\}$ converges to zero as $n \to \infty$.

3 Viscosity approximation iteration algorithms

In this section, we present the strong convergence theorems of Moudafi's viscosity approximation implicit and explicit iteration algorithms for two one-parameter continuous semigroups of nonexpansive mappings $\mathcal{T} := \{T(t) : t \in \mathbb{R}^+\}$ and $\mathcal{S} := \{S(s) : s \in \mathbb{R}^+\}$ in CAT(0) spaces.

Before proving main results, we need the following two vital lemmas.

Lemma 3.1 [10, 21] *Let X be a complete CAT(0) space. Then, for all $u, x, y \in X$, the following inequality holds*:

$$d^2(x,u) \leq d^2(y,u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Lemma 3.2 *Let X be a complete CAT(0) space. For any $u, v, w \in X$ and $r, s, t \in [0,1]$, $r + s + t = 1$, let $z = ru \oplus sv \oplus tw$. Then, for any $x, y \in X$, the following inequality holds*:

$$\langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \leq r\langle \overrightarrow{ux}, \overrightarrow{zy} \rangle + s\langle \overrightarrow{vx}, \overrightarrow{zy} \rangle + t\langle \overrightarrow{wx}, \overrightarrow{zy} \rangle + rtd^2(u,w) + std^2(v,w).$$

Proof It follows from (2.1) and (2.6) that

$$\begin{aligned}
d^2(u,z) &= d^2\left(u, ru \oplus (1-r)\left(\frac{s}{1-r}v \oplus \frac{t}{1-r}w\right)\right) \\
&= (1-r)^2 d^2\left(u, \frac{s}{1-r}v \oplus \frac{t}{1-r}w\right) \\
&\leq (1-r)^2\left(\frac{s}{1-r}d^2(u,v) + \frac{t}{1-r}d^2(u,w) - \frac{s}{1-r}\cdot\frac{t}{1-r}d^2(v,w)\right) \\
&= (1-r)sd^2(u,v) + (1-r)td^2(u,w) - std^2(v,w).
\end{aligned}$$

Similarly, we can obtain $d^2(v,z) \le (1-s)rd^2(v,u) + (1-s)td^2(v,w) - rtd^2(u,w)$ and $d^2(w,z) \le (1-t)rd^2(w,u) + (1-t)sd^2(w,v) - rsd^2(u,v)$. Therefore, we have

$$\begin{aligned} &rd^2(u,z) + sd^2(v,z) + td^2(w,z) \\ &\quad \le (1-r)rsd^2(u,v) + (1-r)rtd^2(u,w) - rstd^2(v,w) \\ &\qquad + (1-s)srd^2(v,u) + (1-s)std^2(v,w) - rstd^2(u,w) \\ &\qquad + (1-t)trd^2(w,u) + (1-t)tsd^2(w,v) - rstd^2(u,v) \\ &\quad = rsd^2(u,v) + rtd^2(u,w) + std^2(v,w). \end{aligned} \tag{3.1}$$

From (2.6) and (3.1), we have that

$$\begin{aligned} 2\langle \overrightarrow{zx}, \overrightarrow{zy} \rangle &= d^2(z,y) + d^2(x,z) - d^2(x,y) \\ &\le rd^2(u,y) + sd^2(v,y) + td^2(w,y) - rsd^2(u,v) + rd^2(x,z) \\ &\quad + sd^2(x,z) + td^2(x,z) - rd^2(x,y) - sd^2(x,y) - td^2(x,y) \\ &= 2r\langle \overrightarrow{ux}, \overrightarrow{zy} \rangle + 2s\langle \overrightarrow{vx}, \overrightarrow{zy} \rangle + 2t\langle \overrightarrow{wx}, \overrightarrow{zy} \rangle - rsd^2(u,v) \\ &\quad + rd^2(u,z) + sd^2(v,z) + td^2(w,z) \\ &\le 2r\langle \overrightarrow{ux}, \overrightarrow{zy} \rangle + 2s\langle \overrightarrow{vx}, \overrightarrow{zy} \rangle + 2t\langle \overrightarrow{wx}, \overrightarrow{zy} \rangle + rtd^2(u,w) + std^2(v,w), \end{aligned}$$

which is the desired result. □

Now we are in a position to state and prove our main results.

Theorem 3.3 Let C be a closed convex subset of a complete CAT(0) space X, and let $\{T(t)\}$ and $\{S(s)\}$ be two one-parameter continuous semigroups of nonexpansive mappings on C satisfying $\mathcal{F} := F(\mathcal{T}) \cap F(\mathcal{S}) \neq \emptyset$ and both uniformly asymptotically regular (in short, u.a.r.) on C, that is, for all $h, k \ge 0$ and any bounded subset B of C,

$$\lim_{t\to\infty} \sup_{x\in B} d\bigl(T(h)\bigl(T(t)x\bigr), T(t)x\bigr) = 0, \qquad \lim_{s\to\infty} \sup_{x\in B} d\bigl(S(k)\bigl(S(s)x\bigr), S(s)x\bigr) = 0.$$

Let f be a contraction on C with coefficient $\alpha \in (0,1)$. Suppose that the sequence $\{x_n\}$ is given by

$$x_n = \alpha_n f(x_n) \oplus \beta_n T(t_n)x_n \oplus \gamma_n S(s_n)x_n \tag{3.2}$$

for all $n \ge 0$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $t_n, s_n \in [0,\infty)$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\gamma_n = o(\alpha_n)$;
(iii) $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} s_n = \infty$;
(iv) for any bounded subset B of C, $\lim_{n\to\infty} \sup_{x\in B} \langle T(t_n)x, S(s_n)x \rangle = 0$.

Then $\{x_n\}$ converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$, which is equivalent to the following variational inequality:

$$\bigl\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \bigr\rangle \ge 0, \quad \forall x \in \mathcal{F}. \tag{3.3}$$

Proof We shall divide the proof of Theorem 3.3 into five steps.

Step 1. The sequence $\{x_n\}$ defined by (3.2) is well defined for all $n \geq 0$.

In fact, let us define mappings $G, M : C \to C$ by

$$G_n(x) := \alpha_n f(x) \oplus \beta_n T(t_n)x \oplus \gamma_n S(s_n)x, \quad x \in C$$

and

$$M_n(x) := \frac{\beta_n}{1-\alpha_n} T(t_n)x \oplus \frac{\gamma_n}{1-\alpha_n} S(s_n)x, \quad x \in C,$$

respectively. For any $x, y \in C$, from Lemma 2.2, we have

$$\begin{aligned} & d\big(M_n(x), M_n(y)\big) \\ &\quad = d\left(\frac{\beta_n}{1-\alpha_n} T(t_n)x \oplus \frac{\gamma_n}{1-\alpha_n} S(s_n)x, \frac{\beta_n}{1-\alpha_n} T(t_n)y \oplus \frac{\gamma_n}{1-\alpha_n} S(s_n)y\right) \\ &\quad \leq \frac{\beta_n}{1-\alpha_n} d\big(T(t_n)x, T(t_n)y\big) + \frac{\gamma_n}{1-\alpha_n} d\big(S(s_n)x, S(s_n)y\big) \\ &\quad \leq \frac{\beta_n}{1-\alpha_n} d(x,y) + \frac{\gamma_n}{1-\alpha_n} d(x,y) = d(x,y). \end{aligned}$$

Therefore we have that

$$\begin{aligned} & d\big(G_n(x), G_n(y)\big) \\ &\quad = d\big(\alpha_n f(x) \oplus (1-\alpha_n) M_n(x), \alpha_n f(y) \oplus (1-\alpha_n) M_n(y)\big) \\ &\quad \leq \alpha_n d\big(f(x), f(y)\big) + (1-\alpha_n) d\big(M_n(x), M_n(y)\big) \\ &\quad \leq \alpha_n \alpha d(x,y) + (1-\alpha_n) d(x,y) \\ &\quad = \big(1 - \alpha_n(1-\alpha)\big) d(x,y). \end{aligned}$$

This implies that G_n is a contraction mapping. Hence, the sequence $\{x_n\}$ is well defined for all $n \geq 0$.

Step 2. The sequence $\{x_n\}$ is bounded.

For any $p \in \mathcal{F}$, from Lemma 2.3, we have that

$$\begin{aligned} d(x_n, p) &= d\big(\alpha_n f(x_n) \oplus \beta_n T(t_n)x_n \oplus \gamma_n S(s_n)x_n, p\big) \\ &\leq \alpha_n d\big(f(x_n), p\big) + \beta_n d\big(T(t_n)x_n, p\big) + \gamma_n d\big(S(s_n)x_n, p\big) \\ &\leq \alpha_n d\big(f(x_n), p\big) + \beta_n d(x_n, p) + \gamma_n d(x_n, p) \\ &= \alpha_n d\big(f(x_n), p\big) + (1-\alpha_n) d(x_n, p). \end{aligned} \tag{3.4}$$

Then

$$d(x_n, p) \leq d\big(f(x_n), p\big) \leq d\big(f(x_n), f(p)\big) + d\big(f(p), p\big) \leq \alpha d(x_n, p) + d\big(f(p), p\big).$$

This implies that

$$d(x_n, p) \leq \frac{1}{1-\alpha} d\big(f(p), p\big).$$

Hence $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$, $\{S(s_n)x_n\}$ and $\{f(x_n)\}$.

Step 3. For any $h, k \geq 0$, $\lim_{n\to\infty} d(x_n, T(h)x_n) = 0$ and $\lim_{n\to\infty} d(x_n, S(k)x_n) = 0$.

From Lemma 2.3 and condition (ii), we have

$$\begin{aligned} d\big(x_n, T(t_n)x_n\big) &= d\big(\alpha_n f(x_n) \oplus \beta_n T(t_n)x_n \oplus \gamma_n S(s_n)x_n, T(t_n)x_n\big) \\ &\leq \alpha_n d\big(f(x_n), T(t_n)x_n\big) + \gamma_n d\big(S(s_n)x_n, T(t_n)x_n\big) \to 0 \quad (n \to \infty) \end{aligned}$$

and

$$\begin{aligned} d\big(x_n, S(s_n)x_n\big) &= d\big(\alpha_n f(x_n) \oplus \beta_n T(t_n)x_n \oplus \gamma_n S(s_n)x_n, S(s_n)x_n\big) \\ &\leq \alpha_n d\big(f(x_n), S(s_n)x_n\big) + \beta_n d\big(T(t_n)x_n, S(s_n)x_n\big) \to 0 \quad (n \to \infty). \end{aligned}$$

Since $\{T(t)\}$ and $\{S(s)\}$ is u.a.r., we obtain that for all $h, k > 0$,

$$\lim_{n\to\infty} d\big(T(h)\big(T(t_n)x_n\big), T(t_n)x_n\big) \leq \lim_{n\to\infty} \sup_{x\in B} d\big(T(h)\big(T(t_n)x\big), T(t_n)x\big) = 0$$

and

$$\lim_{n\to\infty} d\big(S(k)\big(S(s_n)x_n\big), S(s_n)x_n\big) \leq \lim_{n\to\infty} \sup_{x\in B} d\big(S(k)\big(S(s_n)x\big), S(s_n)x\big) = 0,$$

where B is any bounded subset of C containing $\{x_n\}$. Hence, we have

$$\begin{aligned} &d\big(x_n, T(h)x_n\big) \\ &\quad \leq d\big(x_n, T(t_n)x_n\big) + d\big(T(t_n)x_n, T(h)\big(T(t_n)x_n\big)\big) + d\big(T(h)\big(T(t_n)x_n\big), T(h)x_n\big) \\ &\quad \leq 2d\big(x_n, T(t_n)x_n\big) + d\big(T(t_n)x_n, T(h)\big(T(t_n)x_n\big)\big) \to 0 \quad (n \to \infty) \end{aligned}$$

and

$$\begin{aligned} &d\big(x_n, S(k)x_n\big) \\ &\quad \leq d\big(x_n, S(s_n)x_n\big) + d\big(S(s_n)x_n, S(k)\big(S(s_n)x_n\big)\big) + d\big(S(k)\big(S(s_n)x_n\big), S(k)x_n\big) \\ &\quad \leq 2d\big(x_n, S(s_n)x_n\big) + d\big(S(s_n)x_n, S(k)\big(S(s_n)x_n\big)\big) \to 0 \quad (n \to \infty). \end{aligned}$$

Step 4. The sequence $\{x_n\}$ contains a subsequence converging strongly to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$, which is equivalent to (3.3).

Since $\{x_n\}$ is bounded, by Lemma 2.4, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ (without loss of generality, we denote it by $\{x_j\}$) which Δ-converges to a point $\tilde{x}$.

First we claim that $\tilde{x} \in \mathcal{F} = F(\mathcal{T}) \cap F(\mathcal{S})$. Since every CAT(0) space has Opial's property, for any $h \geq 0$, if $T(h)\tilde{x} \neq \tilde{x}$, we have

$$\begin{aligned}\limsup_{j\to\infty} d\big(x_j, T(h)\tilde{x}\big) &\leq \limsup_{j\to\infty}\big(d\big(x_j, T(h)x_j\big) + d\big(T(h)x_j, T(h)\tilde{x}\big)\big)\\ &\leq \limsup_{j\to\infty}\big(d\big(x_j, T(h)x_j\big) + d(x_j,\tilde{x})\big)\\ &= \limsup_{j\to\infty} d(x_j,\tilde{x})\\ &< \limsup_{j\to\infty} d\big(x_j, T(h)\tilde{x}\big).\end{aligned}$$

This is a contraction, and hence $\tilde{x} \in F(\mathcal{T})$. Similarly, we can obtain that $\tilde{x} \in F(\mathcal{S})$. So we have $\tilde{x} \in \mathcal{F}$.

Next we prove that $\{x_j\}$ converges strongly to $\tilde{x}$. Indeed, it follows from Lemma 3.2 that

$$\begin{aligned}d^2(x_j,\tilde{x}) &= \langle\overrightarrow{x_j\tilde{x}}, \overrightarrow{x_j\tilde{x}}\rangle\\ &\leq \alpha_j\big\langle\overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}}\big\rangle + \beta_j\big\langle\overrightarrow{T(t_j)x_j\tilde{x}}, \overrightarrow{x_j\tilde{x}}\big\rangle + \gamma_j\big\langle\overrightarrow{S(s_j)x_j\tilde{x}}, \overrightarrow{x_j\tilde{x}}\big\rangle + \alpha_j N_j\\ &\leq \alpha_j\big\langle\overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}}\big\rangle + \beta_j d\big(T(t_j)x_j,\tilde{x}\big)d(x_j,\tilde{x}) + \gamma_j d\big(S(s_j)x_j,\tilde{x}\big)d(x_j,\tilde{x}) + \alpha_j N_j\\ &\leq \alpha_j\big\langle\overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}}\big\rangle + (1-\alpha_j)d^2(x_j,\tilde{x}) + \alpha_j N_j,\end{aligned}$$

where $N_j := \frac{\gamma_j}{\alpha_j}\beta_j d^2(T(t_j)x_j, S(s_j)x_j) + \gamma_j d^2(f(x_j), S(s_j)x_j)$. It follows that

$$\begin{aligned}d^2(x_j,\tilde{x}) &\leq \big\langle\overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}}\big\rangle + N_j\\ &= \big\langle\overrightarrow{f(x_j)f(\tilde{x})}, \overrightarrow{x_j\tilde{x}}\big\rangle + \big\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}}\big\rangle + N_j\\ &\leq d\big(f(x_j), f(\tilde{x})\big)d(x_j,\tilde{x}) + \big\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}}\big\rangle + N_j\\ &\leq \alpha d^2(x_j,\tilde{x}) + \big\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}}\big\rangle + N_j,\end{aligned}$$

and thus

$$d^2(x_j,\tilde{x}) \leq \frac{1}{1-\alpha}\big\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}}\big\rangle + \frac{1}{1-\alpha}N_j. \tag{3.5}$$

Since $\{x_j\}$ Δ-converges to $\tilde{x}$, by Lemma 2.6 we have

$$\limsup_{n\to\infty}\big\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}}\big\rangle \leq 0.$$

It follows from (3.5) and $\lim_{j\to\infty} N_j = 0$ that $\{x_j\}$ converges strongly to $\tilde{x}$.

Next we show that $\tilde{x}$ solves the variational inequality (3.3). Applying Lemma 2.3, for any $q \in \mathcal{F}$, we have

$$\begin{aligned}d^2(x_j,q) &= d^2\big(\alpha_j f(x_j) \oplus \beta_j T(t_j)x_j \oplus \gamma_j S(s_j)x_j, q\big)\\ &\leq \alpha_j d^2\big(f(x_j),q\big) + \beta_j d^2\big(T(t_j)x_j,q\big) + \gamma_j d^2\big(S(s_j)x_j,q\big) - \alpha_j\beta_j d^2\big(f(x_j), T(t_j)x_j\big)\\ &\leq \alpha_j d^2\big(f(x_j),q\big) + (1-\alpha_j)d^2(x_j,q) - \alpha_j\beta_j d^2\big(f(x_j), T(t_j)x_j\big).\end{aligned}$$

This implies that

$$d^2(x_j,q) \le d^2\big(f(x_j),q\big) - \beta_j\big(d\big(f(x_j),x_j\big) + d\big(x_j,T(t_j)x_j\big)\big)^2.$$

Taking the limit through $j \to \infty$, we can obtain

$$d^2(\tilde{x},q) \le d^2\big(f(\tilde{x}),q\big) - d^2\big(f(\tilde{x}),\tilde{x}\big). \tag{3.6}$$

On the other hand, from (2.7) we have

$$\langle \overrightarrow{xf(\tilde{x})}, \overrightarrow{q\tilde{x}} \rangle = \frac{1}{2}\big[d^2(\tilde{x},\tilde{x}) + d^2\big(f(\tilde{x}),q\big) - d^2(\tilde{x},q) - d^2\big(f(\tilde{x}),\tilde{x}\big)\big]. \tag{3.7}$$

From (3.6) and (3.7) we have

$$\langle \overrightarrow{xf(\tilde{x})}, \overrightarrow{q\tilde{x}} \rangle \ge 0, \quad \forall q \in \mathcal{F}.$$

That is, $\tilde{x}$ solves inequality (3.3).

Step 5. The sequence $\{x_n\}$ converges strongly to $\tilde{x}$.

Assume that $x_{n_i} \to \hat{x}$ as $n \to \infty$. By the same argument, we get that $\hat{x} \in \mathcal{F}$ and solves the variational inequality (3.3), *i.e.*,

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\hat{x}\tilde{x}} \rangle \le 0 \tag{3.8}$$

and

$$\langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \le 0. \tag{3.9}$$

Adding up (3.8) and (3.9), we get that

$$\begin{aligned}
0 &\ge \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \\
&= \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle + \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\hat{x}\tilde{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \\
&= \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \\
&\ge \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - d\big(f(\hat{x}),f(\tilde{x})\big)d(\hat{x},\tilde{x}) \\
&\ge d^2(\tilde{x},\hat{x}) - \alpha d^2(\hat{x},\tilde{x}) = (1-\alpha)d^2(\tilde{x},\hat{x}).
\end{aligned}$$

Since $0 < \alpha < 1$, we have that $d(\tilde{x},\hat{x}) = 0$, and so $\tilde{x} = \hat{x}$. Hence the sequence $\{x_n\}$ converges strongly to $\tilde{x}$, which is the unique solution to the variational inequality (3.3).

This completes the proof. □

Theorem 3.4 Let C be a closed convex subset of a complete CAT(0) space X, and let $\{T(t)\}$ and $\{S(s)\}$ be two one-parameter continuous semigroups of nonexpansive mappings on C satisfying $\mathcal{F} := F(\mathcal{T}) \cap F(\mathcal{S}) \neq \emptyset$ and both uniformly asymptotically regular on C. Let f be a contraction on C with coefficient $\alpha \in (0,1)$. Suppose that $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n f(x_n) \oplus \beta_n T(t_n)x_n \oplus \gamma_n S(s_n)x_n \tag{3.10}$$

for all $n \geq 0$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $t_n, s_n \in [0,\infty)$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\gamma_n = o(\alpha_n)$;
(iii) for all $n \geq 0$, $\alpha_n < 1 - \alpha$;
(iv) $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} s_n = \infty$;
(iv) for any bounded subset B of C, $\lim_{n\to\infty} \sup_{x\in B} d(T(t_n)x, S(s_n)x) = 0$.

Then $\{x_n\}$ converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$, which is equivalent to the variational inequality (3.3).

Proof We first show that the sequence $\{x_n\}$ is bounded. For any $p \in \mathcal{F}$, we have that

$$\begin{aligned}
d(x_{n+1}, p) &= d\big(\alpha_n f(x_n) \oplus \beta_n T(t_n)x_n \oplus \gamma_n S(s_n)x_n, p\big) \\
&\leq \alpha_n d\big(f(x_n), p\big) + \beta_n d\big(T(t_n)x_n, p\big) + \gamma_n d\big(S(s_n)x_n, p\big) \\
&\leq \alpha_n \big(d\big(f(x_n), f(p)\big) + d\big(f(p), p\big)\big) + \beta_n d(x_n, p) + \gamma_n d(x_n, p) \\
&\leq (\alpha_n \alpha + 1 - \alpha_n) d(x_n, p) + \alpha_n d\big(f(p), p\big) \\
&= \big(1 - \alpha_n(1-\alpha)\big) d(x_n, p) + \alpha_n (1-\alpha) \cdot \frac{1}{1-\alpha} d\big(f(p), p\big) \\
&\leq \max\left\{ d(x_n, p), \frac{1}{1-\alpha} d\big(f(p), p\big) \right\}.
\end{aligned}$$

By induction, we have

$$d(x_n, p) \leq \max\left\{ d(x_0, p), \frac{1}{1-\alpha} d\big(f(p), p\big) \right\}$$

for all $n \geq 0$. Hence $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$, $\{S(s_n)x_n\}$ and $\{f(x_n)\}$.

In view of condition (ii), we have

$$d\big(x_{n+1}, T(t_n)x_n\big) \leq \alpha_n d\big(f(x_n), T(t_n)x_n\big) + \gamma_n d\big(S(s_n)x_n, T(t_n)x_n\big) \to 0 \quad (n \to \infty).$$

Since $\{T(t)\}$ is u.a.r and $\lim_{n\to\infty} t_n = \infty$, then for all $h \geq 0$, we obtain that

$$\lim_{n\to\infty} d\big(T(h)\big(T(t_n)x_n\big), T(t_n)x_n\big) \leq \lim_{n\to\infty} \sup_{x\in B} d\big(T(h)\big(T(t_n)x\big), T(t_n)x\big) = 0,$$

where B is any bounded subset of C containing $\{x_n\}$. Hence

$$\begin{aligned}
d\big(x_{n+1}, T(h)x_{n+1}\big) &\leq d\big(x_{n+1}, T(t_n)x_n\big) + d\big(T(t_n)x_n, T(h)\big(T(t_n)x_n\big)\big) \\
&\quad + d\big(T(h)\big(T(t_n)x_n\big), T(h)x_{n+1}\big) \\
&\leq 2d\big(x_{n+1}, T(t_n)x_n\big) + d\big(T(t_n)x_n, T(h)\big(T(t_n)x_n\big)\big) \\
&\to 0 \quad (n \to \infty).
\end{aligned} \tag{3.11}$$

Similarly, for all $k \geq 0$, we have

$$\lim_{n\to\infty} d\big(x_{n+1}, S(k)x_{n+1}\big) = 0. \tag{3.12}$$

Let $\{z_m\}$ be a sequence in C such that

$$z_m = \alpha_m f(z_m) \oplus \beta_m T(t_m)z_m \oplus \gamma_m S(s_m)z_m.$$

It follows from Theorem 3.3 that $\{z_m\}$ converges strongly to a fixed point $\tilde{x} \in \mathcal{F}$, which solves the variational inequality (3.3).

Now we claim that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \le 0.$$

Indeed, it follows from Lemma 3.2 that

$$\begin{aligned}
d^2(z_m, x_{n+1}) &= \langle \overrightarrow{z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle \\
&\le \alpha_m \langle \overrightarrow{f(z_m)x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle + \beta_m \langle \overrightarrow{T(t_m)z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle \\
&\quad + \gamma_m \langle \overrightarrow{S(s_m)z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m N_m \\
&= \alpha_m \langle \overrightarrow{f(z_m)f(\tilde{x})}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m \langle \overrightarrow{\tilde{x} z_m}, \overrightarrow{z_m x_{n+1}} \rangle \\
&\quad + \alpha_m \langle \overrightarrow{z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle + \beta_m \langle \overrightarrow{T(t_m)z_m T(t_m)x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle \\
&\quad + \beta_m \langle \overrightarrow{T(t_m)x_{n+1}x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle + \gamma_m \langle \overrightarrow{S(s_m)z_m S(s_m)x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle \\
&\quad + \gamma_m \langle \overrightarrow{S(s_m)x_{n+1}x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m N_m \\
&\le \alpha_m \alpha d(z_m, \tilde{x}) d(z_m, x_{n+1}) + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m d(\tilde{x}, z_m) d(z_m, x_{n+1}) \\
&\quad + \alpha_m d^2(z_m, x_{n+1}) + \beta_m d^2(z_m, x_{n+1}) + \beta_m d\bigl(T(t_m)x_{n+1}, x_{n+1}\bigr) d(z_m, x_{n+1}) \\
&\quad + \gamma_m d^2(z_m, x_{n+1}) + \gamma_m d\bigl(S(s_m)x_{n+1}, x_{n+1}\bigr) d(z_m, x_{n+1}) + \alpha_m N_m \\
&\le \alpha_m \alpha d(z_m, \tilde{x}) M + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}} \rangle + \alpha_m d(\tilde{x}, z_m) M + d^2(z_m, x_{n+1}) \\
&\quad + \beta_m d\bigl(T(t_m)x_{n+1}, x_{n+1}\bigr) M + \gamma_m d\bigl(S(s_m)x_{n+1}, x_{n+1}\bigr) M + \alpha_m N_m,
\end{aligned}$$

where

$$N_m := \frac{\gamma_m}{\alpha_m} \beta_m d^2\bigl(T(t_m)z_m, S(s_m)z_m\bigr) + \gamma_m d^2\bigl(f(z_m), S(s_m)z_m\bigr)$$

and

$$M \ge \sup_{m,n\ge 1} \bigl\{ d(z_m, x_n) \bigr\}.$$

This implies that

$$\begin{aligned}
\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1} z_m} \rangle &\le (1+\alpha) M d(z_m, \tilde{x}) + \frac{d(T(t_m)x_{n+1}, x_{n+1})}{\alpha_m} M \\
&\quad + \frac{\gamma_m}{\alpha_m} M d\bigl(S(s_m)x_{n+1}, x_{n+1}\bigr) + N_m.
\end{aligned} \tag{3.13}$$

Taking the upper limit as $n \to \infty$ first, and then $m \to \infty$, from (3.11), (3.12) and $\lim_{m\to\infty} N_m = 0$, we get

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m}\rangle \le 0. \tag{3.14}$$

Since

$$\begin{aligned}\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle &= \langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m\tilde{x}}\rangle \\ &\le \langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m}\rangle + d\big(f(\tilde{x}),\tilde{x}\big)d(z_m,\tilde{x}).\end{aligned}$$

Thus, by taking the upper limit as $n \to \infty$ first, and then $m \to \infty$, it follows from $z_m \to \tilde{x}$ and (3.14) that

$$\limsup_{n\to\infty}\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \le 0.$$

Finally, we prove that $x_n \to \tilde{x}$ as $n \to \infty$. In fact, for any $n \ge 0$, letting

$$y_n = \alpha_n\tilde{x} \oplus \beta_n T(t_n)x_n \oplus \gamma_n S(s_n)x_n,$$

from Lemma 3.1 and Lemma 3.2, we have that

$$\begin{aligned}d^2(x_{n+1},\tilde{x}) &\le d^2(y_n,\tilde{x}) + 2\langle\overrightarrow{x_{n+1}y_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\le \big(\beta_n d\big(T(t_n)x_n,\tilde{x}\big) + \gamma_n d\big(S(s_n)x_n,\tilde{x}\big)\big)^2 + 2\big[\alpha_n\langle\overrightarrow{f(x_n)y_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + \beta_n\langle\overrightarrow{T(t_n)x_ny_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \gamma_n\langle\overrightarrow{S(s_n)x_ny_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\big] \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\big[\alpha_n^2\langle\overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + \alpha_n\beta_n\langle\overrightarrow{f(x_n)T(t_n)x_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \alpha_n\gamma_n\langle\overrightarrow{f(x_n)S(s_n)x_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + \beta_n\alpha_n\langle\overrightarrow{T(t_n)x_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \beta_n^2\langle\overrightarrow{T(t_n)x_nT(t_n)x_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + \beta_n\gamma_n\langle\overrightarrow{T(t_n)x_nS(s_n)x_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \gamma_n\alpha_n\langle\overrightarrow{S(s_n)x_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + \gamma_n\beta_n\langle\overrightarrow{S(s_n)x_nT(t_n)x_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \gamma_n^2\langle\overrightarrow{S(s_n)x_nS(s_n)x_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \alpha_n N_n\big] \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\big[\alpha_n^2\langle\overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \alpha_n\beta_n\langle\overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + \alpha_n\gamma_n\langle\overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \beta_n^2 d\big(T(t_n)x_n, T(t_n)x_n\big)d(x_{n+1},\tilde{x}) \\
&\quad + \gamma_n^2 d\big(S(s_n)x_n, S(s_n)x_n\big)d(x_{n+1},\tilde{x}) + \alpha_n N_n\big] \\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n\big(\langle\overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + N_n\big) \\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n\langle\overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + 2\alpha_n\big(\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + N_n\big) \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n\alpha d(x_n,\tilde{x})d(x_{n+1},\tilde{x}) + 2\alpha_n\big(\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + N_n\big) \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + \alpha_n\alpha\big(d^2(x_n,\tilde{x}) + d^2(x_{n+1},\tilde{x})\big) + 2\alpha_n\big(\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + N_n\big),\end{aligned}$$

where $N_n := \frac{\gamma_n}{\alpha_n}\beta_n d^2(T(t_n)x_n, S(s_n)x_n) + \gamma_n d^2(\tilde{x}, S(s_n)x_n)$. This implies that

$$\begin{aligned} d^2(x_{n+1},\tilde{x}) &\le \frac{1-(2-\alpha)\alpha_n+\alpha_n^2}{1-\alpha\alpha_n} d^2(x_n,\tilde{x}) + \frac{2\alpha_n}{1-\alpha\alpha_n}\bigl(\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + N_n\bigr) \\ &= \left(1-\frac{\alpha_n(2-2\alpha-\alpha_n)}{1-\alpha\alpha_n}\right) d^2(x_n,\tilde{x}) + \frac{2\alpha_n}{1-\alpha\alpha_n}\bigl(\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + N_n\bigr). \end{aligned}$$

Then it follows that

$$d^2(x_{n+1},\tilde{x}) \le \bigl(1-\alpha_n'\bigr) d^2(x_n,\tilde{x}) + \alpha_n'\beta_n',$$

where

$$\alpha_n' = \frac{\alpha_n(2-2\alpha-\alpha_n)}{1-\alpha\alpha_n}, \qquad \beta_n' = \frac{2}{2-2\alpha-\alpha_n}\bigl(\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + N_n\bigr).$$

Applying Lemma 2.7 and $\lim_{n\to\infty} N_n = 0$, we can conclude that $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly to this research work. All authors read and approved the final manuscript.

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Acknowledgements
The authors would like to express their thanks to the editors and referees for their helpful comments and advice. This work was supported by the Scientific Research Fund of Sichuan Provincial Education Department (13ZA0199) and the Scientific Research Fund of Sichuan Provincial Department of Science and Technology (2012JYZ011) and the National Natural Science Foundation of China (Grant No. 11361070).

Received: 6 November 2013 Accepted: 10 July 2014 Published: 15 Aug 2014

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10.1186/1029-242X-2014-283
Cite this article as: Tang and Chang: **Viscosity approximation methods for two nonexpansive semigroups in CAT(0) spaces.** *Journal of Inequalities and Applications* 2014, 2014:283