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On some properties of new paranormed sequence space defined by λ^2-convergent sequences

Naim L Braha*

*Correspondence:
nbraha@yahoo.com
Department of Mathematics and Computer Sciences, University of Prishtina, Avenue 'Mother Teresa' 5, Prishtinë, 10000, Kosovo
Department of Computer Sciences and Applied Mathematics, College Vizioni per Arsim, Ferizaj, 70000, Kosovo

Abstract
In this paper, we introduce the new sequence space $l(\lambda^2, p)$ and we will show some topological properties like completeness, isomorphism, and some inclusion relations between this sequence spaces and some of the other sequence spaces. In addition we will compute the α-, β-, and γ-duals of these spaces. At the end of the article we will show some matrix transformations between the $l(\lambda^2, p)$ space and the other spaces.
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1 Introduction

By w we denote the space of all complex sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=0}^{\infty}$. Also, we shall use the conventions that $e = (1, 1, \dots)$ and $e(n)$ is the sequence whose only non-zero term is 1 in the nth place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$. Any vector subspace of w is called a sequence space. We shall write l_1, c, and c_0 for the sequence spaces of all bounded, convergent, and null sequences, respectively. Further, by l_p $(1 \le p < \infty)$, we denote the sequence space of all p-absolutely convergent series, that is, $l_p = \{x = (x_k) \in w : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$ for $1 \le p < \infty$. Moreover, we write bs, cs, and cs_0 for the sequence spaces of all bounded, convergent, and null series, respectively. A sequence space X is called an FK space if it is a complete linear metric space with continuous coordinates $p_n : X \to \mathbb{C}$ $(n \in \mathbb{N})$, where $\mathbb{C}$ denotes the complex field and $p_n(x) = x_n$ for all $x = (x_n) \in X$ and every $n \in \mathbb{N}$. A normed FK space is called a BK space, that is, a BK space is a Banach sequence space with continuous coordinates.

The sequence spaces l_1, c, and c_0 are BK spaces with the usual sup-norm given by $\|x\|_\infty = \sup_n |x(n)|$. Also, the space l_p is a BK space with the usual l_p-norm defined by

$$\|x\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p\right)^{\frac{1}{p}},$$

where $1 \le p < \infty$. A sequence $(y_n)_{n=0}^{\infty}$ in a normed space X is called a Schauder basis for X if for every $x \in X$ there is a unique sequence $(a_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} a_n y_n$, *i.e.*, $\lim_n \|x - \sum_{k=0}^{n} a_n y_n\| = 0$. The α-, β-, and γ-duals of a sequence space X are, respectively, defined by

$$X^\alpha = \{a = (a_k) \in w : ax = (a_k x_k) \in l_1 \text{ for all } x = (x_k) \in X\},$$

$$X^{\beta} = \{a = (a_k) \in w : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\}$$

and

$$X^{\gamma} = \{a = (a_k) \in w : ax = (a_k x_k) \in bs \text{ for all } x = (x_k) \in X\}.$$

If A is an infinite matrix with complex entries a_{nk} $(n, k \in \mathbb{N})$, then we write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^{\infty}$. Also, we write A_n for the sequence in the nth row of A, that is, $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. Further, if $x = (x_k) \in w$ then we define the A-transform of x as the sequence $Ax = (A_n(x))_{n=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{n,k} x_k \quad (n \in \mathbb{N}) \tag{1}$$

provided the series on (1) is convergent for each $n \in \mathbb{N}$.

Furthermore, the sequence x is said to be A-summable to $a \in \mathbb{C}$ if Ax converges to a which is called the A-limit of x. In addition, let X and Y be sequence spaces. Then we say that A defines a matrix mapping from X into Y if for every sequence $x \in X$ the A-transform of x exists and is in Y. Moreover, we write (X, Y) for the class of all infinite matrices that map X into Y. Thus $A \in (X, Y)$ if and only if $A_n \in X^{\beta}$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$. For an arbitrary sequence space X, the matrix domain of an infinite matrix A in X is defined by

$$X_A = \{x \in w : Ax \in X\}, \tag{2}$$

which is a sequence space. The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors; see for instance [1–18]. In this paper, we introduce the new sequence space $l(\lambda^2, p)$ and we will show some topological properties as completeness, isomorphism, and some inclusion relations between this sequence spaces and some of the other sequence spaces. In addition we will compute the α-, β-, and γ-duals of these spaces.

2 Notion of the λ^2-convergent sequences

Let $\Lambda = \{\lambda_k : k = 0, 1, \ldots\}$ be a nondecreasing sequence of positive numbers tending to ∞, as $k \to \infty$, and $\lambda_{n+1} \geq 2 \cdot \lambda_n$, for each $n \in \mathbb{N}$. From this relation it follows that $\Delta^2 \lambda_n \geq 0$. The first difference is defined as follows: $\Delta \lambda_k = \lambda_k - \lambda_{k-1}$, where $\lambda_{-1} = \lambda_{-2} = 0$, and the second difference is defined as $\Delta^2(\lambda_k) = \Delta(\Delta(\lambda_k)) = \lambda_k - 2\lambda_{k-1} + \lambda_{k-2}$.

Let $x = (x_k)$ be a sequence of complex numbers, such that $x_{-1} = x_{-2} = 0$. In [19] is given the concept of λ^2-convergent sequence as follows: Let $x = (x_k)$ be any given sequence of complex numbers, we will say that it converges λ^2-strongly to number x if

$$\lim_n \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \bigl|\lambda_k(x_k - x) - 2\lambda_{k-1}(x_{k-1} - x) + \lambda_{k-2}(x_{k-2} - x)\bigr| = 0.$$

This generalizes the concept of Λ-strong convergence in [20].

Let us denote

$$\Lambda^2{}_n(x) = \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) x_k \tag{3}$$

for all $n \in \mathbb{N}$. Assume here and below that (p_k), (q_k) are bounded sequences of strictly positive real numbers with $\sup_k p_k = H$ and $\max_k(1, H) = M$, for $1 < p_k < \infty$ for all $k \in \mathbb{N}$. The linear space $l(p)$ as defined by Madoxx [3] is as follows:

$$l(p) = \left\{ x = (x_n) \in w : \sum_{n=0}^{\infty} |x_n|^{p_n} < \infty \right\}, \tag{4}$$

which are complete spaces paranormed by

$$h(x) = \left(\sum_{n=0}^{\infty} |x_n|^{p_n} \right)^{\frac{1}{M}}. \tag{5}$$

3 The sequence space $l(\lambda^2, p)$

In this section we will define the sequence space $l(\lambda^2, p)$ and prove that this sequence space according to its paranorm is a complete linear space. We have

$$l(\lambda^2, p) = \left\{ x = (x_k) \in w : \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) x_k \right|^{p_n} < \infty \right\} \tag{6}$$

and in case where $p_n = p$, for every $n \in \mathbb{N}$ we get

$$l_p^{\lambda^2} = \left\{ x = (x_k) \in w : \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) x_k \right|^{p} < \infty \right\}. \tag{7}$$

Let $x = (x_n) \in w$ be any sequence; we will define the Λ_n^2-transform of the sequence $x = (x_n)$ as follows:

$$\Lambda_n^2(x) = \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) x_k, \quad n \in \mathbb{N}. \tag{8}$$

Theorem 1 *The sequence space $l(\lambda^2, p)$ is the complete linear metric space with respect to the paranorm defined by*

$$g(x) = \left(\sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) x_k \right|^{p_n} \right)^{\frac{1}{M}}. \tag{9}$$

Proof The linearity of $l(\lambda^2, p)$ follows from Minkowski's inequality. In what follows we will prove that $g(x)$ defines a paranorm. In fact, for any $x, y \in l(\lambda^2, p)$ we get

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2})(x_k + y_k) \right|^{p_n} \right)^{\frac{1}{M}} \\ &\quad \le \left(\sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) x_k \right|^{p_n} \right)^{\frac{1}{M}} \\ &\qquad + \left(\sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) y_k \right|^{p_n} \right)^{\frac{1}{M}} \end{aligned} \tag{10}$$

and for any $\alpha \in \mathbb{R}$

$$|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}. \tag{11}$$

It is clear that $g(\theta) = 0$, $g(x) = g(-x)$ for all $x \in l(\lambda^2, p)$. From inequalities (10) and (11) we find the subadditivity of $g(x)$ and hence $g(\alpha x) \leq \max\{1, |\alpha|^M\} g(x)$. Let x_m be any sequence of points $\{x^m\} \in l(\lambda^2, p)$ such that $g(x^m - x) \to 0$ and (α_m) also any sequence of scalars such that $\alpha_m \to \alpha$. Then, since the inequality

$$g(x^m) \leq g(x) + g(x^m - x) \tag{12}$$

holds by the subadditivity of g, we find that $\{g(x^m)\}$ is bounded and we thus have

$$\begin{aligned} g(\alpha_m x^m - \alpha x) &= \left(\sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2})(\alpha_m x_k^m - \alpha x_k) \right|^{p_n} \right)^{\frac{1}{M}} \\ &\leq |\alpha_m - \alpha|^{\frac{1}{M}} g(x^m) + |\alpha|^{\frac{1}{M}} g(x^m - x), \end{aligned} \tag{13}$$

which tends to zero as $n \to \infty$. Therefore, the scalar multiplication is continuous. Hence g is a paranorm on the space $l(\lambda^2, p)$. It remains to prove the completeness of the space $l(\lambda^2, p)$. Let $\{x^j\}$ be any Cauchy sequence in the space $l(\lambda^2, p)$, where $x^j = \{x_0^j, x_1^j, \ldots\}$. Then, for a given $\epsilon > 0$, there exists a positive integer $m_0(\epsilon)$ such that $g(x^j - x^i) < \frac{\epsilon}{2}$ for all $i, j > m_0(\epsilon)$. Using the definition of g, we obtain for each fixed $n \in \mathbb{N}$

$$\left| \Lambda_n^2(x^j) - \Lambda_n^2(x^i) \right| \leq \left(\sum_{n=0}^{\infty} \left| \Lambda_n^2(x^j) - \Lambda_n^2(x^i) \right|^{p_n} \right)^{\frac{1}{M}} < \frac{\epsilon}{2} \tag{14}$$

for every $i, j > m_0(\epsilon)$, which leads to the fact that $\{\Lambda_n^2(x^0), \Lambda_n^2(x^1), \ldots\}$ is a Cauchy sequence of real numbers for every fixed $n \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $\Lambda_n^2(x^i) \to \Lambda_n^2(x)$ as $i \to \infty$. Using these infinitely many limits, we may write the sequence $\{\Lambda_n^2(x), \Lambda_n^2(x), \ldots\}$. From (14) as $i \to \infty$, we have

$$\left| \Lambda_n^2(x^j) - \Lambda_n^2(x) \right| < \frac{\epsilon}{2}, \quad j \geq m_0(\epsilon) \tag{15}$$

for every fixed $n \in \mathbb{N}$. By using (14) and boundedness of the Cauchy sequence, we have

$$\left(\sum_{n=0}^{\infty} \left| \Lambda_n^2(x) \right|^{p_n} \right)^{\frac{1}{M}} \leq \left(\sum_{n=0}^{\infty} \left| \Lambda_n^2(x^j) - \Lambda_n^2(x) \right|^{p_n} \right)^{\frac{1}{M}} + \left(\sum_{n=0}^{\infty} \left| \Lambda_n^2(x^j) \right|^{p_n} \right)^{\frac{1}{M}} < \infty. \tag{16}$$

Hence, we get $x \in l(\lambda^2, p)$. Therefore, the space $l(\lambda^2, p)$, is complete. □

Theorem 2 *The sequence space $l(\lambda^2, p)$ is a BK space.*

Proof Let us denote by $\Lambda^2 = (\lambda_{n,k}^2)_{n,k=0}^{\infty}$ the following matrix:

$$(\lambda_{n,k}{}^2) = \begin{cases} \frac{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}{\lambda_n - \lambda_{n-1}}, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}$$

for all $n,k \in \mathbb{N}$. Then the Λ^2-transform of a sequence $x \in w$ is the sequence $\Lambda^2(x) = (\Lambda_n^2(x))_{n=0}^{\infty}$, where $\Lambda_n^2(x)$ is given by (8) for every $n \in \mathbb{N}$. Thus

$$\|x\|_{l(\lambda^2,p)} = \left\| \Lambda^2(x) \right\|_{l(p)}.$$

Now the proof of the theorem follows from Theorem 4.3.12 given in [21]. □

Theorem 3 *The sequence space $l(\lambda^2,p)$ is linearly isomorphic to the space $l(p)$, where $0 < p_k \le H < \infty$.*

Proof Let $T : l(\lambda^2,p) \to l(p)$ be an operator defined by $x \to y = \Lambda^2(x)$, where Λ^2 is given by (8). The operator T is linear and injective, from $T(x) = 0$ it follows that $x = 0$. In what follows we will prove that T is surjective. Let $y \in l(p)$ be any element; we define $x = (x_n)$ by

$$x_n(\lambda) = \frac{\Delta\lambda_n y_n - \Delta\lambda_{n-1} y_{n-1}}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}}; \tag{17}$$

then we get

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) x_k \right|^{p_n} \right)^{\frac{1}{M}} \\
&\quad = \left(\sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\Delta\lambda_k y_k - \Delta\lambda_{k-1} y_{k-1}) \right|^{p_n} \right)^{\frac{1}{M}} \\
&\quad = \left(\sum_{n=0}^{\infty} |y_n|^{p_n} \right)^{\frac{1}{M}}.
\end{aligned}$$

□

As a consequence of Theorem 2 and Theorem 3 we get the following result.

Corollary 1 *Define the sequence $e_{\lambda^2}^{(n)} \in c_0(\lambda^2,p)$ for every fixed $n \in \mathbb{N}$ by*

$$e_{\lambda^2}^{(n)} = \begin{cases} (-1)^{n-k} \frac{\lambda_n - \lambda_{n-1}}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} & (k \le n \le k+1), \\ 0 & \textit{otherwise}, \end{cases}$$

where $k \in \mathbb{N}$. Then we have the following.

(1) *The sequence $(e_{\lambda^2}^{(0)}, e_{\lambda^2}^{(1)}, e_{\lambda^2}^{(2)}, \ldots)$ is a Schauder basis for the space $c_0(\lambda^2,p)$ and every $x \in c_0(\lambda^2,p)$ has a unique representation: $x = \sum_{n=0}^{\infty} \Lambda_n^2(x) e_{\lambda^2}^{(n)}$.*

(2) *The sequence $(e, e_{\lambda^2}^{(0)}, e_{\lambda^2}^{(1)}, e_{\lambda^2}^{(2)}, \ldots)$ is a Schauder basis for the space $c(\lambda^2,p)$ and every $x \in c(\lambda^2,p)$ has a unique representation: $x = le + \sum_{n=0}^{\infty} (\Lambda_n^2(x) - l) e_{\lambda^2}^{(n)}$, where $l = \lim_n \Lambda_n^2(x)$.*

Proof Since $\Lambda^2(e) = e$ and $\Lambda^2(e_{\lambda^2}^{(n)}) = e^{(n)}$ for every $n \in \mathbb{N}$, the proof of the theorem follows from Corollary 2.3 given in [22]. □

Theorem 4 *The inclusion $l(\lambda^2,p) \subset c_0(\lambda^2,p)$ holds. The inclusion is strict.*

Proof Let $x \in l(\lambda^2, p)$; then it follows that $\Lambda^2(x) \in l(p)$, from which follows that

$$\sum_{n=0}^{\infty} \left|\Lambda_n^2(x)\right|^{p_n} < \infty.$$

From the last relation we get $\lim_n \Lambda_n^2(x) \to 0$, as $n \to \infty$, respectively, $\Lambda^2(x) \in c_0(p) \Rightarrow x \in c_0(\Lambda^2, p)$. To prove that the inclusion is strict we will show the following.

Example 1 Let $1 < p_n < 2$, $\forall n \in \mathbb{N}$, $x_n = (\frac{2^n\sqrt{n}}{n+1} - \frac{2^{n+1}\sqrt{n+1}}{n+2}) \cdot \frac{1}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}}$, and $\lambda_n = 2^n$. Then it follows that

$$\begin{aligned}\frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) x_k &= \frac{1}{2^{n-1}} \sum_{k=0}^{n} \left(\frac{2^k\sqrt{k}}{k+1} - \frac{2^{k+1}\sqrt{k+1}}{k+2}\right) \\ &= \frac{-2\sqrt{n}}{n+1} \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Hence $x = (x_n) \in c_0(\lambda^2, p)$. On the other hand $x = (x_n) \notin l(\lambda^2, p)$, for $1 < p_n < 2$, $\forall n \in \mathbb{N}$. With which we have proved the theorem. □

Theorem 5 *The inclusions $c_0(\lambda^2, p) \subset c(\lambda^2, p) \subset l_\infty(\lambda^2, p)$ strictly hold.*

Proof It is clear that the inclusion $c_0(\lambda^2, p) \subset c(\lambda^2, p) \subset l_\infty(\lambda^2, p)$ holds. Further, since $c_0 \subset c$ is strict, from Lemmas 1 and 2 from [19] it follows that $c_0(\lambda^2, p) \subset c(\lambda^2, p)$ is also strict. In what follows we will show that the last inclusion is strict, too. For this reason we will show the following.

Example 2 Let

$$x_n = (-1)^n \frac{\lambda_n - \lambda_{n-2}}{\lambda_n - 2\lambda_{n-2} + \lambda_{n-2}},$$

then it follows that

$$\begin{aligned}\Lambda_n^2(x) &= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) x_k \\ &= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (-1)^k (\lambda_k - \lambda_{k-2}) = (-1)^n.\end{aligned}$$

From the last relation it follows that $x = (x_k) \in l_\infty(\lambda^2, p) \setminus c(\lambda^2, p)$. □

Theorem 6 *The inclusion $l(\lambda^2, p) \subset l(p)$ holds if and only if $S(x) \in l(p)$ for every sequence $x \in l(\lambda^2, p)$, where $1 \le p_k \le H$. Here $S(x) = \{S_n(x)\}$ and $S_n(x) = x_n - \Lambda_n^2(x)$.*

Proof The proof of the theorem is similar to Theorem 3.3 given in [6]. □

Theorem 7

(1) *If $p_n > 1$ for all $n \in \mathbb{N}$, then the inclusion $l_p^{\lambda^2} \subset l(\lambda^2, p)$ holds.*
(2) *If $p_n < 1$ for all $n \in \mathbb{N}$, then the inclusion $l(\lambda^2, p) \subset l_p^{\lambda^2}$ holds.*

Proof (1) Let $p = (p_n) > 1$ for all $n \in \mathbb{N}$ and $x \in l_p^{\lambda^2}$. Then it follows that $\Lambda^2(x) \in l(p)$. Hence

$$\lim_n \left|\Lambda_n^2(x)\right| = 0,$$

we find $n_0 \in \mathbb{N}$ such that $|\Lambda^2(x)| < 1$, for every $n > n_0$, respectively,

$$\left|\Lambda_n^2(x)\right|^{p_n} < \left|\Lambda_n^2(x)\right| \quad \text{for } p_n > 1, \forall n > n_0.$$

From the last relation we get $x \in l(\lambda^2, p)$.

(2) Let us suppose that $x \in l(\lambda^2, p)$. Then $\Lambda^2(x) \in l(p)$, $\exists n_1 \in \mathbb{N}$, such that

$$\forall n > n_1 \quad \Rightarrow \quad \left|\Lambda_n^2(x)\right|^{p_n} < 1,$$
$$\left|\Lambda_n^2(x)\right| = \left(\left|\Lambda_n^2(x)\right|^{p_n}\right)^{\frac{1}{p_n}} < \left|\Lambda_n^2(x)\right|^{p_n}$$

for all $n > n_1$. Hence $x \in l_p^{\Lambda^2}$. □

4 Duals of the space $l(\lambda^2, p)$

In this section we will give the theorems in which the α-, β-, and γ-duals are determined of the sequence space $l(\lambda^2, p)$. In proving the theorems we apply the technique used in [1]. Also we will give some matrix transformations from $l(p, \lambda^2, p)$ into $l(q)$ by using the matrix given in [4].

Theorem 8 *Let $K_1 = \{k \in \mathbb{N} : p_k \leq 1\}$ and $K_2 = \{k \in \mathbb{N} : p_k > 1\}$. Define the matrix $M^a = (m_{nk}^a)$ by*

$$m_{nk}^a = \begin{cases} (-1)^{n-k} \cdot \frac{\Delta\lambda_k}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}} \cdot a_n, & n-1 \leq k \leq n, \\ 0, & 0 \leq k \leq n-1 \text{ or } k > n. \end{cases} \tag{18}$$

Then

$$l_{K_1}^a\left(\lambda^2, p\right) = \left\{a = (a_n) \in w : M^a \in \left(l(p); l_\infty\right)\right\}, \tag{19}$$
$$l_{K_2}^a\left(\lambda^2, p\right) = \left\{a = (a_n) \in w : M^a \in \left(l(p); l_1\right)\right\}. \tag{20}$$

Proof Let $x = (x_n) \in l(\lambda^2, p)$, $a = (a_n) \in w$, and $(y_n) = (\Lambda_n^2(x))$. Then we have

$$a_n x_n = a_n \sum_{k=n-1}^{n} \left((-1)^{n-k} \frac{\Delta\lambda_k}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}}\right) y_k = \left(M^a y\right)_n; \quad n \in \mathbb{N}, \tag{21}$$

where M^a is defined by (18). From (21) it follows that $ax = (a_n x_n) \in l_1$ or $ax = (a_n x_n) \in l_\infty$ whenever $x \in l(\lambda^2, p)$ if and only if $M^a y \in l_1$ or $M^a y \in l_\infty$ whenever $y \in l(p)$. This means that $a \in l_{K_1}^\alpha(\lambda^2, p)$ or $a \in l_{K_2}^\alpha(\lambda^2, p)$ if and only if $M^a \in (l(p), l_1)$ or $M^a \in (l(p), l_\infty)$. □

As a direct consequence of Theorem 8, we get the following.

Corollary 2 *Let $K^* = \{k \in \mathbb{N} : n-1 \leq k \leq n\} \cup K$ for every $K \in F$, where F is the collection of all finite subsets of $\mathbb{N}$. Then*

(1) $l^{\alpha}_{K_1}(\lambda^2,p) = \{a=(a_n)\in w : \sup_K \sup_{n\in\mathbb{N}} |\sum_{n\in K^*} m^a_{nk}|^{p_k} < \infty\}$,
(2) $l^{\alpha}_{K_2}(\lambda^2,p) = \bigcup_{M>1}\{a=(a_n)\in w : \sup_{K\in F}\sum_K |\sum_{n\in K^*} m^a_{nk}M^{-1}|^{p_k} < \infty\}$,
for some constant M.

In what follows we will characterize the β- and γ-dual of the sequence space $l(\lambda^2,p)$.

Theorem 9 *Let $K_1 = \{k\in\mathbb{N} : p_k \le 1\}$, $K_2 = \{k\in\mathbb{N} : p_k > 1\}$. Define the sequence $d^1 = (d^1_k)$, $d^2 = (d^2_k)$, and the matrix $D^a = (d^a_{nk})$ by*

$$d^a_{nk} = \begin{cases} d^1_k, & 0\le k\le n-1,\\ d^2_k, & k=n,\\ 0, & k>n,\end{cases}$$

where $d^1_k = (\frac{a_k}{\lambda_k-2\lambda_{k-1}+\lambda_{k-2}} - \frac{a_{k+1}}{\lambda_{k+1}-2\lambda_k+\lambda_{k-1}})\Delta\lambda_k$, $d^2_k = \frac{a_k\Delta\lambda_k}{\lambda_k-2\lambda_{k-1}+\lambda_{k-2}}$. Then

$$l^{\beta}_{K_1}(\lambda^2,p) = l^{\gamma}_{K_1}(\lambda^2,p) = \{a=(a_n)\in w : D^a\in(l(p),l_\infty)\}, \tag{22}$$

$$l^{\beta}_{K_2}(\lambda^2,p) = l^{\gamma}_{K_2}(\lambda^2,p) = \{a=(a_n)\in w : D^a\in(l(p),c)\}. \tag{23}$$

Proof Let $x=(x_n)\in l(\lambda^2,p)$. Then we obtain

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n-1} d^1_k y_k + d^2_n y_n = (D^a y)_n. \tag{24}$$

From (24) it follows that $ax=(a_nx_n)\in cs$ or bs if and only if $D^a(y)\in c$ or l_∞. This means that $a=(a_n)\in\{l^{\beta}_{K_1}(\lambda^2,p)$ or $l^{\beta}_{K_2}(\lambda^2,p)\}$ or $a=(a_n)\in\{l^{\gamma}_{K_1}(\lambda^2,p)$ or $l^{\gamma}_{K_2}(\lambda^2,p)\}$. With which the theorem is proved. □

As an immediate result of the above theorem, we get the following.

Corollary 3 *Let (p'_k) be a conjugate sequence of numbers (p_k), it means that $\frac{1}{p_k}+\frac{1}{p'_k}=1$ and $1<p_k<\infty$ for all $k\in\mathbb{N}$. Then*
(1) $l^{\beta}_{K_1}(\lambda^2,p) = l^{\gamma}_{K_1}(\lambda^2,p) = \{a=(a_n)\in w : d^1_k, d^2_k\in l_\infty(p)\}$,
(2) $l^{\beta}_{K_2}(\lambda^2,p) = l^{\gamma}_{K_2}(\lambda^2,p) = \bigcup_{M>1}\{a=(a_n)\in w : d^1_kM^{-1}, d^2_kM^{-1}\in l_\infty(p)\cap l(p')\}$.

5 Some matrix transformations related to sequence space $l(\lambda^2,p)$

In this section we will show some matrix transformations between the sequence space $l(\lambda^2,p)$ and sequence spaces $l(q)$, $c_0(q)$, $c(q)$, and $l_\infty(q)$ where $q=(q_n)$ is a sequence of nondecreasing, bounded positive real numbers. Let $x,y\in w$ be connected by the relation $y=\Lambda^2(x)$. For an infinite matrix $A=(a_{nk})$, taking into consideration Theorem 3, we get

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m-1}\overline{a_{nk}}y_k + \frac{a_{nm}\Delta\lambda_n}{\lambda_n-2\lambda_{n-1}+\lambda_{n-2}} \quad (m,n\in\mathbb{N}), \tag{25}$$

where

$$\overline{a_{nk}} = \left(\frac{a_{nk}}{\lambda_k-2\lambda_{k-1}+\lambda_{k-2}} - \frac{a_{n,k+1}}{\lambda_{k+1}-2\lambda_k+\lambda_{k-1}}\right)\Delta\lambda_k \quad (n,k\in\mathbb{N}). \tag{26}$$

Let N, K be a finite subsets of the natural numbers $\mathbb{N}$ and L, T be a natural numbers. $K_1 = \{k \in \mathbb{N} : p_k \leq 1\}$ and $K_2 = \{k \in \mathbb{N} : p_k > 1\}$ and also let (p'_k) be a conjugate sequence of numbers (p_k). Prior to giving the theorems, let us suppose that (q_n) is a nondecreasing bounded sequence of positive real numbers and consider the following conditions:

$$\sup_n \sup_{k\in K_1} \left|\sum_{n\in N} \overline{a_{nk}}\right|^{q_n} < \infty, \tag{27}$$

$$\exists T \sup_N \sum_{k\in K_2} \left|\sum_n \overline{a_{nk}} T^{-1}\right|^{p_k} < \infty, \tag{28}$$

$$\exists T \sup_k \sum_n \left|\overline{a_k} T^{\frac{-1}{p_k}}\right|^{q_n} < \infty, \tag{29}$$

$$\lim_n |\overline{a_{nk}}|^{q_n} = 0 \quad (\forall k \in \mathbb{N}), \tag{30}$$

$$\forall L, \quad \sup_n \sup_{K_1} \left|\overline{a_{nk}} L^{\frac{1}{q_n}}\right| < \infty, \tag{31}$$

$$\forall L, \quad \exists T \sup_n \sum_{K_2} \left|\overline{a_k} L^{\frac{1}{q_n}} T^{-1}\right|^{p_k} < \infty, \tag{32}$$

$$\sup_n \sup_{K_1} |\overline{a_{nk}}|^{p_k} < \infty, \tag{33}$$

$$\exists T \sup_n \sum_{K_2} \left|\overline{a_k} T^{-1}\right|^{p_k} < \infty, \tag{34}$$

$$\forall L, \quad \sup_n \sup_{K_1} \left(|\overline{a_k} - \overline{a_k}| L^{\frac{1}{q_n}}\right)^{p_k} < \infty, \tag{35}$$

$$\lim_n |\overline{a_k} - \overline{a_k}|^{q_n} = 0, \quad \forall k, \tag{36}$$

$$\forall L, \quad \exists T \sup_n \sum_{K_2} \left(|\overline{a_{nk}} - \overline{a_k}| L^{\frac{1}{q_n}} T^{-1}\right)^{p_k} < \infty, \tag{37}$$

$$\exists L, \quad \sup_n \sup_{K_1} \left|\overline{a_{nk}} L^{\frac{-1}{q_n}}\right|^{p_k} < \infty, \tag{38}$$

$$\exists L, \quad \sup_n \sum_{K_2} \left|\overline{a_{nk}} L^{\frac{-1}{q_n}}\right|^{p_k} < \infty, \tag{39}$$

$$\left(\frac{a_{nk}\Delta\lambda_k}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}\right)_{k=0}^{\infty} \in c_0(q), \quad \forall n \in \mathbb{N}, \tag{40}$$

$$\left(\frac{a_{nk}\Delta\lambda_k}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}\right)_{k=0}^{\infty} \in c(q), \quad \forall n \in \mathbb{N}, \tag{41}$$

$$\left(\frac{a_{nk}\Delta\lambda_k}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}\right)_{k=0}^{\infty} \in l_\infty(q), \quad \forall n \in \mathbb{N}. \tag{42}$$

From the above conditions we get the following.

Theorem 10

$A \in (l(\lambda^2, p); l(q))$ if and only if (27), (28), (29), *and* (40) *hold*,
$A \in (l(\lambda^2, p); c_0(q))$ if and only if (30), (31), (31), *and* (40) *hold*,
$A \in (l(\lambda^2, p); l(q))$ if and only if (33), (34), (35), (36), (37), *and* (41) *hold*,
$A \in (l(\lambda^2, p); l(q))$ if and only if (38), (39), *and* (42) *hold*.

Competing interests
The author declares to have no competing interests.

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