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On a multidimensional version of the Hilbert-type inequality in the whole plane

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Abstract
The main objective of this paper is a study of some new multidimensional Hilbert-type inequalities with a general homogeneous kernel in the whole plane. We derive a pair of equivalent inequalities, and we also establish the conditions under which the constant factors included in the obtained inequalities are the best possible. Some applications in particular settings are also considered.
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1 Introduction

Hilbert's inequality is one of the most significant weighted inequalities in mathematical analysis and its applications. Through the years, Hilbert-type inequalities were discussed by numerous authors, who either reproved them using various techniques, or applied and generalized them in many different ways. For more details as regards Hilbert's inequality the reader is referred to [1] or [2].

Although classical, Hilbert's inequality and its generalizations and modifications are still of a great interest. Xin and Yang in [3] proved Hilbert-type inequalities with the homogeneous kernel of degree −2.

If $p > 1$, $1/p + 1/q = 1$, $|\lambda| < 1$, $0 < \alpha_1 < \alpha_2 < \pi$, $f, g \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} |x|^{-p\lambda-1} f^p(x)\,dx < \infty$ and $0 < \int_{-\infty}^{\infty} |y|^{q\lambda-1} g^q(y)\,dy < \infty$, then we have

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\} f(x)g(y)\,dx\,dy \\ &\quad< k(\lambda)\left(\int_{-\infty}^{\infty} |x|^{-p\lambda-1} f^p(x)\,dx\right)^{1/p}\left(\int_{-\infty}^{\infty} |y|^{q\lambda-1} g^q(y)\,dy\right)^{1/q}, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} |y|^{p(1-\lambda)-1}\left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\} f(x)\,dx\right)^p dy \\ &\quad< k^p(\lambda)\int_{-\infty}^{\infty} |x|^{-p\lambda-1} f^p(x)\,dx, \end{aligned} \tag{1.2}$$

where the constant factors

$$k(\lambda) = \frac{\pi}{\sin\lambda\pi}\left[\frac{\sin\lambda\alpha_1}{\sin\alpha_1} + \frac{\sin\lambda(\pi-\alpha_2)}{\sin\alpha_2}\right] \quad \bigl(0 < |\lambda| < 1\bigr)$$

and $k^p(\lambda)$ are the best possible. Inequalities (1.1) and (1.2) are equivalent.

Our main objective is to emphasize the previous result. Our generalization will include a multidimensional version of the Hilbert-type inequality in the whole plane.

Some of the recent results concerning Hilbert's inequality include extension to multidimensional case, equipped with conjugate exponents p_i, that is, $\sum_{i=1}^n 1/p_i = 1$, $p_i > 1$, $n \geq 2$ (see [4–6]). Here we refer to [7], which provides a unified treatment of the multidimensional Hilbert-type inequality in the setting with conjugate exponents. Suppose $(\Omega_i, \Sigma_i, \mu_i)$ are σ-finite measure spaces and $K : \prod_{i=1}^n \Omega_i \to \mathbb{R}$, $\phi_{ij} : \Omega_j \to \mathbb{R}$, $f_i : \Omega_i \to \mathbb{R}$, $i, j = 1, 2, \ldots, n$, are non-negative measurable functions. If $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$, then the following inequalities hold and are equivalent:

$$\int_{\boldsymbol{\Omega}} K(\mathbf{x}) \prod_{i=1}^n f_i(x_i)\, d\mu(\mathbf{x}) \leq \prod_{i=1}^n \|\phi_{ii}\omega_i f_i\|_{p_i} \tag{1.3}$$

and

$$\left[\int_{\Omega_n} h(x_n) \left(\int_{\hat{\boldsymbol{\Omega}}^n} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, d\hat{\mu}^n(\mathbf{x})\right)^q d\mu(x_n)\right]^{1/q} \leq \prod_{i=1}^{n-1} \|\phi_{ii}\omega_i f_i\|_{p_i}, \tag{1.4}$$

where

$$\omega_i(x_i) = \left[\int_{\hat{\boldsymbol{\Omega}}^i} K(\mathbf{x}) \prod_{j=1, j\neq i}^n \phi_{ij}^{p_i}(x_j)\, d\hat{\mu}^i(\mathbf{x})\right]^{1/p_i} \tag{1.5}$$

and

$$\begin{aligned} &1/q = \sum_{i=1}^{n-1}(1/p_i), \qquad \boldsymbol{\Omega} = \prod_{i=1}^n \Omega_i, \qquad \hat{\boldsymbol{\Omega}}^i = \prod_{j=1, j\neq i}^n \Omega_j, \qquad \mathbf{x} = (x_1, x_2, \ldots, x_n), \\ &h = \phi_{nn}^{-q}\omega_n^{1-q}, \qquad d\mu(\mathbf{x}) = \prod_{i=1}^n d\mu_i(x_i), \qquad d\hat{\mu}^i(\mathbf{x}) = \prod_{j=1, j\neq i}^n d\mu_j(x_j). \end{aligned} \tag{1.6}$$

The abbreviations as in (1.6) will be used throughout the whole paper. Also note that $\|\cdot\|_{p_i}$ denotes the usual norm in $L^{p_i}(\Omega_i)$, that is,

$$\|\phi_{ii}\omega_i f_i\|_{p_i} = \left[\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i)\, d\mu_i(x_i)\right]^{1/p_i}, \quad i = 1, 2, \ldots, n.$$

Our results will be based on the mentioned results of Yang *et al.* In what follows, without further explanation, we assume that all integrals exist on the respective domains of their definitions.

2 Main results

In this section we develop an unified treatment of the Hilbert and Hardy-Hilbert-type inequalities with general homogeneous kernel. Further, regarding the notations from the previous section, we assume that $\Omega_i = \mathbb{R}_+$, equipped with the non-negative Lebesgue measures $d\mu_i(x_i) = dx_i$, $i = 1, 2, \ldots, n$. In addition, we have $\mathbf{\Omega} = \mathbb{R}_+^n$ and $d\mathbf{x} = dx_1\, dx_2 \cdots dx_n$.

Recall that the function $K : \mathbb{R}_+^n \to \mathbb{R}$ is said to be homogeneous of degree $-s$, $s > 0$, if $K(t\mathbf{x}) = t^{-s}K(\mathbf{x})$ for all $t > 0$. Furthermore, for $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, we define

$$k_i(\mathbf{a}) = \int_{\mathbb{R}^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^{n} u_j^{a_j}\, \hat{d}^i\mathbf{u}, \quad i = 1, 2, \ldots, n, \tag{2.1}$$

where $\hat{\mathbf{u}}^i = (u_1, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_n)$, $\hat{d}^i\mathbf{u} = du_1 \cdots du_{i-1}\, du_{i+1} \cdots du_n$, and provided that the above integral converges. Note that the constant factor $k_i(\mathbf{a})$ does not depend on the component a_i. Thus, the component a_i can be replaced with an arbitrary real number. This fact will sometimes be used in the sequel, for reasons of simpler notation. Further, by using the substitutions $t_i = -u_i$, $i \neq 1$, we obtain the following identity:

$$\int_{\mathbb{R}^{n-1}} K(-u_1, \ldots, -u_{i-1}, 1, -u_{i+1}, \ldots, -u_n) \prod_{j=1, j\neq i}^{n} |u_j|^{a_j}\, \hat{d}^i\mathbf{u} = k_i(\mathbf{a}) \tag{2.2}$$

for $i = 1, 2, \ldots, n$, where we assume that the above integral converges.

Utilizing inequalities (1.3) and (1.4) we obtain the following theorem.

Theorem 2.1 *Let $p_i > 1$, $i = 1, 2, \ldots, n$, be conjugate exponents and let A_{ij}, $i, j = 1, 2, \ldots, n$, be the real parameters such that $\sum_{i=1}^n A_{ij} = 0$, $j = 1, 2, \ldots, n$. If $K : \mathbb{R}^n \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-\lambda$, $\lambda = 2m$, $m \in \mathbb{N}$, and $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, n$, are non-negative measurable functions, then the following inequalities hold and are equivalent*:

$$\int_{\mathbb{R}^n} K(\mathbf{x}) \prod_{i=1}^n f_i(x_i)\, d\mathbf{x} \le \prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A}_i) \prod_{i=1}^n \bigl\| |x_i|^{(n-1-\lambda)/p_i + \alpha_i} f_i \bigr\|_{p_i} \tag{2.3}$$

and

$$\begin{aligned} &\left[\int_{\mathbb{R}} |x_n|^{(1-q)(n-1-\lambda) - q\alpha_n} \left(\int_{\mathbb{R}^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n\mathbf{x} \right)^q dx_n \right]^{1/q} \\ &\quad \le \prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A}_i) \prod_{i=1}^{n-1} \bigl\| |x_i|^{(n-1-\lambda)/p_i + \alpha_i} f_i \bigr\|_{p_i}, \end{aligned} \tag{2.4}$$

where $1/q = \sum_{i=1}^{n-1}(1/p_i)$, $\alpha_i = \sum_{j=1}^n A_{ij}$, $\mathbf{A}_i = (A_{i1}, A_{i2}, \ldots, A_{in})$, $i = 1, 2, \ldots, n$, and $k_i(\cdot)$, $i = 1, 2, \ldots, n$, is defined by (2.1).

Proof Rewrite inequality (1.3) for the functions $\phi_{ij}(x_j) = |x_j|^{A_{ij}}$, $i, j = 1, 2, \ldots, n$. Clearly, the set of the above defined power functions satisfies the condition

$$\prod_{i,j=1}^n \phi_{ij}(x_j) = \prod_{j=1}^n \prod_{i=1}^n x_j^{A_{ij}} = \prod_{j=1}^n x_j^{\sum_{i=1}^n A_{ij}} = 1,$$

since $\sum_{i=1}^n A_{ij} = 0$.

Obviously, it is enough to calculate the functions $\omega_i(x_i)$, $i = 1, 2, \ldots, n$. Without loss of generality, we will take into account the function $\omega_1(x_1)$.

Now, when we express the function $\omega_1(x_1)$ in terms of the integral formula (2.1), we will use the following identity:

$$\int_{-\infty}^{\infty} (\phi_{11}\omega_1 f_1)^{p_1}(x_1)\,dx_1 = \int_{-\infty}^{0} |x_1|^{p_1 A_{11}} \omega_1^{p_1}(x_1) f_1^{p_1}(x_1)\,dx_1 + \int_0^{\infty} x_1^{p_1 A_{11}} \omega_1^{p_1}(x_1) f_1^{p_1}(x_1)\,dx_1. \tag{2.5}$$

In the case when $x_1 \le 0$, $\lambda = 2m$, $m \in \mathbb{N}$, it follows that $x_1^{-\lambda} = |x_1|^{-\lambda}$. By using the substitutions $x_i = |x_1| u_i$, $i \neq 1$, the identity (2.2) and definition (2.1), we obtain the expression

$$\begin{aligned}
\omega_1^{p_1}(x_1) &= \int_{\mathbb{R}^{n-1}} K(\mathbf{x}) \prod_{j=2}^{n} |x_j|^{p_1 A_{1j}}\,\hat{d}^1\mathbf{x} \\
&= |x_1|^{n-1-\lambda+p_1(\alpha_1 - A_{11})} \int_{\mathbb{R}^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{j=2}^{n} |u_j|^{p_1 A_{1j}}\,\hat{d}^1\mathbf{u} \\
&= |x_1|^{n-1-\lambda+p_1(\alpha_1 - A_{11})} k_1(p_1\mathbf{A}_1).
\end{aligned} \tag{2.6}$$

Now, from (2.5) and (2.6) we get

$$\|\phi_{11}\omega_1 f_1\|_{p_1} = k_1(p_1\mathbf{A}_1) \int_{-\infty}^{\infty} |x_1|^{n-1-\lambda+p_1\alpha_1} f_1^{p_1}(x_1)\,dx_1.$$

Similarly to the first part of the proof we obtain the identity

$$\|\phi_{ii}\omega_i f_i\|_{p_i} = k_i(p_i\mathbf{A}_i) \int_{-\infty}^{\infty} |x_i|^{n-1-\lambda+p_i\alpha_i} f_i^{p_i}(x_i)\,dx_i, \quad i = 1, \ldots, n.$$

Finally, inequality (1.3) yields inequality (2.3). In the same way inequality (2.4) follows directly from (1.4). □

The main idea in obtaining the best possible constant factor in inequalities (2.3) and (2.4) is a reduction of the constant in the form without exponents, by an appropriate choice of the parameters $\mathbf{A}_i$, $i = 1, \ldots, n$. For that reason, we assume

$$k_1(p_1\mathbf{A}_1) = k_2(p_2\mathbf{A}_2) = \cdots = k_n(p_n\mathbf{A}_n). \tag{2.7}$$

If we use the change of variables $u_1 = 1/t_2, u_3 = t_3/t_2, u_4 = t_4/t_2, \ldots, u_n = t_n/t_2$, which provides the Jacobian of the transformation

$$\left| \frac{\partial(u_1, u_3, \ldots, u_n)}{\partial(t_2, t_3, \ldots, t_n)} \right| = t_2^{-n},$$

we have

$$\begin{aligned}
k_2(p_2\mathbf{A}_2) &= \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{t}}^1) t_2^{\lambda-n-p_2(\alpha_2 - A_{22})} \prod_{j=3}^{n} t_j^{p_2 A_{2j}}\,\hat{d}^1\mathbf{t} \\
&= k_1\bigl(p_1 A_{11}, \lambda - n - p_2(\alpha_2 - A_{22}), p_2 A_{23}, \ldots, p_2 A_{2n}\bigr).
\end{aligned}$$

According to (2.7), we have $p_1A_{12} = \lambda - n - p_2(\alpha_2 - A_{22}), p_1A_{13} = p_2A_{23}, \dots, p_1A_{1n} = p_2A_{2n}$. In a similar manner we express $k_i(p_i\mathbf{A}_i)$, $i = 3,\dots,n$, in terms of $k_1(\cdot)$. To obtain a case of the best inequality it is natural to impose the following conditions on the parameters A_{ij}:

$$p_jA_{ji} = \lambda - n - p_i(\alpha_i - A_{ii}), \quad j \neq i, i, j \in \{1,2,\dots,n\}. \tag{2.8}$$

In that case the constant factor from Theorem 2.1 is simplified to the following form:

$$L^* = k_1(\widetilde{\mathbf{A}}), \tag{2.9}$$

where $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2, \dots, \widetilde{A}_n)$ and

$$\widetilde{A}_i = p_1A_{1i} \quad \text{for } i \neq 1 \quad \text{and} \quad \widetilde{A}_1 = p_nA_{n1}. \tag{2.10}$$

Further, by using (2.8) and (2.9), inequalities (2.3) and (2.4) with the parameters A_{ij}, satisfying (2.8), become

$$\int_{\mathbb{R}^n} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\, d\mathbf{x} \leq L^* \prod_{i=1}^{n} \bigl\| |x_i|^{-\widetilde{A}_i - 1/p_i} f_i \bigr\|_{p_i} \tag{2.11}$$

and

$$\left[\int_{\mathbb{R}} |x_n|^{(1-q)(-1-p_n\widetilde{A}_n)} \left(\int_{\mathbb{R}^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n\mathbf{x}\right)^q dx_n\right]^{1/q} \\ \leq L^* \prod_{i=1}^{n-1} \bigl\| |x_i|^{-\widetilde{A}_i - 1/p_i} f_i \bigr\|_{p_i}. \tag{2.12}$$

To prove the main result we need the next lemma.

Lemma 2.2 *Let $K : \mathbb{R}_+^n \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-\lambda$, $\lambda = 2m$, $m \in \mathbb{N}$, such that for every $i = 2,3,\dots,n$,*

$$K(1, t_2, \dots, t_i, \dots, t_n) \leq CK(1, t_2, \dots, 0, \dots, t_n), \quad -1 \leq t_i \leq 1, \tag{2.13}$$

where C is a positive constant. Let the parameters $\widetilde{A}_i$, $i = 1,\dots,n$, be defined by (2.10) *and $0 < \varepsilon < \min_{1\leq i\leq n}\{p_i + p_i\widetilde{A}_i\}$. Then we have*

$$\int_{|x_1|\geq 1} |x_1|^{\widetilde{A}_1 - \varepsilon/p_1} \left(\sum_{i=2}^{n} \int_{D_i^-} K(\mathbf{x}) \prod_{j=2}^{n} |x_j|^{\widetilde{A}_j - \varepsilon/p_j}\, \hat{d}^1\mathbf{x}\right) dx_1 \\ + \int_{|x_1|\geq 1} |x_1|^{\widetilde{A}_1 - \varepsilon/p_1} \left(\sum_{i=2}^{n} \int_{D_i^+} K(\mathbf{x}) \prod_{j=2}^{n} |x_j|^{\widetilde{A}_j - \varepsilon/p_j}\, \hat{d}^1\mathbf{x}\right) dx_1 \leq O(1), \tag{2.14}$$

where

$$D_i^- = \bigl\{(x_2,\dots,x_n) : -1 \leq x_i < 0, x_j \in \mathbb{R}, j \neq i\bigr\}, \\ D_i^+ = \bigl\{(x_2,\dots,x_n) : 0 < x_i \leq 1, x_j \in \mathbb{R}, j \neq i\bigr\}, \quad i = 2,\dots,n. \tag{2.15}$$

Proof First, we define the integral $I_j(x_1)$, $j = 2, \ldots, n$, by the formulas

$$I_j(x_1) = \int_{D_j^-} K(\mathbf{x}) \prod_{i=2}^n |x_i|^{\widetilde{A}_i - \varepsilon/p_i} \, \hat{d}^1\mathbf{x},$$

where D_j^- is defined by (2.15). Without loss of generality we only estimate the integral $I_2(x_1)$. By using homogeneity of the function K, the substitutions $t_i = x_i/|x_1|$, $i \neq 1$, and the condition (2.13), we obtain

$$\begin{aligned}
I_2(x_1) &= |x_1|^{\sum_{i=2}^n (\widetilde{A}_i - \varepsilon/p_i) + n - 1 - \lambda} \int_{\mathbb{R}^{n-2}} \int_0^{1/|x_1|} K(\hat{\mathbf{t}}^1) \prod_{i=2}^n |t_i|^{\widetilde{A}_i - \varepsilon/p_i} \, \hat{d}^1\mathbf{t} \\
&\leq C|x_1|^{\sum_{i=2}^n (\widetilde{A}_i - \varepsilon/p_i) + n - 1 - \lambda} \int_{\mathbb{R}^{n-2}} K(1, 0, t_3, \ldots, t_n) \prod_{i=3}^n |t_i|^{\widetilde{A}_i - \varepsilon/p_i} \, dt_3 \cdots dt_n \\
&\quad \times \int_0^{1/|x_1|} t_2^{\widetilde{A}_2 - \varepsilon/p_2} \, dt_2 \\
&= C(1 - \varepsilon/p_2 + \widetilde{A}_2)^{-1} |x_1|^{\varepsilon/p_2 - \widetilde{A}_2 - 1} k_1(\widetilde{A}_1 - \varepsilon/p_1, \widetilde{A}_3 - \varepsilon/p_3, \ldots, \widetilde{A}_n - \varepsilon/p_n) \\
&\quad \times |x_1|^{\sum_{i=2}^n (\widetilde{A}_i - \varepsilon/p_i) + n - 1 - \lambda}.
\end{aligned}$$

Hence, we have

$$I_j(x_1) \leq |x_1|^{\varepsilon/p_j - \widetilde{A}_j - 1} |x_1|^{\sum_{i=2}^n (\widetilde{A}_i - \varepsilon/p_i) + n - 1 - \lambda} O_j(1)$$

for $\varepsilon \to 0^+$, $j \in \{2, \ldots, n\}$, and consequently

$$\int_{|x_1| \geq 1} |x_1|^{\widetilde{A}_1 - \varepsilon/p_1} \sum_{j=2}^n I_j(x_1) \, dx_1 \leq O(1). \tag{2.16}$$

In a similar manner we obtain inequality (2.16) when in the definition of the integral $I_j(x_1)$ the space is D_j^- replaced by D_j^+. Finally, from (2.16) we get (2.15). □

Now, we are ready to state and prove the main result, concerning the best possible constant factor in inequalities (2.11) and (2.12).

Theorem 2.3 *Let the kernel K and the parameters A_{ij}, $i, j = 1, \ldots, n$, be defined as in Theorem* 2.1. *If the kernel K and the parameters A_{ij} satisfy the conditions* (2.13) *and* (2.8), *respectively, then the constant L^* is the best possible in inequalities* (2.11) *and* (2.12).

Proof Let us suppose that the constant factor L^* given by (2.9) is not the best possible in inequality (2.11). Then there exists a positive constant $L_1 < L^*$, such that (2.11) is still valid when we replace L^* by L_1.

We define the real functions $\tilde{f}_{i,\varepsilon} : \mathbb{R}^n \mapsto \mathbb{R}$ by the formulas

$$\tilde{f}_{i,\varepsilon}(x_i) = \begin{cases} 0, & |x_i| < 1, \\ |x_i|^{\widetilde{A}_i - \varepsilon/p_i}, & |x_i| \geq 1, \end{cases} \qquad i = 1, \ldots, n,$$

where $0 < \varepsilon < \min_{1\le i\le n}\{p_i + p_i\widetilde{A}_i\}$. Now, we shall put these functions in inequality (2.11). Then the right-hand side of inequality (2.11) becomes

$$L_1 \prod_{i=1}^{n}\left[\int_{|x_i|\ge 1} |x_i|^{-1-\varepsilon}\,dx_i\right]^{1/p_i} = 2L_1\int_1^\infty t^{-1-\varepsilon}\,dt = \frac{2L_1}{\varepsilon}. \tag{2.17}$$

Further, let J denotes the left-hand side of inequality (2.11), for the above choice of the functions $\tilde{f}_{i,\varepsilon}$. Now, it is easy to see that the following inequality holds:

$$\begin{aligned} J \ge{}& \int_{|x_1|\ge 1} |x_1|^{\widetilde{A}_1-\varepsilon/p_1}\left(\int_{\mathbb{R}^{n-1}} K(\mathbf{x})\prod_{j=2}^{n}|x_j|^{\widetilde{A}_j-\varepsilon/p_j}\,\hat{d}^1\mathbf{x}\right)dx_1 \\ &- \int_{|x_1|\ge 1} |x_1|^{\widetilde{A}_1-\varepsilon/p_1}\left(\sum_{i=2}^{n}\int_{D_i^-} K(\mathbf{x})\prod_{j=2}^{n}|x_j|^{\widetilde{A}_j-\varepsilon/p_j}\,\hat{d}^1\mathbf{x}\right)dx_1 \\ &- \int_{|x_1|\ge 1} |x_1|^{\widetilde{A}_1-\varepsilon/p_1}\left(\sum_{i=2}^{n}\int_{D_i^+} K(\mathbf{x})\prod_{j=2}^{n}|x_j|^{\widetilde{A}_j-\varepsilon/p_j}\,\hat{d}^1\mathbf{x}\right)dx_1, \end{aligned}$$

where D_i^-, D_i^+, $i = 2,\ldots,n$, are defined by (2.15). By using the substitutions $u_i = x_i/|x_1|$, $i \ne 1$, and Lemma 2.2 we obtain

$$\begin{aligned} J &\ge \int_{|x_1|\ge 1} |x_1|^{\widetilde{A}_1-\varepsilon/p_1}\left(\int_{\mathbb{R}^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{j=2}^{n}|u_j|^{\widetilde{A}_j-\varepsilon/p_j}\,\hat{d}^1\mathbf{u}\right)dx_1 - O(1) \\ &= \frac{2}{\varepsilon}k_1(\widetilde{\mathbf{A}} - \varepsilon\mathbf{1}/\mathbf{p}) - O(1), \end{aligned} \tag{2.18}$$

where $\widetilde{\mathbf{A}} - \varepsilon\mathbf{1}/\mathbf{p} = (\widetilde{A}_1 - \varepsilon/p_1, \widetilde{A}_2 - \varepsilon/p_2, \ldots, \widetilde{A}_n - \varepsilon/p_n)$. From (2.11), (2.17), and (2.18) we get

$$k_1(\widetilde{\mathbf{A}} - \varepsilon\mathbf{1}/\mathbf{p}) - o(1) \le L_1.$$

Now, by letting $\varepsilon \to 0^+$ we obtain $L^* \le L_1$ which contradicts with the assumption $L_1 < L^*$. Thus, the constant L^* is the best possible.

Finally, the equivalence of inequalities (2.11) and (2.12) means that the constant L^* is also the best possible in inequality (2.12). That completes the proof. □

It is easy to see that the parameters A_{ij}, $i,j = 1,\ldots,n$, defined by

$$A_{ij} = \frac{\lambda - p_j}{p_i p_j}, \quad i \ne j \quad \text{and} \quad A_{ii} = \frac{(\lambda - p_i)(1 - p_i)}{p_i^2}, \tag{2.19}$$

satisfy the conditions (2.8).

Setting $n = 3$ and the parameters $\widetilde{A}_1 = (\lambda - p_1)/p_1$, $\widetilde{A}_2 = (\lambda - p_2)/p_2$, $\widetilde{A}_3 = (\lambda - p_3)/p_3$ in inequalities (2.11) and (2.12) we obtain the following result.

Corollary 2.4 *Let p_1, p_2, p_3 be conjugate parameters such that $p_i > 1$, $i = 1,2,3$, and let $1/q = 1/p_1 + 1/p_2$. Let $K : \mathbb{R}^3 \to \mathbb{R}$ be non-negative measurable homogeneous function of* *degree* $-\lambda$, $\lambda = 2m$, $m \in \mathbb{N}$, *satisfying condition* (2.13). *If* $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$, *are non-negative measurable functions, then the following inequalities hold and are equivalent*:

$$\int_{\mathbb{R}^3} K(x_1, x_2, x_3) \prod_{i=1}^{3} f_i(x_i)\, dx_1\, dx_2\, dx_3 \le M \prod_{i=1}^{3} \bigl\| |x_i|^{(p_i - \lambda - 1)/p_i} f_i \bigr\|_{p_i} \tag{2.20}$$

and

$$\left[\int_{\mathbb{R}} |x_3|^{\lambda/(p_3-1)-1} \left(\int_{\mathbb{R}^2} K(x_1, x_2, x_3) f_1(x_1) f_2(x_2)\, dx_1\, dx_2\right)^q dx_3\right]^{1/q} \le M \prod_{i=1}^{2} \bigl\| |x_i|^{(p_i - \lambda - 1)/p_i} f_i \bigr\|_{p_i}, \tag{2.21}$$

where the constant

$$M = k\left(\frac{\lambda - p_1}{p_1}, \frac{\lambda - p_2}{p_2}, \frac{\lambda - p_3}{p_3}\right)$$

is the best possible in inequalities (2.20) *and* (2.21).

Remark 2.5 Note that $K(x_1, x_2, x_3) = (|x_1| + |x_2| + |x_3|)^{-\lambda}$, $\lambda = 2m$, $m \in \mathbb{N}$, is a homogeneous function of degree $-\lambda$. In this case using Corollary 2.4 and the formula (see [8])

$$\int_{(0,\infty)^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{r_i - 1}}{(1 + \sum_{i=1}^{n-1} u_i)^{\sum_{i=1}^{n} r_i}}\, du_1 \cdots du_{n-1} = \frac{\prod_{i=1}^{n} \Gamma(r_i)}{\Gamma(\sum_{i=1}^{n} r_i)}, \quad r_i > 0, i = 1, \ldots, n,$$

we obtain the best possible constant $M = 4/\Gamma(\lambda) \prod_{i=1}^{3} \Gamma(\lambda/p_i)$.

Remark 2.6 The kernel $K_1(x, y) = \min_{i \in \{1,2\}} \{1/(x^2 + 2xy \cos\alpha_i + y^2)\}$, $0 < \alpha_1 < \alpha_2 < \pi$, is homogeneous function of degree -2. By putting the kernel $K_1(x, y)$ and the parameters $A_1 = -\lambda/q$ and $A_2 = \lambda/p$ in inequalities (2.11) and (2.12) we obtain the result of Xin *et al.* (see also [3]).

Competing interests

The author declares that they have no competing interests.

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