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A note on Hardy's inequality

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Abstract

In this paper, we prove that the inequality $\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}a_k)^p \le (\frac{p}{p-1})^p \sum_{n=1}^{\infty}(1-\frac{d(p)}{(n-1/2)^{1-1/p}})a_n^p$ holds for $p \le -1$ and $d(p)=(1+(2^{-1/p}-1)p)/[8(1+(2^{-1/p}-1)p)+2]$ if $a_n>0$ $(n=1,2,\ldots)$, and $\sum_{n=1}^{\infty}a_n^p<+\infty$.

MSC: 26D15

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1 Introduction

Let $p>1$ and $a_n>0$ $(n=1,2,\ldots)$ with $\sum_{n=1}^{\infty}a_n{}^p<+\infty$, then Hardy's well-known inequality [1] is given by

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}a_n^p. \tag{1.1}$$

Recently, the refinement, improvement, generalization, extension, and application for Hardy's inequality have attracted the attention of many researchers [2–10].

Yang and Zhu [11] presented an improvement of Hardy's inequality (1.1) for $p=2$ as follows:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^2 < 4\sum_{n=1}^{\infty}\left(1-\frac{1}{3\sqrt{n}+5}\right)a_n^2.$$

For $7/6\le p\le 2$, Huang [12] proved that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}\left(1-\frac{15}{196(n^{1-1/p}+3{,}436)}\right)a_n^p.$$

In [13], Wen and Zhang proved that the inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}\left(1-\frac{C_p}{2n^{1-1/p}}\right)a_n^p \tag{1.2}$$

holds for $p>1$ if $a_n>0$ $(n=1,2,\ldots)$, with $\sum_{n=1}^{\infty}a_n^p<+\infty$, where $C_p=1-(1-1/p)^{p-1}$ for $p\ge 2$ and $C_p=1-1/p$ for $1<p\le 2$.

Xu *et al.* [14] gave a further improvement of the inequality (1.2):

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p<\left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}\left(1-\frac{Z_p}{2(n-1)^{1-1/p}}\right)a_n^p,$$

where $Z_p=p-1-\frac{(p-1)^2}{p}2^{1/p}$ for $1<p\le 2$ and $Z_p=1-(\frac{p-1}{p})^{p-1}2^{\frac{p-1}{p}}$ for $p>2$.

For the special parameter $p=5/4$, Deng *et al.* [15] established

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{5/4}\le 5^{5/4}\sum_{n=1}^{\infty}\left(1-\frac{1}{10(n^{1/5}+\eta_{5/4})}\right)a_n^{5/4},$$

where $\eta_{5/4}=5^{5/4}/[10(5^{5/4}-(\sum_{n=1}^{\infty}\frac{1}{n^{5/4}}(\sum_{m=1}^{n}m^{-4/5})^{1/4}))-1]=0.46\cdots$.

In [16], Long and Linh discussed Hardy's inequality with the parameter $p<0$, and proved that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p<\left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n^p \tag{1.3}$$

for $p\le -1$ and

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p<\frac{2^{1-p}}{1-p}\sum_{n=1}^{\infty}a_n^p$$

for $-1<p<0$ if $a_n>0$ $(n=1,2,\ldots)$ with $\sum_{n=1}^{\infty}a_n^p<+\infty$.

It is the aim of this paper to present an improvement of inequality (1.3) for the parameter $p\le -1$. Our main result is Theorem 1.1.

Theorem 1.1 *Let* $p\le -1$, $d(p)=(1+(2^{-1/p}-1)p)/[8(1+(2^{-1/p}-1)p)+2]$ *and* $a_n>0$ $(n=1,2,\ldots)$ *with* $\sum_{n=1}^{\infty}a_n^p<+\infty$, *then*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p\le\left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}\left(1-\frac{d(p)}{(n-1/2)^{1-1/p}}\right)a_n^p.$$

2 Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

Lemma 2.1 (see [17, Corollary 1.3]) *Suppose that* $a,b\in\mathbb{R}$ *with* $a<b$, $f:[a,b]^n\to\mathbb{R}$ *has continuous partial derivatives and*

$$D_m=\left\{x=(x_1,x_2,\ldots,x_n)\big|a\le\min_{1\le k\le n}\{x_k\}<x_m=\max_{1\le k\le n}\{x_k\}\le b\right\},\quad m=1,2,\ldots,n.$$

If $\frac{\partial f(x)}{\partial x_m}>0$ *holds for all* $x=(x_1,x_2,\ldots,x_n)\in D_m$ *and* $m=1,2,\ldots,n$, *then*

$$f(x_1,x_2,\ldots,x_n)\ge f(x_{\min},x_{\min},\ldots,x_{\min})$$

for all $x_m\in[a,b]$ $(m=1,2,\ldots,n)$, *where* $x_{\min}=\min_{1\le k\le n}\{x_k\}$.

Lemma 2.2 *Let $n \in \mathbb{R}$ be a positive natural number and $r \in \mathbb{R}$ with $r \geq 1$. Then*

$$\sum_{k=1}^{n}\left(k-\frac{1}{2}\right)^{1/r} \geq \frac{r}{r+1}\left(n^{1+1/r}-1\right)+2^{-1/r}. \tag{2.1}$$

Proof We use mathematical induction to prove inequality (2.1). We clearly see that inequality (2.1) becomes equality for $n = 1$. We assume that inequality (2.1) holds for $n = i$ ($i \in \mathbb{N}$, $i \geq 1$), namely

$$\sum_{k=1}^{i}\left(k-\frac{1}{2}\right)^{1/r} \geq \frac{r}{r+1}\left(i^{1+1/r}-1\right)+2^{-1/r}.$$

Then for $n = i + 1$ we have

$$\begin{aligned}\sum_{k=1}^{i+1}\left(k-\frac{1}{2}\right)^{1/r} &= \sum_{k=1}^{i}\left(k-\frac{1}{2}\right)^{1/r}+\left(i+\frac{1}{2}\right)^{1/r} \\ &\geq \frac{r}{r+1}\left(i^{1+1/r}-1\right)+2^{-1/r}+\left(i+\frac{1}{2}\right)^{1/r} \\ &= \frac{r}{r+1}\left[(i+1)^{1+1/r}-1\right]+2^{-1/r}+\left(i+\frac{1}{2}\right)^{1/r}-\int_{i}^{i+1}x^{1/r}\,dx. \end{aligned} \tag{2.2}$$

Note that $x^{1/r}$ ($r \geq 1$) is concave on $(0, +\infty)$, therefore Hermite-Hadamard's inequality implies that

$$\left(i+\frac{1}{2}\right)^{1/r} \geq \int_{i}^{i+1}x^{1/r}\,dx. \tag{2.3}$$

From (2.2) and (2.3) we know that inequality (2.1) holds for $n = i + 1$. □

Remark 2.1 The inequality

$$2^{-1/r} \geq \frac{r}{r+1} \tag{2.4}$$

holds for all $r \geq 1$ with equality if and only if $r = 1$.

Proof We clearly see that inequality (2.4) becomes equality for $r = 1$.

If $r > 1$, then it is well known that the function $(1 + 1/r)^r$ is strictly increasing on $(1, +\infty)$, so we get

$$\left(1+\frac{1}{r}\right)^{r} > 2. \tag{2.5}$$

Therefore, inequality (2.4) follows from (2.5). □

Lemma 2.3 *The inequality*

$$\frac{(3r+1)2^{-1-1/r}}{r+1} > \left(2^{-1/r}-\frac{r}{r+1}\right)\frac{3r^2+1}{r} \tag{2.6}$$

holds for all $r \geq 1$.

Proof Let $r \geq 1$, then we clearly see that

$$(6\log 2 - 3)r^2 + r + 2\log 2 - 2 \geq 4(2\log 2 - 1) > 0. \tag{2.7}$$

Inequality (2.7) leads to

$$e^{(3r^2-r+2)/(6r^2+2)} < 2. \tag{2.8}$$

It follows from the well-known inequality $(1+x)^{1/x} < e$ $(x > 0)$ that

$$e > \left(1 + \frac{3r^2 - r + 2}{6r^3 + 2r}\right)^{(6r^3+2r)/(3r^2-r+2)}. \tag{2.9}$$

From (2.8) and (2.9) we have

$$2^{-1/r} < \frac{6r^3 + 2r}{6r^3 + 3r^2 + r + 2}. \tag{2.10}$$

Therefore, inequality (2.6) follows easily from (2.10). □

Lemma 2.4 *Let* $r \geq 1$ *and*

$$f(x) = \frac{x^r}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}}. \tag{2.11}$$

Then f *is convex on* $[1/2, +\infty)$.

Proof From (2.11) we have

$$\begin{aligned} f'(x) &= \frac{-\frac{2r+1}{r+1}x^{r+1/r} + (2^{-1/r} - \frac{r}{r+1})rx^{r-1}}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+2}}, \\ f''(x) &= \frac{\frac{(3r+1)(2r+1)}{(r+1)^2}x^{2+2/r} - (2^{-1/r} - \frac{r}{r+1})\frac{(3r^2+1)(2r+1)}{r(r+1)}x^{1+1/r} + r(r-1)(2^{-1/r} - \frac{r}{r+1})^2}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+3}}x^{r-2}. \end{aligned} \tag{2.12}$$

It follows from Lemma 2.3 and (2.12) that

$$f''(x) \geq \frac{\frac{(3r+1)(2r+1)}{(r+1)^2}2^{-1-1/r} - (2^{-1/r} - \frac{r}{r+1})\frac{(3r^2+1)(2r+1)}{r(r+1)}}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+3}}x^{r+1/r-1} > 0 \tag{2.13}$$

for all $x \in [1/2, +\infty)$.

Therefore, Lemma 2.4 follows from inequality (2.13). □

Lemma 2.5 *Let* $r \geq 1$, $0 \leq t \leq 4$ *and* $c = (r + 1 - 2^{1/r}r)/[8(r + 1 - 2^{1/r}r) + 2]$, *then*

$$(r+1)\left(2^{-1/r} - \frac{r}{r+1}\right)(1 - ct)t \geq 1 - \left[1 + \left(\frac{1+r}{2^{1/r}r} - 1\right)t\right]^{-r}. \tag{2.14}$$

Proof If $r = 1$, then we clearly see that inequality (2.14) becomes equality. Next, we assume that $r > 1$. Let

$$f(t) = (r+1)\left(2^{-1/r} - \frac{r}{r+1}\right)(1-ct)t - 1 + \left[1 + \left(\frac{1+r}{2^{1/r}r} - 1\right)t\right]^{-r}. \tag{2.15}$$

Then simple computations lead to

$$f(0) = 0, \tag{2.16}$$

$$f'(t) = \frac{(r+1)(2^{-1/r} - \frac{r}{r+1})}{[1 + (\frac{r+1}{2^{1/r}r} - 1)t]^{r+1}}\left\{(1-2ct)\left[1 + \left(\frac{r+1}{2^{1/r}r} - 1\right)t\right]^{r+1} - 1\right\}. \tag{2.17}$$

Note that

$$1 - 2ct \geq 1 - 8c = \frac{1}{4(r+1-2^{1/r}r)+1} > 0, \tag{2.18}$$

$$\left[1 + \left(\frac{r+1}{2^{1/r}r} - 1\right)t\right]^{r+1} \geq 1 + (r+1)\left(\frac{r+1}{2^{1/r}r} - 1\right)t. \tag{2.19}$$

It follows from Remark 2.1 and (2.17)-(2.19) that

$$f'(t) \geq \frac{(r+1)(2^{-1/r} - \frac{r}{r+1})}{[1 + (\frac{r+1}{2^{1/r}r} - 1)t]^{r+1}}\left\{(1-2ct)\left[1 + (r+1)\left(\frac{r+1}{2^{1/r}r} - 1\right)t\right] - 1\right\}. \tag{2.20}$$

Let

$$g(t) = (1-2ct)\left[1 + (r+1)\left(\frac{r+1}{2^{1/r}r} - 1\right)t\right] - 1. \tag{2.21}$$

Then from $g(0) = 0$ and $g(4) = 4(r+1-2^{1/r}r)^2/[2^{1/r}r(4(r+1-2^{1/r}r)+1)] \geq 0$ together with the fact that $g(t)$ is a concave parabola we know that

$$g(t) \geq 0 \tag{2.22}$$

for $t \in [0,4]$.

Therefore, Lemma 2.5 follows easily from (2.15) and (2.16) together with (2.20)-(2.22). □

Lemma 2.6 *Let* $r \geq 1$, $c = (r+1-2^{1/r}r)/[8(r+1-2^{1/r}r)+2]$, N *is a positive natural number,* $a_k > 0$ $(k = 1,2,\ldots,N)$ *and* $B_N = \min_{1\leq k\leq N}\{(k-1/2)^{1/r}a_k\}$, *then*

$$\begin{aligned}
&\left(\frac{r+1}{r}\right)^r \sum_{n=1}^{N}\left(1 - \frac{c}{(n-1/2)^{1+1/r}}\right)a_n^r - \sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n} 1/a_k}\right)^r \\
&\quad\geq B_N^r\left[\left(\frac{r+1}{r}\right)^r \sum_{n=1}^{N}\left(1 - \frac{c}{(n-1/2)^{1+1/r}}\right)\frac{1}{n-1/2}\right. \\
&\quad\quad\left. - \sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n}(k-1/2)^{1/r}}\right)^r\right].
\end{aligned} \tag{2.23}$$

Proof Let $a_k = b_k/(k-1/2)^{1/r}$ $(k = 1, 2, \dots, N)$, then $B_N = \min_{1\le k\le N}\{b_k\}$ and inequality (2.23) becomes

$$\begin{aligned}&\left(\frac{r+1}{r}\right)^r \sum_{n=1}^{N}\left(1-\frac{c}{(n-1/2)^{1+1/r}}\right)\frac{b_n^r}{n-1/2} - \sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n}\frac{(k-1/2)^{1/r}}{b_k}}\right)^r \\ &\quad\ge B_N^r\left[\left(\frac{r+1}{r}\right)^r \sum_{n=1}^{N}\left(1-\frac{c}{(n-1/2)^{1+1/r}}\right)\frac{1}{n-1/2}\right. \\ &\qquad\left. - \sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n}(k-1/2)^{1/r}}\right)^r\right]. \end{aligned} \tag{2.24}$$

Let $D_m = \{\mathbf{b} = (b_1, b_2, \dots, b_N) | b_m = \max_{1\le k\le N}\{b_k\} > \min_{1\le k\le N}\{b_k\}\}$ $(m = 1, 2, \dots, N)$, and

$$\begin{aligned} f(b_1, b_2, \dots, b_N) &= \left(\frac{r+1}{r}\right)^r \sum_{n=1}^{N}\left(1-\frac{c}{(n-1/2)^{1+1/r}}\right)\frac{b_n^r}{n-1/2} \\ &\quad - \sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n}\frac{(k-1/2)^{1/r}}{b_k}}\right)^r. \end{aligned} \tag{2.25}$$

Then for any $\mathbf{b} \in D_m$ $(m = 1, 2, \dots, N)$ we have

$$\begin{aligned} \frac{\partial f(\mathbf{b})}{\partial b_m} &= \left(1-\frac{c}{(m-1/2)^{1+1/r}}\right)\frac{(r+1)^r b_m^{r-1}}{(m-1/2)r^{r-1}} \\ &\quad - \frac{r(m-1/2)^{1/r}}{b_m^2}\sum_{n=m}^{N}\frac{n^r}{(\sum_{k=1}^{n}\frac{(k-1/2)^{1/r}}{b_k})^{r+1}} \\ &> \left(1-\frac{c}{(m-1/2)^{1+1/r}}\right)\frac{(r+1)^r b_m^{r-1}}{(m-1/2)r^{r-1}} \\ &\quad - r(m-1/2)^{1/r} b_m^{r-1}\sum_{n=m}^{+\infty}\frac{n^r}{(\sum_{k=1}^{n}(k-1/2)^{1/r})^{r+1}}. \end{aligned} \tag{2.26}$$

From Lemma 2.2 and (2.26) one has

$$\begin{aligned} \frac{1}{r(m-1/2)^{1/r} b_m^{r-1}}\frac{\partial f(\mathbf{b})}{\partial b_m} &> \left(\frac{r+1}{r}\right)^r\left(1-\frac{c}{(m-1/2)^{1+1/r}}\right)\frac{1}{(m-1/2)^{1+1/r}} \\ &\quad - \sum_{n=m}^{+\infty}\frac{n^r}{(\frac{r}{r+1}n^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}}. \end{aligned} \tag{2.27}$$

It clearly follows from Lemma 2.4 and the Hermite-Hadamard inequality that

$$\int_{m-1/2}^{m+1/2}\frac{x^r}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}} \ge \frac{m^r}{(\frac{r}{r+1}m^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}}$$

and

$$\int_{m-1/2}^{+\infty}\frac{x^r}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}} \ge \sum_{n=m}^{+\infty}\frac{n^r}{(\frac{r}{r+1}n^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}}. \tag{2.28}$$

Note that

$$\int_{m-1/2}^{+\infty} \frac{x^r}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}} = \left(\frac{1+r}{r}\right)^r \frac{2^{1/r}}{r+1-2^{1/r}r}\left\{1 - \left[1 + \left(\frac{r+1}{2^{1/r}r} - 1\right)(m-1/2)^{-1-1/r}\right]^{-r}\right\}, \tag{2.29}$$

$$0 < (m-1/2)^{-1-1/r} \le 2^{1+1/r} \le 4. \tag{2.30}$$

From Lemma 2.5 and (2.30) one has

$$\left(\frac{r+1}{r}\right)^r \left(1 - \frac{c}{(m-1/2)^{1+1/r}}\right)\frac{1}{(m-1/2)^{1+1/r}} \ge \left(\frac{1+r}{r}\right)^r \frac{2^{1/r}}{r+1-2^{1/r}r}\left\{1 - \left[1 + \left(\frac{r+1}{2^{1/r}r} - 1\right)(m-1/2)^{-1-1/r}\right]^{-r}\right\}. \tag{2.31}$$

Inequalities (2.27), (2.28), and (2.31) together with (2.29) lead to the conclusion that

$$\frac{\partial f(\mathbf{b})}{\partial b_m} > 0 \tag{2.32}$$

for any $\mathbf{b} = (b_1, b_2, \dots, b_N) \in D_m$ and $m = 1, 2, \dots, N$.

It follows from Lemma 2.1 and (2.32) that

$$f(b_1, b_2, \dots, b_N) \ge f(B_N, B_N, \dots, B_N). \tag{2.33}$$

Therefore, inequality (2.24) follows from (2.25) and (2.33). □

Lemma 2.7 *Let* $r \ge 1$, $c = (r+1-2^{1/r}r)/[8(r+1-2^{1/r}r)+2]$, *then*

$$\left(\frac{r+1}{r}\right)\left(1 - 2^{1+1/r}c\right) > 1. \tag{2.34}$$

Proof We clearly see that inequality (2.34) holds for $r = 1$. Next, we assume that $r > 1$, let $t = 2^{1+1/r}$, then $0 < t < 4$ and Lemma 2.5 leads to

$$\begin{aligned}\left(\frac{r+1}{r}\right)^r\left(1 - 2^{1+1/r}c\right) - 1 &\ge \frac{(\frac{r+1}{r})^r}{2(r+1-2^{1/r}r)}\left[1 - \left(1 + \frac{2(r+1-2^{1/r}r)}{r}\right)^{-r}\right] - 1 \\ &\ge \frac{(\frac{r+1}{r})^r}{2(r+1-2^{1/r}r)}\frac{2(r+1-2^{1/r}r)}{1+2(r+1-2^{1/r}r)} - 1 \\ &= \frac{(\frac{r+1}{r})^r}{1+2(r+1-2^{1/r}r)} - 1.\end{aligned} \tag{2.35}$$

Note that

$$r + 1 - 2^{1/r}r < 1 - \log 2 \tag{2.36}$$

for all $r \ge 1$. In fact, let $x \ge 1$ and

$$f(x) = x - 2^{1/x}x + 1. \tag{2.37}$$

Then

$$f'(x) = 1 + \left(\frac{\log 2}{x} - 1\right)2^{1/x}, \tag{2.38}$$

$$f''(x) = -\frac{(\log 2)^2}{x^3}2^{1/x} < 0. \tag{2.39}$$

It follows from (2.38) and (2.39) that

$$f'(x) > \lim_{x\to+\infty}\left[1 + \left(\frac{\log 2}{x} - 1\right)2^{1/x}\right] = 0. \tag{2.40}$$

Equation (2.37) and inequality (2.40) lead to the conclusion that

$$f(x) < \lim_{x\to+\infty}\left(x - 2^{1/x}x + 1\right) = 1 - \log 2. \tag{2.41}$$

From (2.35) and (2.36) together with the fact that $[(r+1)/r]^r \geq 2$ we have

$$\left(\frac{r+1}{r}\right)^r\left(1 - 2^{1+1/r}c\right) - 1 > \frac{2}{1 + 2(1 - \log 2)} - 1 = \frac{2\log 2 - 1}{3 - 2\log 2} > 0. \tag{2.42}$$

Therefore, inequality (2.34) follows from (2.42). □

Lemma 2.8 *Let* $r \geq 1$, $c = (r + 1 - 2^{1/r}r)/[8(r + 1 - 2^{1/r}r) + 2]$, N *is a positive natural number,* $a_k > 0$ $(k = 1, 2, \ldots, N)$ *and* $B_N = \min_{1\leq k\leq N}\{(k - 1/2)^{1/r}a_k\}$, *then*

$$\left(\frac{r+1}{r}\right)^r\sum_{n=1}^{N}\left(1 - \frac{c}{(n-1/2)^{1+1/r}}\right)a_n^r - \sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n}1/a_k}\right)^r$$
$$\geq 2B_N^r\left[\left(\frac{r+1}{r}\right)^r\left(1 - 2^{1+1/r}c\right) - 1\right]. \tag{2.43}$$

Proof Let $m \in \{1, 2, \ldots, N\}$, $f(0) = 0$ and

$$f(m) = \left(\frac{r+1}{r}\right)^r\sum_{n=1}^{m}\left(1 - \frac{c}{(n-1/2)^{1+1/r}}\right)\frac{1}{n - 1/2} - \sum_{n=1}^{m}\left(\frac{n}{\sum_{k=1}^{n}(k-1/2)^{1/r}}\right)^r. \tag{2.44}$$

Then

$$f(1) = 2\left[\left(\frac{1+r}{r}\right)^r\left(1 - 2^{1+1/r}c\right) - 1\right], \tag{2.45}$$

$$f(m) - f(m-1) = \frac{(\frac{1+r}{r})^r}{m - 1/2}\left(1 - \frac{c}{(m-1/2)^{1+1/r}}\right) - \left(\frac{m}{\sum_{k=1}^{m}(k-1/2)^{1/r}}\right)^r. \tag{2.46}$$

It follows from Lemma 2.2 and (2.46) together with Remark 2.1 that

$$f(m) - f(m-1)$$
$$\geq \frac{(\frac{1+r}{r})^r}{m - 1/2}\left(1 - \frac{c}{(m-1/2)^{1+1/r}}\right) - \left(\frac{m}{\frac{r}{r+1}(m^{1+1/r} - 1) + 2^{-1/r}}\right)^r$$

$$\begin{aligned}&\geq \frac{(\frac{1+r}{r})^r}{m-1/2}\left(1-\frac{c}{(m-1/2)^{1+1/r}}\right)-\left(\frac{m}{\frac{r}{r+1}m^{1+1/r}}\right)^r\\&=\frac{(\frac{1+r}{r})^r[(4(r+1-2^{1/r}r)+1)(m-1/2)^{1+1/r}-m(r+1-2^{1/r}r)]}{m(m-1/2)^{2+1/r}[8(r+1-2^{1/r}r)+2]}.\end{aligned}\tag{2.47}$$

Let

$$g(t)=\left[4\left(r+1-2^{1/r}r\right)+1\right](t-1/2)^{1+1/r}-\left(r+1-2^{1/r}r\right)t.\tag{2.48}$$

Then

$$\begin{aligned}g(1)&=\left[4\left(r+1-2^{1/r}r\right)+1\right]2^{-1-1/r}-\left(r+1-2^{1/r}r\right)\\&>\left(2^{1-1/r}-1\right)\left(r+1-2^{1/r}r\right)\geq 0,\end{aligned}\tag{2.49}$$

$$\begin{aligned}g'(t)&=\left(1+\frac{1}{r}\right)\left[4\left(r+1-2^{1/r}r\right)+1\right](t-1/2)^{1/r}-\left(r+1-2^{1/r}r\right)\\&>\left(2^{2-1/r}-1\right)\left(r+1-2^{1/r}r\right)\geq 0\end{aligned}\tag{2.50}$$

for $t\geq 1$.

From (2.47)-(2.50) we get

$$f(1)<f(2)<\cdots<f(N-1)<f(N).\tag{2.51}$$

Therefore, Lemma 2.8 follows easily from Lemma 2.6, (2.44), (2.45), and (2.51). □

3 Proof of Theorem 1.1

Let $r=-p$, $c=c(r)=d(-r)$ and $b_n=1/a_n$ $(n=1,2,\ldots)$, then $r\geq 1$, $c=(r+1-2^{1/r}r)/[8(r+1-2^{1/r}r)+2]$, $b_n>0$ and $\sum_{n=1}^{\infty}b_n^r<+\infty$.

It follows from Lemmas 2.7 and 2.8 that one has

$$\sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n}1/b_k}\right)^r\leq\left(\frac{r+1}{r}\right)^r\sum_{n=1}^{N}\left(1-\frac{c}{(n-1/2)^{1+1/r}}\right)b_n^r.\tag{3.1}$$

Letting $n\to+\infty$, (3.1) leads to

$$\sum_{n=1}^{\infty}\left(\frac{n}{\sum_{k=1}^{n}1/b_k}\right)^r\leq\left(\frac{r+1}{r}\right)^r\sum_{n=1}^{\infty}\left(1-\frac{c}{(n-1/2)^{1+1/r}}\right)b_n^r.\tag{3.2}$$

Therefore, Theorem 1.1 follows immediately from (3.2) and $r=-p$ together with $b_n=1/a_n$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Y-MC provided the main idea and carried out the proof of Lemmas 2.1 and 2.2. QX carried out the proof of Lemmas 2.3-2.5 and Theorem 1.1. X-MZ carried out the proof of Lemmas 2.6-2.8. All authors read and approved the final manuscript.

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