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Multidimensional fixed points for generalized ψ-quasi-contractions in quasi-metric-like spaces

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Abstract
In this paper, we introduce the concept of a quasi-metric-like space, and by defining the w-compatibility of two mappings, we obtain multidimensional coincidence point and multidimensional fixed point theorems for generalized ψ-quasi-contractions in quasi-metric-like spaces. Our results extend the fixed point theorems in Vetro and Radenović (Appl. Math. Comput. 219:1594-1600, 2012) and references therein.

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Keywords: quasi-metric-like space; w-compatibility; coincidence point; fixed point

1 Introduction and preliminaries

In 1987, Guo and Lakshmikantham [1] initiated the study of the coupled fixed point. In 2010, Samet and Vetro [2] presented the concept of a fixed point of N-order as an extension of the coupled fixed point.

Definition 1.1 ([2]) Let X be a non-empty set and let $F : X^N \to X$ $(N \geq 2)$ be a given mapping. An element $(x_1, x_2, \dots, x_N) \in X^N$ is called a fixed point of N-order of the mapping F if

$$\begin{aligned} &F(x_1, x_2, \dots, x_{N-1}, x_N) = x_1, \\ &F(x_2, x_3, \dots, x_N, x_1) = x_2, \\ &\vdots \\ &F(x_N, x_1, x_2, \dots, x_{N-1}) = x_N. \end{aligned}$$

Subsequently, a number of papers occurred on tripled fixed point and quadruple fixed point theory (see, *e.g.*, [3–10]). Berzig and Samet [11] discussed the existence of the fixed point of N-order for m-mixed monotone mappings in complete ordered metric spaces. Very recently, Roldán *et al.* [12] extended the notion of the fixed point of N-order to the Φ-fixed point and obtained some Φ-fixed point theorems for a mixed monotone mapping in partially ordered complete metric spaces. Afterward, many results on multidimensional fixed points have been established (see, *e.g.*, [13–18]).

Matthews [19] introduced the notion of a partial metric space where the self-distance does not need to be zero. By generalizing the partial metric, Hitzler and Seda [20] presented the concept of a dislocated metric which was redefined as a metric-like by Amini-Harandi [21]. The existence of fixed points in dislocated metric (metric-like) spaces has been discussed by many authors (see, *e.g.*, [22–30]).

Definition 1.2 ([20, 21]) A mapping $\sigma : X \times X \to [0, +\infty)$, where X is a nonempty set, is said to be a dislocated metric (metric-like) on X if, for any $x, y, z \in X$, the following three conditions hold true:

(σ1) $\sigma(x,y) = \sigma(y,x) = 0 \Rightarrow x = y$;
(σ2) $\sigma(x,y) = \sigma(y,x)$;
(σ3) $\sigma(x,z) \le \sigma(x,y) + \sigma(y,z)$.

The pair (X, σ) is then called a dislocated metric (metric-like) space.

Karapınar *et al.* [31] introduced the notion of quasi-partial metric spaces and studied some fixed point theorems on quasi-partial metric spaces.

Definition 1.3 ([31]) A quasi-partial metric on a nonempty set X is a function $q : X \times X \to R^+$ which satisfies:

(QPM_1) If $0 \le q(x,x) = q(x,y) = q(y,y)$, then $x = y$,
(QPM_2) $q(x,x) \le q(x,y)$,
(QPM_3) $q(x,x) \le q(y,x)$, and
(QPM_4) $q(x,z) + q(y,y) \le q(x,y) + q(y,z)$,

for all $x, y, z \in X$. The pair (X, q) is called a quasi-partial metric space.

In this paper, similar to the notation of Amini-Harandi [21], we define a quasi-metric-like space generalizing the metric-like space and the quasi-partial metric space. Furthermore, we discuss the existence and uniqueness of a multidimensional fixed point for a generalized g-ψ-quasi-contractive mapping in quasi-metric-like spaces using the new w-compatibility of two mappings.

2 A quasi-metric-like space

Definition 2.1 A mapping $\rho : X \times X \to [0, +\infty)$, where X is a nonempty set, is said to be a quasi-metric-like on X if, for any $x, y, z \in X$, the following conditions hold:

(ρ1) $\rho(x,y) = 0 \Rightarrow x = y$;
(ρ2) $\rho(x,z) \le \rho(x,y) + \rho(y,z)$.

The pair (X, ρ) is called a quasi-metric-like space.

Definition 2.2 Let (X, ρ) be a quasi-metric-like space. Then

(1) A sequence $\{x_n\}$ converges to a point $x \in X$ if and only if

$$\lim_{n\to+\infty} \rho(x_n, x) = \lim_{n\to+\infty} \rho(x, x_n) = \rho(x, x).$$

In this case, x is called a ρ-limit of $\{x_n\}$.

(2) A sequence $\{x_n\}$ is called a Cauchy sequence in (X,ρ) if $\lim_{m,n\to+\infty}\rho(x_m,x_n)$ and $\lim_{m,n\to+\infty}\rho(x_n,x_m)$ exist and are finite.
(3) The quasi-metric-like-like space (X,ρ) is called complete if, for every Cauchy sequence $\{x_n\}$ in X, there is some $x\in X$ such that

$$\begin{aligned}\lim_{n\to+\infty}\rho(x_n,x) = \lim_{n\to+\infty}\rho(x,x_n) = \rho(x,x)\\ = \lim_{m,n\to+\infty}\rho(x_m,x_n) = \lim_{m,n\to+\infty}\rho(x_n,x_m).\end{aligned}$$

Every quasi-partial metric space is a quasi-metric-like space. Below we give an example of a quasi-metric-like space.

Example 2.3 Let $X=\{0,1\}$, and let

$$\rho(x,y)=\begin{cases}2, & \text{if } x=y=0;\\ 1, & \text{if } x=0, y=1;\\ \frac{3}{2}, & \text{if } x=1, y=0;\\ 0, & \text{if } x=y=1.\end{cases}$$

Then (X,ρ) is a quasi-metric-like space, but $\rho(0,0)\nleq\rho(1,0)$, so (X,ρ) is not a quasi-partial metric space.

Remark 2.4 Every metric-like space is a quasi-metric-like space. Because the limit of a convergent sequence in metric-like space is not necessarily unique [25], the ρ-limit of a convergent sequence in quasi-metric-like spaces is not necessarily unique.

3 Main results

In this section, we establish the coincidence point and fixed point of r-order theorems, and an illustrative example is employed to show the validity of our results.

Definition 3.1 Let X be a nonempty set, and let $g:X\to X$ and let $F:X^r\to X$ ($r\geq 2$) be two given mappings. An element $(x_1,x_2,\ldots,x_r)\in X^r$ is called a coincidence point of r-order of $F:X^r\to X$ and $g:X\to X$ if

$$\begin{aligned}g(x_1) &= F(x_1,x_2,\ldots,x_{r-1},x_r),\\ g(x_2) &= F(x_2,x_3,\ldots,x_r,x_1),\\ &\vdots\\ g(x_r) &= F(x_r,x_1,x_2,\ldots,x_{r-1}).\end{aligned}$$

If g is the identity mapping on X, then $(x_1,x_2,\ldots,x_r)\in X^r$ is a fixed point of r-order of the mapping F.

Throughout this paper, we denote all of the coincidence points of r-order of $F:X^r\to X$ and $g:X\to X$ by $C(F,g,r)$.

By Ψ we denote the set of real functions $\psi:[0,+\infty)\to[0,+\infty)$ which have the following properties:

(i) ψ is nondecreasing;
(ii) $\psi(0) = 0$;
(iii) $\lim_{t\to+\infty}(t - \psi(t)) = +\infty$;
(iv) $\lim_{s\to t^+} \psi(s) < t$ for all $t > 0$.

From (iv) and $\psi(t) \le \lim_{s\to t^+} \psi(s) < t$, we deduce that $\psi(t) < t$ for all $t > 0$ [32].

Vetro and Radenović [32] introduced the concept of a g-ψ-quasi-contraction. We present the following definition as a generalization of the g-ψ-quasi-contraction.

Definition 3.2 Let (X,ρ) be a quasi-metric-like space, $g : X \to X$ and let $F : X^r \to X$ $(r \ge 2)$. F is called a generalized g-ψ-quasi-contraction if there exists $\psi : [0,+\infty) \to [0,+\infty)$ such that

$$\rho\bigl(F(x_1,x_2,\dots,x_r),F(y_1,y_2,\dots,y_r)\bigr) \le \psi\bigl(M(x_1,x_2,\dots,x_r;y_1,y_2,\dots,y_r)\bigr), \tag{1}$$

where

$$\begin{aligned}
&M(x_1,x_2,\dots,x_r;y_1,y_2,\dots,y_r)\\
&\quad= \max\bigl\{\rho(gx_1,gy_1),\rho(gx_2,gy_2),\dots,\\
&\qquad \rho(gx_r,gy_r),\rho\bigl(gx_1,F(x_1,x_2,\dots,x_r)\bigr),\rho\bigl(gx_2,F(x_2,x_3,\dots,x_r,x_1)\bigr),\dots,\\
&\qquad \rho\bigl(gx_r,F(x_r,x_1,\dots,x_{r-1})\bigr),\rho\bigl(gy_1,F(y_1,y_2,\dots,y_r)\bigr),\\
&\qquad \rho\bigl(gy_2,F(y_2,y_3,\dots,y_r,y_1)\bigr),\dots,\rho\bigl(gy_r,F(y_r,y_1,\dots,y_{r-1})\bigr),\\
&\qquad \rho\bigl(gx_1,F(y_1,y_2,\dots,y_r)\bigr),\rho\bigl(gx_2,F(y_2,y_3,\dots,y_r,y_1)\bigr),\dots,\\
&\qquad \rho\bigl(gx_r,F(y_r,y_1,\dots,y_{r-1})\bigr),\rho\bigl(gy_1,F(x_1,x_2,\dots,x_r)\bigr),\\
&\qquad \rho\bigl(gy_2,F(x_2,x_3,\dots,x_r,x_1)\bigr),\dots,\rho\bigl(gy_r,F(x_r,x_1,\dots,x_{r-1})\bigr)\bigr\},
\end{aligned} \tag{2}$$

for any $(x_1,x_2,\dots,x_r),(y_1,y_2,\dots,y_r) \in X^r$.

If g is the identity mapping, then F is a generalized ψ-quasi-contraction.

Definition 3.3 Let X be a nonempty set. The mappings $g : X \to X$ and $F : X^r \to X$ $(r \ge 2)$ are called w-compatible if

$$F\bigl(g(x_1),g(x_2),\dots,g(x_r)\bigr) = g\bigl(F(x_1,x_2,\dots,x_r)\bigr),$$

whenever $(x_1,x_2,\dots,x_r) \in C(F,g,r)$.

Theorem 3.4 *Let (X,ρ) be a quasi-metric-like space, $g : X \to X$ and $F : X^r \to X$ $(r \ge 2)$. Suppose that F is a generalized g-ψ-quasi-contraction with $\psi \in \Psi$. If $F(X^r) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then $C(F,g,r)$ is nonempty.*

Proof Let $(x_1^0,x_2^0,\dots,x_r^0) \in X^r$. Since $F(X^r) \subseteq g(X)$, we can construct a sequence $\{(x_1^n,x_2^n,\dots,x_r^n)\}$ such that

$$g\bigl(x_i^n\bigr) = F\bigl(x_i^{n-1},x_{i+1}^{n-1},\dots,x_r^{n-1},x_1^{n-1},\dots,x_{i-1}^{n-1}\bigr) \quad \text{for } i = 1,2,\dots,r.$$

Define

$$\begin{aligned}
&O_n(x_1^0,x_2^0,\ldots,x_r^0)=\{gx_1^0,gx_2^0,\ldots,gx_r^0,gx_1^1,gx_2^1,\ldots,gx_r^1,\ldots,gx_1^n,gx_2^n,\ldots,gx_r^n\},\\
&O(x_1^0,x_2^0,\ldots,x_r^0)=\{gx_1^0,gx_2^0,\ldots,gx_r^0,gx_1^1,gx_2^1,\ldots,gx_r^1,\ldots,gx_1^n,gx_2^n,\ldots,gx_r^n,\ldots\},\\
&\delta_n(x_1^0,x_2^0,\ldots,x_r^0)=\operatorname{diam}(O_n(x_1^0,x_2^0,\ldots,x_r^0))=\sup\{\rho(x,y):x,y\in O_n(x_1^0,x_2^0,\ldots,x_r^0)\}.
\end{aligned}$$

If there exists $n_0\in N$ such that $\delta_{n_0}(x_1^0,x_2^0,\ldots,x_r^0)=0$, then for any $0\le k\le n_0-1$, $(x_1^k,x_2^k,\ldots,x_r^k)\in C(F,g,r)$.

We suppose that $\delta_n(x_1^0,x_2^0,\ldots,x_r^0)>0$, for all $n\in N$.

Step 1. We shall prove that for each $n\in N$,

$$\delta_n(x_1^0,x_2^0,\ldots,x_r^0)=\max\Big\{\sup_{1\le i,l\le r,0\le s\le n}\rho(gx_i^0,gx_l^s),\sup_{1\le i,l\le r,0\le s\le n}\rho(gx_l^s,gx_i^0)\Big\}. \tag{3}$$

In fact, for any $1\le i,l\le r$, $1\le j,s\le n$, we have

$$\begin{aligned}
\rho(gx_i^j,gx_l^s)&=\rho\big(F(x_i^{j-1},x_{i+1}^{j-1},\ldots,x_r^{j-1},x_1^{j-1},\ldots,x_{i-1}^{j-1}),\\
&\qquad F(x_l^{s-1},x_{l+1}^{s-1},\ldots,x_r^{s-1},x_1^{s-1},\ldots,x_{l-1}^{s-1})\big)\\
&\le\psi\big(M(x_i^{j-1},\ldots,x_r^{j-1},x_1^{j-1},\ldots,x_{i-1}^{j-1};\\
&\qquad x_l^{s-1},\ldots,x_r^{s-1},x_1^{s-1},\ldots,x_{l-1}^{s-1})\big),
\end{aligned}\tag{4}$$

where

$$\begin{aligned}
&M(x_i^{j-1},\ldots,x_r^{j-1},x_1^{j-1},\ldots,x_{i-1}^{j-1};x_l^{s-1},\ldots,x_r^{s-1},x_1^{s-1},\ldots,x_{l-1}^{s-1})\\
&\quad=\max\{\rho(gx_i^{j-1},gx_l^{s-1}),\rho(gx_{i+1}^{j-1},gx_{l+1}^{s-1}),\ldots,\rho(gx_{i-1}^{j-1},gx_{l-1}^{s-1}),\\
&\qquad\rho(gx_i^{j-1},F(x_i^{j-1},x_{i+1}^{j-1},\ldots,x_r^{j-1},x_1^{j-1},\ldots,x_{i-1}^{j-1})),\\
&\qquad\rho(gx_l^{s-1},F(x_l^{s-1},x_{l+1}^{s-1},\ldots,x_r^{s-1},x_1^{s-1},\ldots,x_{l-1}^{s-1})),\\
&\qquad\rho(gx_{i+1}^{j-1},F(x_{i+1}^{j-1},x_{i+2}^{j-1},\ldots,x_r^{j-1},x_1^{j-1},\ldots,x_i^{j-1})),\\
&\qquad\rho(gx_{l+1}^{s-1},F(x_{l+1}^{s-1},x_{l+2}^{s-1},\ldots,x_r^{s-1},x_1^{s-1},\ldots,x_l^{s-1})),\ldots,\\
&\qquad\rho(gx_{i-1}^{j-1},F(x_{i-1}^{j-1},x_i^{j-1},\ldots,x_r^{j-1},x_1^{j-1},\ldots,x_{i-2}^{j-1})),\\
&\qquad\rho(gx_{l-1}^{s-1},F(x_{l-1}^{s-1},x_l^{s-1},\ldots,x_r^{s-1},x_1^{s-1},\ldots,x_{l-2}^{s-1})),\\
&\qquad\rho(gx_l^{s-1},F(x_i^{j-1},x_{i+1}^{j-1},\ldots,x_r^{j-1},x_1^{j-1},\ldots,x_{i-1}^{j-1})),\\
&\qquad\rho(gx_i^{j-1},F(x_l^{s-1},x_{l+1}^{s-1},\ldots,x_r^{s-1},x_1^{s-1},\ldots,x_{l-1}^{s-1})),\\
&\qquad\rho(gx_{l+1}^{s-1},F(x_{i+1}^{j-1},x_{i+2}^{j-1},\ldots,x_r^{j-1},x_1^{j-1},\ldots,x_i^{j-1})),\\
&\qquad\rho(gx_{i+1}^{j-1},F(x_{l+1}^{s-1},x_{l+2}^{s-1},\ldots,x_r^{s-1},x_1^{s-1},\ldots,x_l^{s-1})),\ldots,\\
&\qquad\rho(gx_{l-1}^{s-1},F(x_{i-1}^{j-1},x_i^{j-1},\ldots,x_r^{j-1},x_1^{j-1},\ldots,x_{i-2}^{j-1})),\\
&\qquad\rho(gx_{i-1}^{j-1},F(x_{l-1}^{s-1},x_l^{s-1},\ldots,x_r^{s-1},x_1^{s-1},\ldots,x_{l-2}^{s-1}))\}\\
&\quad=\max\{\rho(gx_i^{j-1},gx_l^{s-1}),\rho(gx_{i+1}^{j-1},gx_{l+1}^{s-1}),\ldots,\rho(gx_{i-1}^{j-1},gx_{l-1}^{s-1}),
\end{aligned}$$

$$\rho(gx_i^{j-1}, gx_i^j), \rho(gx_l^{s-1}, gx_l^s), \rho(gx_{i+1}^{j-1}, gx_{i+1}^j), \rho(gx_{l+1}^{s-1}, gx_{l+1}^s), \ldots,$$
$$\rho(gx_{i-1}^{j-1}, gx_{i-1}^j), \rho(gx_{l-1}^{s-1}, gx_{l-1}^s), \rho(gx_l^{s-1}, gx_i^j), \rho(gx_i^{j-1}, gx_l^s),$$
$$\rho(gx_{l+1}^{s-1}, gx_{i+1}^j), \rho(gx_{i+1}^{j-1}, gx_{l+1}^s), \ldots, \rho(gx_{l-1}^{s-1}, gx_{i-1}^j), \rho(gx_{i-1}^{j-1}, gx_{l-1}^s)\}.$$

So, for $1 \le i, l \le r$, $1 \le j, s \le n$, we have

$$\rho(gx_i^j, gx_l^s) \le \psi(\delta_n(x_1^0, x_2^0, \ldots, x_r^0)) < \delta_n(x_1^0, x_2^0, \ldots, x_r^0). \tag{5}$$

Hence, equation (3) is true.

Step 2. Now, we claim that for each $n \in N$, $\lim_{n\to+\infty} \delta_n(x_1^0, x_2^0, \ldots, x_r^0) < +\infty$. For this, we distinguish three cases.

Since the sequence $\{\delta_n(x_1^0, x_2^0, \ldots, x_r^0)\}$ is nondecreasing, there exists $\lim_{n\to+\infty} \delta_n(x_1^0, x_2^0, \ldots, x_r^0)$.

Case 1. If, for all $n \in N$, $\delta_n(x_1^0, x_2^0, \ldots, x_r^0) = \operatorname{diam}\{gx_1^0, gx_2^0, \ldots, gx_r^0\}$, then the claim holds.

Case 2. Suppose that there exist $n_0 \in N$, $1 \le i_0, l_0 \le r$, and $1 \le s_0 \le n_0$ such that

$$\delta_{n_0}(x_1^0, x_2^0, \ldots, x_r^0) = \rho(gx_{i_0}^0, gx_{l_0}^{s_0}),$$

then, for any $n \ge n_0$, there exist $1 \le i, l \le r$, and $1 \le s \le n$ such that

$$\delta_n(x_1^0, x_2^0, \ldots, x_r^0) \le \rho(gx_i^0, gx_l^s).$$

By equation (5), we obtain

$$\begin{aligned}\delta_n(x_1^0, x_2^0, \ldots, x_r^0) &\le \rho(gx_i^0, gx_l^1) + \rho(gx_l^1, gx_l^s) \\ &\le \rho(gx_i^0, gx_l^1) + \psi(\delta_n(x_1^0, x_2^0, \ldots, x_r^0)),\end{aligned}$$

which implies that

$$\delta_n(x_1^0, x_2^0, \ldots, x_r^0) - \psi(\delta_n(x_1^0, x_2^0, \ldots, x_r^0)) \le \rho(gx_i^0, gx_l^1). \tag{6}$$

Suppose that $\lim_{n\to+\infty} \delta_n(x_1^0, x_2^0, \ldots, x_r^0) = +\infty$, from the property (iii) of ψ, we have

$$\lim_{n\to+\infty} (\delta_n(x_1^0, x_2^0, \ldots, x_r^0) - \psi(\delta_n(x_1^0, x_2^0, \ldots, x_r^0))) = +\infty.$$

Nevertheless, by equation (6), we get

$$\lim_{n\to+\infty} (\delta_n(x_1^0, x_2^0, \ldots, x_r^0) - \psi(\delta_n(x_1^0, x_2^0, \ldots, x_r^0))) \le \rho(gx_i^0, gx_l^1),$$

which is a contradiction. Thus, $\lim_{n\to+\infty} \delta_n(x_1^0, x_2^0, \ldots, x_r^0) < +\infty$.

Case 3. If there exist $n_1 \in N$, $1 \le i_1, l_1 \le r$, and $1 \le s_1 \le n_1$ such that

$$\delta_{n_1}(x_1^0, x_2^0, \ldots, x_r^0) = \rho(gx_{l_1}^{s_1}, gx_{i_1}^0),$$

the proof is similar to Case 2.

Step 3. Next, we prove that, for every $1 \leq i \leq r$, $\{gx_i^n\}$ is a Cauchy sequence in (X,ρ).
Let

$$O(gx_1^p, gx_2^p, \ldots, gx_r^p) = \{gx_1^p, gx_2^p, \ldots, gx_r^p, gx_1^{p+1}, gx_2^{p+1}, \ldots, gx_r^{p+1}, \ldots\},$$

and let

$$\delta(x_1^p, x_2^p, \ldots, x_r^p) = \operatorname{diam}(O(gx_1^p, gx_2^p, \ldots, gx_r^p)), \quad p = 0, 1, 2, \ldots.$$

Then,

$$\begin{aligned}\delta(x_1^p, x_2^p, \ldots, x_r^p) &\leq \delta(x_1^0, x_2^0, \ldots, x_r^0) \\ &= \lim_{n\to+\infty} \delta_n(x_1^0, x_2^0, \ldots, x_r^0) < +\infty, \quad p = 0, 1, 2, \ldots.\end{aligned}$$

Since

$$0 \leq \delta(x_1^{p+1}, x_2^{p+1}, \ldots, x_r^{p+1}) \leq \delta(x_1^p, x_2^p, \ldots, x_r^p), \quad p = 0, 1, 2, \ldots,$$

there exists $\delta \geq 0$ such that

$$\lim_{p\to+\infty} \delta(x_1^p, x_2^p, \ldots, x_r^p) = \delta.$$

If $\delta > 0$, using the monotonicity of $\{\delta(x_1^p, x_2^p, \ldots, x_r^p)\}$ and the property (iv) of ψ, we conclude that

$$\lim_{p\to+\infty} \psi(\delta(x_1^p, x_2^p, \ldots, x_r^p)) = \lim_{\delta(x_1^p, x_2^p, \ldots, x_r^p)\to\delta^+} \psi(\delta(x_1^p, x_2^p, \ldots, x_r^p)) < \delta. \tag{7}$$

However, by equation (4), we have, for any $p \geq 0$,

$$\delta(x_1^{p+1}, x_2^{p+1}, \ldots, x_r^{p+1}) \leq \psi(\delta(x_1^p, x_2^p, \ldots, x_r^p)),$$

which implies that

$$\delta = \lim_{p\to+\infty} \delta(x_1^{p+1}, x_2^{p+1}, \ldots, x_r^{p+1}) \leq \lim_{p\to+\infty} \psi(\delta(x_1^p, x_2^p, \ldots, x_r^p)),$$

which contradicts equation (7). Therefore, $\lim_{p\to+\infty} \delta(x_1^p, x_2^p, \ldots, x_r^p) = \delta = 0$, that is, for every $1 \leq i \leq r$, $\{gx_i^n\}$ is a Cauchy sequence in (X,ρ).

Step 4. Finally, we prove that $C(F,g,r)$ is nonempty.

Since $g(X)$ is a complete subspace of X, there exist $u_i = gx_i^*$, $i = 1, 2, \ldots, r$, such that

$$\begin{aligned}\lim_{n\to+\infty} \rho(gx_i^*, gx_i^n) &= \lim_{n\to+\infty} \rho(gx_i^n, gx_i^*) = \lim_{m,n\to+\infty} \rho(gx_i^n, gx_i^m) \\ &= \lim_{m,n\to+\infty} \rho(gx_i^m, gx_i^n) = \rho(gx_i^*, gx_i^*) = \rho(u_i, u_i) = 0.\end{aligned} \tag{8}$$

For $1 \leq i \leq r$, $n \in N$, from

$$\begin{aligned}&\rho\big(gx_i^{n+1}, F\big(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\big)\big)\\&\quad \leq \rho\big(gx_i^{n+1}, gx_i^*\big) + \rho\big(gx_i^*, F\big(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\big)\big)\end{aligned}$$

and

$$\begin{aligned}&\rho\big(gx_i^*, F\big(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\big)\big) - \rho\big(gx_i^*, g_i^{n+1}\big)\\&\quad \leq \rho\big(gx_i^{n+1}, F\big(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\big)\big)\end{aligned}$$

we get

$$\begin{aligned}&\lim_{n\to+\infty} \rho\big(gx_i^{n+1}, F\big(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\big)\big)\\&\quad = \rho\big(gx_i^*, F\big(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\big)\big).\end{aligned} \tag{9}$$

For any $1 \leq i \leq r$, $n \in N$, we have

$$\begin{aligned}&\rho\big(gx_i^{n+1}, F\big(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\big)\big)\\&\quad = \rho\big(F\big(x_i^n, x_{i+1}^n, \ldots, x_r^n, x_1^n, \ldots, x_{i-1}^n\big), F\big(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\big)\big)\\&\quad \leq \psi\big(M\big(x_i^n, x_{i+1}^n, \ldots, x_r^n, x_1^n, \ldots, x_{i-1}^n; x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\big)\big),\end{aligned} \tag{10}$$

where

$$\begin{aligned}&M\big(x_i^n, x_{i+1}^n, \ldots, x_r^n, x_1^n, \ldots, x_{i-1}^n; x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\big)\\&\quad = \max\big\{\rho\big(gx_i^n, gx_i^*\big), \rho\big(gx_{i+1}^n, gx_{i+1}^*\big), \ldots, \rho\big(gx_{i-1}^n, gx_{i-1}^*\big),\\&\qquad \rho\big(gx_i^n, F\big(x_i^n, x_{i+1}^n, \ldots, x_r^n, x_1^n, \ldots, x_{i-1}^n\big)\big),\\&\qquad \rho\big(gx_i^*, F\big(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\big)\big),\\&\qquad \rho\big(gx_{i+1}^n, F\big(x_{i+1}^n, x_{i+2}^n, \ldots, x_r^n, x_1^n, \ldots, x_i^n\big)\big),\\&\qquad \rho\big(gx_{i+1}^*, F\big(x_{i+1}^*, x_{i+2}^*, \ldots, x_r^*, x_1^*, \ldots, x_i^*\big)\big), \ldots,\\&\qquad \rho\big(gx_r^n, F\big(x_r^n, x_1^n, \ldots, x_{r-1}^n\big)\big), \rho\big(gx_r^*, F\big(x_r^*, x_1^*, \ldots, x_{r-1}^*\big)\big),\\&\qquad \rho\big(gx_1^n, F\big(x_1^n, x_2^n, \ldots, x_r^n\big)\big), \rho\big(gx_1^*, F\big(x_1^*, x_2^*, \ldots, x_r^*\big)\big), \ldots,\\&\qquad \rho\big(gx_{i-1}^n, F\big(x_{i-1}^n, x_i^n, \ldots, x_r^n, x_1^n, \ldots, x_{i-2}^n\big)\big),\\&\qquad \rho\big(gx_{i-1}^*, F\big(x_{i-1}^*, x_i^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-2}^*\big)\big),\\&\qquad \rho\big(gx_i^*, F\big(x_i^n, x_{i+1}^n, \ldots, x_r^n, x_1^n, \ldots, x_{i-1}^n\big)\big),\\&\qquad \rho\big(gx_i^n, F\big(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\big)\big),\\&\qquad \rho\big(gx_{i+1}^*, F\big(x_{i+1}^n, x_{i+2}^n, \ldots, x_r^n, x_1^n, \ldots, x_i^n\big)\big),\\&\qquad \rho\big(gx_{i+1}^n, F\big(x_{i+1}^*, x_{i+2}^*, \ldots, x_r^*, x_1^*, \ldots, x_i^*\big)\big), \ldots,\\&\qquad \rho\big(gx_r^*, F\big(x_r^n, x_1^n, \ldots, x_{r-1}^n\big)\big), \rho\big(gx_r^n, F\big(x_r^*, x_1^*, \ldots, x_{r-1}^*\big)\big),\end{aligned}$$

$$\rho\left(gx_1^*, F\left(x_1^n, x_2^n, \ldots, x_r^n\right)\right), \rho\left(gx_1^n, F\left(x_1^*, x_2^*, \ldots, x_r^*\right)\right), \ldots,$$
$$\rho\left(gx_{i-1}^*, F\left(x_{i-1}^n, x_i^n, \ldots, x_r^n, x_1^n, \ldots, x_{i-2}^n\right)\right),$$
$$\rho\left(gx_{i-1}^n, F\left(x_{i-1}^*, x_i^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-2}^*\right)\right)\Big\}.$$

By equations (8) and (9), for any $\varepsilon > 0$, there exists $n_0 \in N$, and, for every $n > n_0$ and $1 \leq i \leq r$, we have

$$\begin{aligned}
&\max\Big\{\rho\left(gx_i^*, F\left(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\right)\right), \ldots,\\
&\quad \rho\left(gx_r^*, F\left(x_r^*, x_1^*, \ldots, x_{r-1}^*\right)\right), \rho\left(gx_1^*, F\left(x_1^*, x_2^*, \ldots, x_r^*\right)\right), \ldots,\\
&\quad \rho\left(gx_{i-1}^*, F\left(x_{i-1}^*, x_i^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-2}^*\right)\right)\Big\}\\
&\leq M\left(x_i^n, x_{i+1}^n, \ldots, x_r^n, x_1^n, \ldots, x_{i-1}^n; x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\right)\\
&\leq \max\Big\{\rho\left(gx_i^*, F\left(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\right)\right), \ldots,\\
&\quad \rho\left(gx_r^*, F\left(x_r^*, x_1^*, \ldots, x_{r-1}^*\right)\right), \rho\left(gx_1^*, F\left(x_1^*, x_2^*, \ldots, x_r^*\right)\right), \ldots,\\
&\quad \rho\left(gx_{i-1}^*, F\left(x_{i-1}^*, x_i^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-2}^*\right)\right)\Big\} + \varepsilon.
\end{aligned} \tag{11}$$

Thus, for each $1 \leq i \leq r$,

$$\begin{aligned}
&\lim_{n\to+\infty} M\left(x_i^n, x_{i+1}^n, \ldots, x_r^n, x_1^n, \ldots, x_{i-1}^n; x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\right)\\
&= \max\Big\{\rho\left(gx_i^*, F\left(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\right)\right), \ldots,\\
&\quad \rho\left(gx_r^*, F\left(x_r^*, x_1^*, \ldots, x_{r-1}^*\right)\right), \rho\left(gx_1^*, F\left(x_1^*, x_2^*, \ldots, x_r^*\right)\right), \ldots,\\
&\quad \rho\left(gx_{i-1}^*, F\left(x_{i-1}^*, x_i^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-2}^*\right)\right)\Big\}.
\end{aligned} \tag{12}$$

If

$$\begin{aligned}
&\max\Big\{\rho\left(gx_i^*, F\left(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\right)\right), \ldots,\\
&\quad \rho\left(gx_r^*, F\left(x_r^*, x_1^*, \ldots, x_{r-1}^*\right)\right), \rho\left(gx_1^*, F\left(x_1^*, x_2^*, \ldots, x_r^*\right)\right), \ldots,\\
&\quad \rho\left(gx_{i-1}^*, F\left(x_{i-1}^*, x_i^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-2}^*\right)\right)\Big\} > 0,
\end{aligned}$$

using equations (9), (10), (11), and (12) and the property (iv) of ψ, we obtain, for every $1 \leq i \leq r$,

$$\begin{aligned}
&\rho\left(gx_i^*, F\left(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\right)\right)\\
&= \lim_{n\to+\infty} \rho\left(gx_i^{n+1}, F\left(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\right)\right)\\
&\leq \lim_{n\to+\infty} \psi\left(M\left(x_i^n, \ldots, x_r^n, x_1^n, \ldots, x_{i-1}^n; x_i^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\right)\right)\\
&< \max\Big\{\rho\left(gx_i^*, F\left(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*\right)\right), \ldots,\\
&\quad \rho\left(gx_r^*, F\left(x_r^*, x_1^*, \ldots, x_{r-1}^*\right)\right), \rho\left(gx_1^*, F\left(x_1^*, x_2^*, \ldots, x_r^*\right)\right), \ldots,\\
&\quad \rho\left(gx_{i-1}^*, F\left(x_{i-1}^*, x_i^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-2}^*\right)\right)\Big\}.
\end{aligned} \tag{13}$$

By the arbitrariness of i in equation (13), we deduce that

$$\begin{aligned}
&\max\bigl\{\rho\bigl(gx_i^*,F\bigl(x_i^*,x_{i+1}^*,\ldots,x_r^*,x_1^*,\ldots,x_{i-1}^*\bigr)\bigr),\ldots,\\
&\quad\rho\bigl(gx_r^*,F\bigl(x_r^*,x_1^*,\ldots,x_{r-1}^*\bigr)\bigr),\rho\bigl(gx_1^*,F\bigl(x_1^*,x_2^*,\ldots,x_r^*\bigr)\bigr),\ldots,\\
&\quad\rho\bigl(gx_{i-1}^*,F\bigl(x_{i-1}^*,x_i^*,\ldots,x_r^*,x_1^*,\ldots,x_{i-2}^*\bigr)\bigr)\bigr\}\\
&\quad<\max\bigl\{\rho\bigl(gx_i^*,F\bigl(x_i^*,x_{i+1}^*,\ldots,x_r^*,x_1^*,\ldots,x_{i-1}^*\bigr)\bigr),\ldots,\\
&\quad\rho\bigl(gx_r^*,F\bigl(x_r^*,x_1^*,\ldots,x_{r-1}^*\bigr)\bigr),\rho\bigl(gx_1^*,F\bigl(x_1^*,x_2^*,\ldots,x_r^*\bigr)\bigr),\ldots,\\
&\quad\rho\bigl(gx_{i-1}^*,F\bigl(x_{i-1}^*,x_i^*,\ldots,x_r^*,x_1^*,\ldots,x_{i-2}^*\bigr)\bigr)\bigr\},
\end{aligned}$$

which is a contradiction. Therefore,

$$\begin{aligned}
&\max\bigl\{\rho\bigl(gx_i^*,F\bigl(x_i^*,x_{i+1}^*,\ldots,x_r^*,x_1^*,\ldots,x_{i-1}^*\bigr)\bigr),\ldots,\\
&\rho\bigl(gx_r^*,F\bigl(x_r^*,x_1^*,\ldots,x_{r-1}^*\bigr)\bigr),\rho\bigl(gx_1^*,F\bigl(x_1^*,x_2^*,\ldots,x_r^*\bigr)\bigr),\ldots,\\
&\rho\bigl(gx_{i-1}^*,F\bigl(x_{i-1}^*,x_i^*,\ldots,x_r^*,x_1^*,\ldots,x_{i-2}^*\bigr)\bigr)\bigr\}=0,
\end{aligned}$$

which implies that $\rho(gx_i^*,F(x_i^*,x_{i+1}^*,\ldots,x_r^*,x_1^*,\ldots,x_{i-1}^*))=0$, $i=1,2,\ldots,r$.

Thus,

$$gx_i^*=F\bigl(x_i^*,x_{i+1}^*,\ldots,x_r^*,x_1^*,\ldots,x_{i-1}^*\bigr),\quad i=1,2,\ldots,r,$$

that is, $(x_1^*,x_2^*,\ldots,x_r^*)\in C(F,g,r)$. □

Theorem 3.5 *Let (X,ρ) be a quasi-metric-like space. Let $g:X\to X$ and let $F:X^r\to X$ $(r\geq 2)$ be mappings satisfying all the conditions of Theorem* 3.4. *If F and g are w-compatible, then F and g have a unique coincidence point of r-order, which is a fixed point of g and a fixed point of r-order of F. Moreover, the coincidence point of r-order is of the form $(u^*,u^*,\ldots,u^*)$ for some $u^*\in X$.*

Proof Suppose that there exist $(x_1^*,x_2^*,\ldots,x_r^*),(x_1^{**},x_2^{**},\ldots,x_r^{**})\in C(F,g,r)$, that is, for each $1\leq i\leq r$,

$$gx_i^*=F\bigl(x_i^*,x_{i+1}^*,\ldots,x_r^*,x_1^*,\ldots,x_{i-1}^*\bigr),\tag{14}$$

$$gx_i^{**}=F\bigl(x_i^{**},x_{i+1}^{**},\ldots,x_r^{**},x_1^{**},\ldots,x_{i-1}^{**}\bigr).\tag{15}$$

First, we prove that, for any $1\leq i,j,k\leq r$,

$$gx_i^*=gx_j^*=gx_k^{**}.\tag{16}$$

By equations (1), (14), and (15), for $1\leq i\leq r-1$, we have

$$\begin{aligned}
\rho\bigl(gx_i^*,gx_{i+1}^{**}\bigr)&=\rho\bigl(F\bigl(x_i^*,x_{i+1}^*,\ldots,x_r^*,x_1^*,\ldots,x_{i-1}^*\bigr),\\
&\qquad F\bigl(x_{i+1}^{**},x_{i+2}^{**},\ldots,x_r^{**},x_1^{**},\ldots,x_i^{**}\bigr)\bigr)\\
&\leq\psi\bigl(M\bigl(x_i^*,x_{i+1}^*,\ldots,x_r^*,x_1^*,\ldots,x_{i-1}^*;x_{i+1}^{**},x_{i+2}^{**},\ldots,x_r^{**},x_1^{**},\ldots,x_i^{**}\bigr)\bigr),
\end{aligned}\tag{17}$$

where

$$\begin{aligned}
&M\bigl(x_i^*, x_{i+1}^*, \dots, x_r^*, x_1^*, \dots, x_{i-1}^*; x_{i+1}^{**}, x_{i+2}^{**}, \dots, x_r^{**}, x_1^{**}, \dots, x_i^{**}\bigr) \\
&\quad = \max\bigl\{\rho\bigl(gx_1^*, gx_2^{**}\bigr), \dots, \rho\bigl(gx_{r-1}^*, gx_r^{**}\bigr), \rho\bigl(gx_r^*, gx_1^{**}\bigr), \\
&\qquad \rho\bigl(gx_2^{**}, gx_1^*\bigr), \rho\bigl(gx_3^{**}, gx_2^*\bigr), \dots, \rho\bigl(gx_r^{**}, gx_{r-1}^*\bigr), \rho\bigl(gx_1^{**}, gx_r^*\bigr), \\
&\qquad \rho\bigl(gx_1^*, gx_1^*\bigr), \dots, \rho\bigl(gx_r^*, gx_r^*\bigr), \rho\bigl(gx_1^{**}, gx_1^{**}\bigr), \dots, \rho\bigl(gx_r^{**}, gx_r^{**}\bigr)\bigr\}. \qquad (18)
\end{aligned}$$

Set

$$\begin{aligned}
\zeta &= \max\bigl\{\rho\bigl(gx_1^*, gx_2^{**}\bigr), \dots, \rho\bigl(gx_{r-1}^*, gx_r^{**}\bigr), \rho\bigl(gx_r^*, gx_1^{**}\bigr), \\
&\quad \rho\bigl(gx_2^{**}, gx_1^*\bigr), \rho\bigl(gx_3^{**}, gx_2^*\bigr), \dots, \rho\bigl(gx_r^{**}, gx_{r-1}^*\bigr), \rho\bigl(gx_1^{**}, gx_r^*\bigr), \\
&\quad \rho\bigl(gx_1^*, gx_1^*\bigr), \dots, \rho\bigl(gx_r^*, gx_r^*\bigr), \rho\bigl(gx_1^{**}, gx_1^{**}\bigr), \dots, \rho\bigl(gx_r^{**}, gx_r^{**}\bigr)\bigr\}. \qquad (19)
\end{aligned}$$

Similarly, we have

$$\rho\bigl(gx_r^*, gx_1^{**}\bigr) \le \psi(\zeta), \qquad \rho\bigl(gx_1^{**}, g_r^*\bigr) \le \psi(\zeta) \qquad (20)$$

and

$$\rho\bigl(gx_{i+1}^{**}, gx_i^*\bigr) \le \psi(\zeta), \quad i = 1, 2, \dots, r-1. \qquad (21)$$

By equations (1), (14), (15), and the monotonicity of ψ, for $1 \le i \le r$, we also have

$$\rho\bigl(gx_i^*, gx_i^*\bigr) \le \psi\bigl(\max\bigl\{\rho\bigl(gx_1^*, gx_1^*\bigr), \dots, \rho\bigl(gx_r^*, gx_r^*\bigr)\bigr\}\bigr) \le \psi(\zeta) \qquad (22)$$

and

$$\rho\bigl(gx_i^{**}, gx_i^{**}\bigr) \le \psi\bigl(\max\bigl\{\rho\bigl(gx_1^{**}, gx_1^{**}\bigr), \dots, \rho\bigl(gx_r^{**}, gx_r^{**}\bigr)\bigr\}\bigr) \le \psi(\zeta). \qquad (23)$$

From equations (17) to (23), we can conclude that

$$\zeta \le \psi(\zeta),$$

which is a contradiction, unless $\zeta = 0$. So

$$\begin{aligned}
&\max\bigl\{\rho\bigl(gx_1^*, gx_2^{**}\bigr), \dots, \rho\bigl(gx_{r-1}^*, gx_r^{**}\bigr), \rho\bigl(gx_r^*, gx_1^{**}\bigr), \\
&\quad \rho\bigl(gx_2^{**}, gx_1^*\bigr), \rho\bigl(gx_3^{**}, gx_2^*\bigr), \dots, \rho\bigl(gx_r^{**}, gx_{r-1}^*\bigr), \rho\bigl(gx_1^{**}, gx_r^*\bigr), \\
&\quad \rho\bigl(gx_1^*, gx_1^*\bigr), \dots, \rho\bigl(gx_r^*, gx_r^*\bigr), \rho\bigl(gx_1^{**}, gx_1^{**}\bigr), \dots, \rho\bigl(gx_r^{**}, gx_r^{**}\bigr)\bigr\} = 0,
\end{aligned}$$

that is,

$$gx_i^* = gx_{i+1}^{**}, \quad i = 1, 2, \dots, r-1, \qquad (24)$$

$$gx_r^* = gx_1^{**}. \qquad (25)$$

On the other hand, for any $1 \le i \le r$, we obtain

$$\begin{aligned}
&\rho(gx_i^*, gx_i^{**}) \\
&\quad \le \psi(M(x_i^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*; x_i^{**}, \ldots, x_r^{**}, x_1^{**}, \ldots, x_{i-1}^{**})) \\
&\quad = \psi(\max\{\rho(gx_i^*, gx_i^{**}), \ldots, \rho(gx_r^*, gx_r^{**}), \rho(gx_1^*, gx_1^{**}), \ldots, \\
&\qquad \rho(gx_{i-1}^*, gx_{i-1}^{**}), \rho(gx_1^*, gx_1^*), \ldots, \rho(gx_r^*, gx_r^*), \rho(gx_1^{**}, gx_1^{**}), \ldots, \\
&\qquad \rho(gx_r^{**}, gx_r^{**}), \rho(gx_i^{**}, gx_i^*), \ldots, \rho(gx_r^{**}, gx_r^*), \rho(gx_1^{**}, gx_1^*), \ldots, \\
&\qquad \rho(gx_{i-1}^{**}, gx_{i-1}^*)\}).
\end{aligned} \tag{26}$$

Set

$$\begin{aligned}
\lambda = \max\{&\rho(gx_i^*, gx_i^{**}), \ldots, \rho(gx_r^*, gx_r^{**}), \rho(gx_1^*, gx_1^{**}), \ldots, \\
&\rho(gx_{i-1}^*, gx_{i-1}^{**}), \rho(gx_1^*, gx_1^*), \ldots, \rho(gx_r^*, gx_r^*), \rho(gx_1^{**}, gx_1^{**}), \ldots, \\
&\rho(gx_r^{**}, gx_r^{**}), \rho(gx_i^{**}, gx_i^*), \ldots, \rho(gx_r^{**}, gx_r^*), \rho(gx_1^{**}, gx_1^*), \ldots, \\
&\rho(gx_{i-1}^{**}, gx_{i-1}^*)\}.
\end{aligned} \tag{27}$$

Similarly, for any

$$\begin{aligned}
\xi \in \{&\rho(gx_1^*, gx_1^*), \ldots, \rho(gx_r^*, gx_r^*), \rho(gx_1^{**}, gx_1^{**}), \ldots, \rho(gx_r^{**}, gx_r^{**}) \\
&\rho(gx_i^{**}, gx_i^*), \ldots, \rho(gx_r^{**}, gx_r^*), \rho(gx_1^{**}, gx_1^*), \ldots, \rho(gx_{i-1}^{**}, gx_{i-1}^*)\},
\end{aligned}$$

we have

$$\xi \le \psi(\lambda). \tag{28}$$

By equations (26), (27), and (28), we get

$$\lambda \le \psi(\lambda),$$

which is a contradiction, unless $\lambda = 0$. That is,

$$gx_i^* = gx_i^{**}, \quad i = 1, 2, \ldots, r. \tag{29}$$

Therefore, equations (24), (25), and (29) imply that equation (16) is true.

Next, we prove that the coincidence point of r-order is unique.

In view of equation (16), let $gx_i^* = u^*$, $i = 1, 2, \ldots, r$.

Using the w-compatibility of F and g, we conclude that

$$\begin{aligned}
gu^* &= g(gx_i^*) = g(F(x_i^*, x_{i+1}^*, \ldots, x_r^*, x_1^*, \ldots, x_{i-1}^*)) \\
&= F(gx_i^*, gx_{i+1}^*, \ldots, gx_r^*, gx_1^*, \ldots, gx_{i-1}^*) = F(u^*, u^*, \ldots, u^*).
\end{aligned}$$

So, $u^* \in C(F, g, r)$. By equation (16), we can deduce that $gu^* = gx_i^*$, $i = 1, 2, \ldots, r$.

Thus,

$$u^* = gx_i^* = gu^* = F(u^*, u^*, \dots, u^*). \tag{30}$$

Moreover, equations (16) and (30) imply that $(u^*, u^*, \dots, u^*)$ is the unique coincidence point of r-order of F and g, u^* is a fixed point of g, and $(u^*, u^*, \dots, u^*)$ is a fixed point of r-order of F. □

For each $a \in (0,1)$, setting $\psi(t) = at$ in Theorem 3.4 and Theorem 3.5, we obtain the following results.

Corollary 3.6 *Let (X, ρ) be a quasi-metric-like space, let $g : X \to X$ and let $F : X^r \to X$ $(r \geq 2)$. Suppose there exists $a \in (0,1)$ such that F is a generalized g-ψ-quasi-contraction with $\psi(t) = at$. If $F(X^r) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then $C(F, g, r)$ is nonempty.*

Corollary 3.7 *Let (X, ρ) be a quasi-metric-like space. Let $g : X \to X$ and let $F : X^r \to X$ $(r \geq 2)$ be mappings satisfying all the conditions of Corollary 3.6. If F and g are w-compatible, then F and g have a unique coincidence point of r-order, which is a fixed point of g and a fixed point of r-order of F. Moreover, the coincidence point of r-order is of the form $(u^*, u^*, \dots, u^*)$ for some $u^* \in X$.*

Example 3.8 Let $X = \{0, 1, 2\}$, Define $\rho : X \times X \to [0, +\infty)$ as follows:

$$\begin{aligned} &\rho(0,0) = 0, \qquad \rho(1,1) = 3, \qquad \rho(2,2) = \frac{1}{2}, \\ &\rho(0,1) = 3, \qquad \rho(0,2) = \frac{3}{2}, \qquad \rho(1,0) = \frac{5}{2}, \\ &\rho(2,0) = 3, \qquad \rho(1,2) = \frac{4}{5}, \qquad \rho(2,1) = 4. \end{aligned}$$

Then (X, ρ) is a complete quasi-metric-like space.

Define $g : X \times X \to X$ by

$$g0 = 1, \qquad g1 = 2, \qquad g2 = 0,$$

and $F : X^r \to X$ $(r \geq 2)$ by

$$F(x_1, x_2, \dots, x_r) = \begin{cases} 0, & \text{if } x_1 = x_2 = \cdots = x_r; \\ \min\{x_1, x_2, \dots, x_r\}, & \text{otherwise.} \end{cases}$$

It is easy to prove that g and F satisfy all conditions of Theorem 3.4 by taking $\psi(t) = \frac{5}{6}t$, and the proof would be lengthy.

Here, $(2, 2, \dots, 2) \in C(F, g, r)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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