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A family of refinements of Heinz inequalities of matrices

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Abstract

For any unitarily invariant norm $|||\cdot|||$, the Heinz inequalities for operators assert that $2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \le |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}||| \le |||AX + XB|||$, for A, B, and X any operators on a complex separable Hilbert space such that A, B are positive and $\nu \in [0,1]$. In this paper, we obtain a family of refinements of these norm inequalities by using the convexity of the function $f(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ and the Hermite-Hadamard inequality.

Keywords: Heinz inequality; convex function; Hermite-Hadamard inequality; unitarily invariant norm

1 Introduction

Let $M_n(\mathbf{C})$ be the algebra of $n \times n$ complex matrices. We denote by $H_n(\mathbf{C})$ the set of all Hermitian matrices in $M_n(\mathbf{C})$. The set of all positive semi-definite matrices in $M_n(\mathbf{C})$ shall be denoted by $H_n^+(\mathbf{C})$. A norm $|||\cdot|||$ on $M_n(\mathbf{C})$ is called unitarily invariant or symmetric if

$$|||UAV||| = |||A|||$$

for all $A \in M_n(\mathbf{C})$ and for all unitaries $U, V \in M_n(\mathbf{C})$.

The arithmetic-geometric mean inequality for two nonnegative real numbers a and b is

$$\sqrt{ab} \le \frac{a+b}{2},$$

which has been generalized to the context of matrices as follows:

$$2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \le |||AX + XB|||,$$

where $A, B \in H_n^+(\mathbf{C})$, $X \in M_n$, and $|||\cdot|||$ is a unitarily invariant norm on $M_n(\mathbf{C})$.

For $\nu \in [0,1]$ and two nonnegative numbers a and b, the Heinz mean is defined as

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu}b^\nu}{2}.$$

Clearly the Heinz mean interpolates between the geometric mean and the arithmetic mean:

$$\sqrt{ab} \le H_\nu(a,b) \le \frac{a+b}{2}.$$

The function $H_\nu(a,b)$ has the following properties: it is convex, attains its minimum at $\nu = \frac{1}{2}$, its maximum at $\nu = 0$ and $\nu = 1$, and $H_\nu(a,b) = H_{1-\nu}(a,b)$ for $0 \le \nu \le 1$. The generalization of the above inequalities to matrices is due to Bhatia and Davis [1] as follows:

$$2\left|\left|\left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|\right|\right| \le \left|\left|\left| A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu \right|\right|\right| \le |||AX + XB|||, \tag{1.1}$$

where $A, B \in H_n^+(\mathbf{C})$, $X \in M_n(\mathbf{C})$, and $\nu \in [0,1]$. For a historical background and proofs of these norm inequalities as well as their refinements, and diverse applications, we refer the reader to the [2–8], and the references therein. Indeed, it has been proved, in [1], that $f(\nu) = |||A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu|||$ is a convex function of ν on $[0,1]$ with symmetry about $\nu = \frac{1}{2}$, and attains its minimum there and it has a maximum at $\nu = 0$ and $\nu = 1$. Moreover, it increases on $[0, \frac{1}{2}]$ and decreases on $[\frac{1}{2}, 1]$.

In [4, 5], (1.1) is refined by using the so-called Hermite-Hadamard inequality:

$$g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b g(t)\,dt \le \frac{g(a)+g(b)}{2},$$

where g is a convex function on $[a,b]$.

Recently, in [3] and [7], respectively, the following inequalities were used to get new refinements of (1.1):

$$g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b g(t)\,dt \le \frac{1}{4}\left(g(a) + 2g\left(\frac{a+b}{2}\right) + g(b)\right) \le \frac{g(a)+g(b)}{2},$$
$$g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b g(t)\,dt \le \frac{1}{32}\left(15g(a) + 2g\left(\frac{a+b}{2}\right) + 15g(b)\right) \le \frac{g(a)+g(b)}{2}.$$

The purpose of this note is to obtain a family of new refinements of Heinz inequalities for matrices. Also the two refinements, given in [3] and [7], are two special cases of this new family.

2 Main results

We start by the following key lemma which plays a central role in our investigation to obtain a further series of refinements of the Heinz inequalities.

Lemma 1 *Let g be a convex function on the interval $[a,b]$. Then for any positive integer n, we have*

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b g(t)\,dt \le \frac{1}{4n}\left[(2n-1)g(a) + 2g\left(\frac{a+b}{2}\right) + (2n-1)g(b)\right] \\ &\le \frac{g(a)+g(b)}{2}. \end{aligned}$$

Proof Since g is convex on $[a,b]$, we have

$$g\left(\frac{a+b}{2}\right) \le \frac{g(a)+g(b)}{2}.$$

Thus

$$(2n-1)g(a)+2g\left(\frac{a+b}{2}\right)+(2n-1)g(b)\le 2ng(a)+2ng(b),$$

whence

$$\frac{1}{4n}\left((2n-1)g(a)+2g\left(\frac{a+b}{2}\right)+(2n-1)g(b)\right)\le\frac{g(a)+g(b)}{2}.$$

To prove the middle inequality, we start by

$$\begin{aligned}\frac{1}{b-a}\int_a^b g(t)\,dt &= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} g(t)\,dt+\int_{\frac{a+b}{2}}^b g(t)\,dt\right]\\ &\le \frac{1}{b-a}\left[\frac{g(\frac{a+b}{2})+g(a)}{2}\cdot\frac{b-a}{2}+\frac{g(b)+g(\frac{a+b}{2})}{2}\cdot\frac{b-a}{2}\right]\\ &= \frac{1}{4}\left[g(a)+2g\left(\frac{a+b}{2}\right)+g(b)\right]\\ &= \frac{1}{4n}\left[ng(a)+2ng\left(\frac{a+b}{2}\right)+ng(b)\right]\\ &= \frac{1}{4n}\left[ng(a)+2g\left(\frac{a+b}{2}\right)+(2n-2)g\left(\frac{a+b}{2}\right)+ng(b)\right]\\ &\le \frac{1}{4n}\left[ng(a)+2g\left(\frac{a+b}{2}\right)+(2n-2)\left[\frac{g(a)+g(b)}{2}\right]+ng(b)\right]\\ &= \frac{1}{4n}\left[ng(a)+2g\left(\frac{a+b}{2}\right)+(n-1)g(a)+(n-1)g(b)+ng(b)\right]\\ &= \frac{1}{4n}\left[(2n-1)g(a)+2g\left(\frac{a+b}{2}\right)+(2n-1)g(b)\right].\end{aligned}$$ □

Applying the previous lemma on the convex function defined earlier

$$f(\nu)=|||A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu|||$$

on the interval $[\mu,1-\mu]$ when $0\le\mu\le\frac{1}{2}$ and on the interval $[1-\mu,\mu]$ when $\frac{1}{2}\le\mu\le 1$, we obtain the following refinement of the first inequality (1.1) which is a kind of refinements of Theorem 1 in a paper Kittaneh [5] and Theorem 1 in a paper of Feng [3].

Theorem 1 *Let $A,B\in H_n^+(\mathbf{C})$, and $X\in M_n(\mathbf{C})$. Let n be any positive integer. Then for any $\mu\in[0,1]$, and for every unitarily invariant norm $|||\cdot|||$ on $M_n(\mathbf{C})$, we have*

$$\begin{aligned}2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\le \frac{1}{|1-2\mu|}\left|\int_\mu^{1-\mu}|||A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu|||\,d\nu\right|\\ &\le \frac{1}{2n}\left[(2n-1)|||A^\mu XB^{1-\mu}+A^{1-\mu}XB^\mu|||+2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right]\\ &\le |||A^\mu XB^{1-\mu}+A^{1-\mu}XB^\mu|||.\end{aligned}\tag{2.1}$$

Proof First assume that $0 \leq \mu \leq \frac{1}{2}$. Then it follows from Lemma 1 that

$$\begin{aligned} f\left(\frac{\mu+1-\mu}{2}\right) &\leq \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} f(t)\,dt \\ &\leq \frac{1}{4n}\left[(2n-1)f(\mu)+2f\left(\frac{1-\mu+\mu}{2}\right)+(2n-1)f(1-\mu)\right] \\ &\leq \frac{f(\mu)+f(1-\mu)}{2}=f(\mu). \end{aligned}$$

Since $f(\mu)=f(1-\mu)$, we have

$$\begin{aligned} f\left(\frac{1}{2}\right) &\leq \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} f(t)\,dt \leq \frac{1}{4n}\left[(4n-2)f(\mu)+2f\left(\frac{1}{2}\right)\right] \\ &\leq \frac{1}{2n}\left[(2n-1)f(\mu)+f\left(\frac{1}{2}\right)\right] \leq f(\mu). \end{aligned}$$

Thus,

$$\begin{aligned} 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leq \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||\,d\nu \\ &\leq \frac{1}{2n}\left[(2n-1)|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right] \\ &\leq |||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||. \end{aligned} \tag{2.2}$$

Now, assume that $\frac{1}{2} \leq \mu \leq 1$. Then, by applying (2.2) to $1-\mu$, it follows that

$$\begin{aligned} 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leq \frac{1}{2\mu-1}\int_{1-\mu}^{\mu} |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||\,d\nu \\ &\leq \frac{1}{2n}\left[(2n-1)|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right] \\ &\leq |||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||. \end{aligned} \tag{2.3}$$

Since

$$\begin{aligned} &\lim_{\mu\to\frac{1}{2}} \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||\,d\nu\right| \\ &\quad = \lim_{\mu\to\frac{1}{2}} \frac{1}{4n}\left[(4n-2)f(\mu)+f\left(\frac{1}{2}\right)\right] = 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||, \end{aligned}$$

the inequalities in (2.1) follow by combining (2.2) and (2.3) and so the required result is proved. □

Applying Lemma 1 to the function $f(\nu) = |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||$ in the interval $[\mu, \frac{1}{2}]$ on $0 \leq \mu \leq \frac{1}{2}$, and in the interval $[\frac{1}{2}, \mu]$ for $\frac{1}{2} \leq \mu \leq 1$, we obtain the following, which is a kind of refinements of Theorem 2 in a paper Kittaneh [5] and Theorem 2 in a paper of Feng [3].

Theorem 2 *Let $A, B \in H_n^+(\mathbf{C})$, and $X \in M_n(\mathbf{C})$. Then, for any positive integer n, any $\mu \in [0,1]$, and for every unitarily invariant norm $|||\cdot|||$ on $M_n(\mathbf{C})$, we have*

$$\begin{aligned}
&|||A^{\frac{1+2\mu}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{1+2\mu}{4}}||| \\
&\quad\le \frac{2}{|1-2\mu|}\left|\int_{\mu}^{\frac{1}{2}} |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||\,d\nu\right| \\
&\quad\le \frac{1}{4n}\Big[(2n-1)|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}||| + 2|||A^{\frac{1+2\mu}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{1+2\mu}{4}}||| \\
&\qquad + 2(2n-1)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\Big] \\
&\quad\le \frac{1}{2}\Big[|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\Big].
\end{aligned} \tag{2.4}$$

Inequalities (2.4) and the first inequality in (1.1) yield the following refinements of the first inequality in (1.1).

Corollary 1 *Let $A, B \in H_n^+(\mathbf{C})$, and $X \in M_n(\mathbf{C})$. Then, for any positive integer n, any $\mu \in [0,1]$, and for every unitarily invariant norm $|||\cdot|||$ on $X \in M_n(\mathbf{C})$, we have*

$$\begin{aligned}
2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\le |||A^{\frac{1+2\mu}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{1+2\mu}{4}}||| \\
&\le \frac{2}{|1-2\mu|}\left|\int_{\mu}^{\frac{1}{2}} |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||\,d\nu\right| \\
&\le \frac{1}{4n}\Big[(2n-1)|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}||| \\
&\quad + 2|||A^{\frac{1+2\mu}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{1+2\mu}{4}}||| + 2(2n-1)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\Big] \\
&\le \frac{1}{2}\Big[|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\Big] \\
&\le |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||.
\end{aligned} \tag{2.5}$$

Applying the Lemma 1 to the function $f(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ on the interval $[\mu, \frac{1}{2}]$ when $0 \le \mu \le \frac{1}{2}$, and on the interval $[\frac{1}{2}, \mu]$ when $\frac{1}{2} \le \mu \le 1$, we obtain the following theorem, which is a kind of refinements of Theorem 3 in a paper Kittaneh [5] and Theorem 3 in a paper of Feng [3].

Theorem 3 *Let $A, B \in H_n^+(\mathbf{C})$, and $X \in M_n(\mathbf{C})$ and let n be a positive integer. Then*:

(1) *for any $0 \le \mu \le \frac{1}{2}$ and for every unitarily invariant norm $|||\cdot|||$, we have*

$$\begin{aligned}
&|||A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}||| \\
&\quad\le \frac{1}{\mu}\int_0^{\mu} |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||\,d\nu \\
&\quad\le \frac{1}{4n}\Big[(2n-1)|||AX + XB||| + 2|||A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}||| \\
&\qquad + (2n-1)|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\Big] \\
&\quad\le \frac{1}{2}\Big[|||AX + XB||| + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\Big];
\end{aligned} \tag{2.6}$$

(2) *for any* $\frac{1}{2} \le \mu \le 1$ *and for every unitarily invariant norm* $|||\cdot|||$, *we have*

$$\begin{aligned}
&\big|\big|\big|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\big|\big|\big| \\
&\quad \le \frac{1}{1-\mu}\int_{\mu}^{1}\big|\big|\big|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\big|\big|\big|\,d\nu \\
&\quad \le \frac{1}{4n}\Big[(2n-1)|||AX+XB||| + 2\big|\big|\big|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\big|\big|\big|\Big] \\
&\quad \le \frac{1}{2}\Big[|||AX+XB||| + \big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big|\Big]. \qquad (2.7)
\end{aligned}$$

Since the function $f(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ is decreasing on the interval $[0, \frac{1}{2}]$ and increasing on the interval $[\frac{1}{2}, 1]$, and using the inequalities (2.6) and (2.7), we obtain a family of refinements of second inequality in (1.1).

Corollary 2 *Let* $A, B \in H_n^{+}(\mathbf{C})$, *and* $X \in M_n(\mathbf{C})$ *and let* n *be a positive integer. Then*:

(1) *for any* $0 \le \mu \le \frac{1}{2}$ *and for every unitarily invariant norm* $|||\cdot|||$, *we have*

$$\begin{aligned}
&\big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big| \\
&\quad \le \big|\big|\big|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\big|\big|\big| \\
&\quad \le \frac{1}{\mu}\int_{0}^{\mu}\big|\big|\big|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\big|\big|\big|\,d\nu \\
&\quad \le \frac{1}{4n}\Big[(2n-1)|||AX+XB||| + 2\big|\big|\big|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\big|\big|\big| \\
&\qquad + (2n-1)\big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big|\Big] \\
&\quad \le \frac{1}{2}\Big[|||AX+XB||| + \big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big|\Big] \\
&\quad \le |||AX+XB|||; \qquad (2.8)
\end{aligned}$$

(2) *for any* $\frac{1}{2} \le \mu \le 1$ *and for every unitarily invariant norm* $|||\cdot|||$, *we have*

$$\begin{aligned}
&\big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big| \\
&\quad \le \big|\big|\big|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\big|\big|\big| \\
&\quad \le \frac{1}{1-\mu}\int_{\mu}^{1}\big|\big|\big|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\big|\big|\big|\,d\nu \\
&\quad \le \frac{1}{4n}\Big[(2n-1)|||AX+XB||| + 2\big|\big|\big|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\big|\big|\big| \\
&\qquad + (2n-1)\big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big|\Big] \\
&\quad \le \frac{1}{2}\Big[|||AX+XB||| + \big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big|\Big] \\
&\quad \le |||AX+XB|||. \qquad (2.9)
\end{aligned}$$

It should be noted that in inequalities (2.8) and (2.9), we have

$$\lim_{\mu\to 0}\frac{1}{\mu}\int_0^{\mu}|||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||\,d\nu$$

$$=\lim_{\mu\to 1}\frac{1}{1-\mu}\int_{\mu}^{1}|||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||\,d\nu=|||AX+XB|||.$$

Remark 1 The two special values $n = 1$ and $n = 8$ give the refinements of Heinz inequalities obtained in [3] and [7], respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the manuscript and read and approved the final manuscript.

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