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Boundedness of Marcinkiewicz integrals with mixed homogeneity along compound surfaces

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Abstract

In this note we establish the L^p boundedness of Marcinkiewicz integrals with mixed homogeneity along compound surfaces, which improve and extend some previous results. The main ingredient is to present a systematic treatment with several singular integral operators.

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1 Introduction

Let \mathbb{R}^n , $n \geq 2$, be the n -dimensional Euclidean space and S^{n-1} denote the unit sphere in \mathbb{R}^n equipped with the induced Lebesgue measure $d\sigma$. Let $\alpha_j \geq 1$ ($j = 1, \dots, n$) be fixed real numbers. Define the function $F: \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ by $F(x, \rho) = \sum_{j=1}^n x_j^2 \rho^{-2\alpha_j}$, $x = (x_1, x_2, \dots, x_n)$. It is clear that, for each fixed $x \in \mathbb{R}^n$, the function $F(x, \rho)$ is a decreasing function in $\rho > 0$. We let $\rho(x)$ denote the unique solution of the equation $F(x, \rho) = 1$. Fabes and Rivière [1] showed that (\mathbb{R}^n, ρ) is a metric space, which is often called the mixed homogeneity space related to $\{\alpha_j\}_{j=1}^n$. For $\lambda > 0$, we let A_λ be the diagonal $n \times n$ matrix $A_\lambda = \text{diag}\{\lambda^{\alpha_1}, \dots, \lambda^{\alpha_n}\}$. Let $\mathbb{R}^+ := (0, \infty)$ and $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, we denote $A_{\varphi(\rho(y))}y'$ by $A_\varphi(y)$ for $y \in \mathbb{R}^n$, where $y' = A_{\rho(y)^{-1}}y \in S^{n-1}$.

The change of variables related to the spaces (\mathbb{R}^n, ρ) is given by the transformation

$$x_1 = \rho^{\alpha_1} \cos \theta_1 \cdots \cos \theta_{n-2} \cos \theta_{n-1},$$

$$x_2 = \rho^{\alpha_2} \cos \theta_1 \cdots \cos \theta_{n-2} \sin \theta_{n-1},$$

$\dots,$

$$x_{n-1} = \rho^{\alpha_{n-1}} \cos \theta_1 \sin \theta_2,$$

$$x_n = \rho^{\alpha_n} \sin \theta_1.$$

Thus $dx = \rho^{\alpha-1} J(x') d\rho d\sigma(x')$, where $\rho^{\alpha-1} J(x')$ is the Jacobian of the above transform and $\alpha = \sum_{j=1}^n \alpha_j$, $J(x') = \sum_{j=1}^n \alpha_j (x'_j)^2$. Obviously, $J(x') \in C^\infty(S^{n-1})$ and there exists $M > 0$ such

that

$$1 \leq J(x') \leq M, \quad \forall x' \in S^{n-1}.$$

It is easy to see that

$$\rho(x) = |x|, \quad \text{if } \alpha_1 = \alpha_2 = \dots = \alpha_n = 1.$$

Let Ω be integrable on S^{n-1} and satisfy

$$\int_{S^{n-1}} \Omega(u)J(u) d\sigma(u) = 0, \tag{1.1}$$

$$\Omega(A_s x) = \Omega(x), \quad \forall s > 0 \text{ and } x \in \mathbb{R}^n. \tag{1.2}$$

For $d \geq 2$ and a suitable function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^d$, we define the parabolic Marcinkiewicz integral operator \mathcal{M}_Ω^Φ on \mathbb{R}^d by

$$\mathcal{M}_\Omega^\Phi(f)(x) = \left(\int_0^\infty \left| \frac{1}{t} \int_{\rho(y) \leq t} \frac{\Omega(y)}{\rho(y)^{\alpha-1}} f(x - \Phi(y)) dy \right|^2 \frac{dt}{t} \right)^{1/2}. \tag{1.3}$$

When $\alpha_1 = \dots = \alpha_n = 1$, we denote \mathcal{M}_Ω^Φ by μ_Ω^Φ . Clearly, if $n = d$ and $\Phi(y) = y$, the operator μ_Ω^Φ reduces to the classical Marcinkiewicz integral operator denoted by μ_Ω , which was introduced by Stein [2] and investigated by many authors (see [3–9] for example). In particular, Ding *et al.* [5] proved that if $\Omega \in H^1(S^{n-1})$, then μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Subsequently, Chen *et al.* [4] showed that μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $2\beta/(2\beta - 1) < p < 2\beta$ if $\Omega \in \mathcal{F}_\beta(S^{n-1})$ for some $\beta > 1$. Here

$$\begin{aligned} & \mathcal{F}_\beta(S^{n-1}) \\ & := \left\{ \Omega \in L^1(S^{n-1}) : \sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y)| \left(\log \frac{1}{|\xi \cdot y|} \right)^\beta d\sigma(y) < \infty \right\}, \quad \forall \beta > 0. \end{aligned} \tag{1.4}$$

The functions class $\mathcal{F}_\beta(S^{n-1})$ was introduced by Grafakos and Stefanov [10] in the study of L^p boundedness of singular integral operator with rough kernels. It follows from [10] that $\mathcal{F}_{\beta_1}(S^{n-1}) \subsetneq \mathcal{F}_{\beta_2}(S^{n-1})$ for $0 < \beta_2 < \beta_1$, and $\bigcup_{q>1} L^q(S^{n-1}) \subsetneq \mathcal{F}_\beta(S^{n-1})$ for any $\beta > 0$. Moreover,

$$\bigcap_{\beta>1} \mathcal{F}_\beta(S^{n-1}) \not\subseteq H^1(S^{n-1}) \not\subseteq \bigcup_{\beta>1} \mathcal{F}_\beta(S^{n-1})$$

and

$$\bigcap_{\beta>1} \mathcal{F}_\beta(S^{n-1}) \not\subseteq L \log^+ L(S^{n-1}).$$

Later on, Al-Salman *et al.* [11] proved that μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ provided that $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$. It is well known that $L(\log^+ L)^{1/2}(S^{n-1})$ and $H^1(S^{n-1})$ do not contain each other. When $n = d$ and $\Phi(y) = P(|y|)y'$ with $P(y)$ being a real polynomial on \mathbb{R} sat-

isfying $P(0) = 0$, Wu [12] proved that μ_{Ω}^{Φ} is bounded on $L^p(\mathbb{R}^n)$ for $1 + 1/(2\beta) < p < 1 + 2\beta$ provided that $\Omega \in \mathcal{F}_{\beta}(S^{n-1})$ for some $\beta > 1/2$. The L^p boundedness for the Marcinkiewicz integral operator associated to polynomial mappings has also been obtained (see [6, 13]).

When $\alpha_j \geq 1$ ($j = 1, \dots, n$), $n = d$ and $\Phi(y) = y$, we denote $\mathcal{M}_{\Omega}^{\Phi}$ by \mathcal{M}_{Ω} . In 2008, Ding *et al.* [14] proved that \mathcal{M}_{Ω} is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, provided that $\Omega \in L^q(S^{n-1})$ for fixed $q > 1$. Chen and Ding [15] extended the above result to the case $\Omega \in L(\log L)^{1/2}(S^{n-1})$. Later on, Chen and Lu [16] proved that \mathcal{M}_{Ω} is bounded on $L^p(\mathbb{R}^n)$ for $2\beta/(2\beta - 1) < p < 2\beta$, provided that $\Omega \in \mathcal{F}_{\beta}(S^{n-1})$ for some $\beta > 1$. This result was recently refined by Liu and Wu [17], who extended the range of β to the case $\beta > 1/2$ and the range of p to the case $1 + 1/(2\beta) < p < 1 + 2\beta$. When $n = d$ and $\Phi(y) = A_{\varphi}(y)$, Al-Salman [18] obtained the following result.

Theorem A *Let $n = d$ and $\Phi(y) = A_{\varphi}(y)$. Suppose that $\Omega \in \mathcal{F}_{\beta}(S^{n-1})$ for some $\beta > 1$ with satisfying (1.1)-(1.2).*

- (i) *If $\varphi(t) = P(t)$ with P being a real polynomial on \mathbb{R} , then $\mathcal{M}_{\Omega}^{\Phi}$ are bounded on $L^p(\mathbb{R}^n)$ for $2\beta/(2\beta - 1) < p < 2\beta$. The bounds are independent of the coefficients of P .*
- (ii) *If $\varphi \in \mathfrak{F}$, then $\mathcal{M}_{\Omega}^{\Phi}$ are bounded on $L^p(\mathbb{R}^n)$ for $2\beta/(2\beta - 1) < p < 2\beta$. Here \mathfrak{F} is the set of all functions ϕ which satisfy:*
 - (a) *$\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous increasing C^1 function satisfying that ϕ' is monotonous;*
 - (b) *there exist constant C_{ϕ} and c_{ϕ} such that $t\phi'(t) \geq C_{\phi}\phi(t)$ and $\phi(2t) \leq c_{\phi}\phi(t)$ for all $t > 0$.*

Remark 1.1 There are some model examples in the class \mathfrak{F} , such as t^{α} ($\alpha > 0$), $t^{\alpha}(\ln(1+t))^{\beta}$ ($\alpha, \beta > 0$), $t \ln \ln(e+t)$, real-valued polynomials P on \mathbb{R} with positive coefficients and $P(0) = 0$ and so on. For $\varphi \in \mathfrak{F}$, there exists a constant $B_{\varphi} > 1$ such that $\varphi(2t) \geq B_{\varphi}\varphi(t)$ (see [19]).

It is natural to ask whether Theorem A also holds if the range of β is relaxed to $\beta > 1/2$ and the range of p is relaxed to $1 + 1/(2\beta) < p < 1 + 2\beta$. In this paper, we will give an affirmative answer to this question. Our main results can be stated as follows.

Theorem 1.1 *Let $n = d$ and $\Phi(y) = (P_1(\varphi(\rho(y)))y'_1, \dots, P_n(\varphi(\rho(y)))y'_n)$ with $P_i(t)$ being real valued polynomials on \mathbb{R} satisfying $P_i(0) = 0$ and $\varphi \in \mathfrak{F}$. Suppose that $\Omega \in \mathcal{F}_{\beta}(S^{n-1})$ for some $\beta > 1/2$ satisfying (1.1)-(1.2). Then $\mathcal{M}_{\Omega}^{\Phi}$ are bounded on $L^p(\mathbb{R}^n)$ for $1 + 1/(2\beta) < p < 1 + 2\beta$. The bounds are independent of the coefficients of P_j for all $1 \leq j \leq n$ but depend on $\max_{1 \leq j \leq d} \deg(P_j)$ and φ .*

Theorem 1.2 *Let $n = d$ and $\Phi(y) = A_{P_N(\varphi)}(y)$ with $\varphi \in \mathfrak{F}$ and $P_N(t) = \sum_{i=1}^N a_i t^i$ and $P_N(t) > 0$ if $t \neq 0$. Suppose that $\Omega \in \mathcal{F}_{\beta}(S^{n-1})$ for some $\beta > 1/2$ satisfying (1.1)-(1.2). Then $\mathcal{M}_{\Omega}^{\Phi}$ are bounded on $L^p(\mathbb{R}^n)$ for $1 + 1/(2\beta) < p < 1 + 2\beta$. The bounds are independent of the coefficients of P_N but depend on N and φ .*

Remark 1.2 It is clear that Theorem 1.1 implies Theorem 1.2. When $\alpha_1 = \dots = \alpha_n = 1$, $\varphi(t) = t$ and $P_1(t) = \dots = P_n(t) = \sum_{i=1}^N a_i t^i$, Theorem 1.1 implies the result of [12]. In fact, Theorem 1.2 with $\varphi(t) = t$ extends the result of [12] to the mixed case. Comparing Theorem A with Theorem 1.2, the range of β is extended to the case $\beta > 1/2$ and the range of p

is enlarged to the case $1 + 1/(2\beta) < p < 1 + 2\beta$. Thus Theorem 1.2 essentially improves and generalizes the corresponding results in Theorem A. In addition, Theorem 1.2 implies the result [17, Theorem 1.3] when $P_N(t) = \varphi(t) = t$.

When $n = 2$, we have the following result.

Theorem 1.3 *Let $\phi = (\phi_1, \dots, \phi_d)$ be real analytic on S^1 . Let $\Phi(y) = P_N(\varphi(\rho(y)))\phi(y) = (P_N(\varphi(\rho(y)))\phi_1(y), \dots, P_N(\varphi(\rho(y)))\phi_d(y))$ with $P_N(t) = \sum_{i=1}^N a_i t^i$ and $\varphi \in \mathfrak{F}$. Suppose that $\Omega \in \mathcal{F}_\beta(S^1)$ for some $\beta > 1$ satisfying (1.1)-(1.2). Then \mathcal{M}_Ω^Φ are bounded on $L^p(\mathbb{R}^d)$ for $1 + 1/(2\beta) < p < 1 + 2\beta$. The bounds are independent of the coefficients of P_N but depend on φ and N .*

We remark that when $\alpha_1 = \dots = \alpha_n = 1$ and $\varphi(t) = t$, the surface $\{\Phi(y) : y \in \mathbb{R}^n\}$ given as in Theorem 1.3 recovers $\{(P_N(|y|)\phi_1(y'), \dots, P_N(|y|)\phi_d(y')) : y \in \mathbb{R}^n\}$, which was originally introduced by Al-Balushi and Al-Salman [20] in the study of L^p bounds of singular integrals associated to certain surfaces.

The third type of surfaces we consider are polynomial compound subvarieties. To state the rest of our result, we need to recall some notations. Let $\mathcal{A}(n, m)$ be the set of polynomials on \mathbb{R}^n which have real coefficients and degrees not exceeding m , and let $V(n, m)$ be the collection of polynomials in $\mathcal{A}(n, m)$ which are homogeneous of degree m . For $P \in \mathcal{A}(n, m)$, we set

$$\|P\| = \left\| \sum_{|\lambda| \leq m} a_\lambda y^\lambda \right\| = \left(\sum_{|\lambda| \leq m} |a_\lambda|^2 \right)^{1/2}.$$

Definition 1.1 ([21]) Let $n \geq 2$, $m \in \mathbb{N}$ and $\beta > 0$. An integrable function Ω on S^{n-1} is said to be in the space $\mathcal{F}(n, m, \beta)$ if

$$\sup_{P \in V(n, m), \|P\|=1} \int_{S^{n-1}} |\Omega(y)| \left(\log^+ \frac{1}{|P(y)|} \right)^\beta d\sigma(y) < \infty. \tag{1.5}$$

It should be pointed out that the condition (1.5) was introduced by Al-Salman and Pan [21] (also see [22]) in a study of the L^p boundedness of singular integrals with rough kernels. It is easy to check that $\mathcal{F}(n, 1, \beta) = \mathcal{F}_\beta(S^{n-1})$. Moreover, it was shown in [21] that

$$\mathcal{F}_\beta(S^1) = \bigcap_{m=1}^{\infty} \mathcal{F}(2, m, \beta). \tag{1.6}$$

The rest of the results can be stated as follows.

Theorem 1.4 *Let $\mathcal{P} = (P_1, \dots, P_d)$ with $P_j : \mathbb{R}^n \rightarrow \mathbb{R}$ being a polynomial for $1 \leq j \leq d$. Let $\Phi(y) = \mathcal{P}(\varphi(\rho(y))y')$ and $\varphi \in \mathfrak{F}$. Suppose that Ω satisfies (1.1)-(1.2) and $\Omega \in \bigcap_{s=1}^{\infty} \mathcal{F}(n, s, \beta)$ for some $\beta > 1/2$. Then \mathcal{M}_Ω^Φ are bounded on $L^p(\mathbb{R}^d)$ for $1 + 1/(2\beta) < p < 1 + 2\beta$. The bounds are independent of the coefficients of P_j for all $1 \leq j \leq d$ but depend on $\max_{1 \leq j \leq d} \deg(P_j)$ and φ .*

Theorem 1.5 *Let $\mathcal{P} = (P_1, \dots, P_d)$ with $P_j : \mathbb{R}^2 \rightarrow \mathbb{R}$ being a polynomial for $1 \leq j \leq d$. Let $\Phi(y) = \mathcal{P}(\varphi(\rho(y))y')$ and $\varphi \in \mathfrak{F}$. Suppose that Ω satisfies (1.1)-(1.2) and $\Omega \in \mathcal{F}_\beta(S^1)$ for some*

$\beta > 1/2$. Then \mathcal{M}_Ω^Φ are bounded on $L^p(\mathbb{R}^d)$ for $1 + 1/(2\beta) < p < 1 + 2\beta$. The bounds are independent of the coefficients of P_j for all $1 \leq j \leq d$ but depend on $\max_{1 \leq j \leq d} \deg(P_j)$ and φ .

Remark 1.3 When $\alpha_j = 1$ ($j = 1, \dots, n$), Theorem 1.4 implies Theorem 1.2 with $\rho = 1$ in [13]. Obviously, Theorem 1.5 follows from Theorem 1.4 because of (1.6).

The rest of this paper is organized as follows. After recalling some preliminary notations and lemmas in Section 2, we will prove our results in Section 3. We would like to remark that the main methods employed in this paper is a combination of ideas and arguments from [12, 21, 23]. The main ingredient in our proofs is to give a systematic treatment with these operators mentioned above.

Throughout this paper, we let p' satisfy $1/p + 1/p' = 1$. The letter C , sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence, but independent of the essential variables.

2 Preliminaries

Lemma 2.1 Let $\{\sigma_{j,t}\}$ be a family of measures. Suppose that

$$\left\| \sup_{j \in \mathbb{Z}} \sup_{t > 0} |\sigma_{j,t} * g| \right\|_p \leq C \|g\|_p$$

holds for some $p > 1$ and $g \in L^p(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that

$$\left\| \left(\int_1^2 \sum_{j \in \mathbb{Z}} |\sigma_{j,t} * g_j|^2 dt \right)^{1/2} \right\|_p \leq C \left\| \left(\sum_{j \in \mathbb{Z}} |g_j|^2 \right)^{1/2} \right\|_p$$

for arbitrary functions $\{g_j\}_{j \in \mathbb{Z}} \in L^p(\ell^2, \mathbb{R}^n)$.

Proof By the assumption, we have

$$\left\| \sup_{j \in \mathbb{Z}} \sup_{t \in [1,2]} |\sigma_{j,t} * g_j| \right\|_p \leq \left\| \sup_{j \in \mathbb{Z}} \sup_{t > 0} |\sigma_{j,t} * \sup_{j \in \mathbb{Z}} |g_j| \right\|_p \leq C \left\| \sup_{j \in \mathbb{Z}} |g_j| \right\|_p.$$

On the other hand, by the dual argument, there exists a function $h \in L^{p'}(\mathbb{R}^n)$ satisfying $\|h\|_{p'} = 1$ such that

$$\begin{aligned} \left\| \int_1^2 \sum_{j \in \mathbb{Z}} |\sigma_{j,t} * g_j| dt \right\|_p &= \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} \int_1^2 |\sigma_{j,t} * g_j(x)| dt h(x) dx \\ &\leq \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} \int_1^2 |\sigma_{j,t} * |g_j(x)|| dt |h(x)| dx \\ &\leq \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} |g_j(x)| \sup_{j \in \mathbb{Z}} \sup_{t \in [1,2]} |\sigma_{j,t} * |\bar{h}(x)|| dx \\ &\leq \left\| \sum_{j \in \mathbb{Z}} |g_j| \right\|_p \left\| \sup_{j \in \mathbb{Z}} \sup_{t > 0} |\sigma_{j,t} * |\bar{h}(x)|| \right\|_{p'} \\ &\leq C \left\| \sum_{j \in \mathbb{Z}} |g_j| \right\|_p, \end{aligned}$$

where $\bar{h}(x) = h(-x)$. Thus, Lemma 2.1 follows from the standard interpolation arguments. \square

Let $\{a_k\}_{k \in \mathbb{Z}}$ be a sequence of real positive numbers with satisfying $\inf_{k \in \mathbb{Z}} a_{k+1}/a_k = a > 1$. Let $\{\lambda_k\}_{k \in \mathbb{Z}}$ be a collection of $C^\infty(0, \infty)$ functions satisfying the following conditions:

$$\begin{aligned} \text{supp}(\lambda_k) &\subset [a_{k+1}^{-1}, a_{k-1}^{-1}]; \\ 0 \leq \lambda_k \leq 1; \quad \sum_{k \in \mathbb{Z}} \lambda_k^2(t) &= 1; \quad |d\lambda_k(t)/dt| \leq C/t, \end{aligned}$$

where C is independent of t and k . Let $M \in \mathbb{N} \setminus \{0\}$ and $L : \mathbb{R}^n \rightarrow \mathbb{R}^M$ be a linear transformation. For each $k \in \mathbb{Z}$, we define the multiplier operators S_k in \mathbb{R}^n by

$$\widehat{S_k f}(\xi) = \lambda_k(|L(\xi)|) \hat{f}(\xi). \tag{2.1}$$

By an argument which is similar to those used in [8, Proposition 3.1], one can easily get the following lemma. The details are omitted here.

Lemma 2.2 *Let S_k be as in (2.1) and $\{g_{j,k,t}\}$ arbitrary functions on \mathbb{R}^n . Then*

(i) *for each fixed $1 < p < 2$ and $1 < q < p$,*

$$\left\| \left(\sum_{j \in \mathbb{Z}} \int_1^2 \left| \sum_{k \in \mathbb{Z}} S_{j+k} g_{j,k,t} \right|^2 dt \right)^{1/2} \right\|_p^q \leq C \sum_{k \in \mathbb{Z}} \left\| \left(\sum_{j \in \mathbb{Z}} \int_1^2 |g_{j,k,t}|^2 dt \right)^{1/2} \right\|_p^q; \tag{2.2}$$

(ii) *for each fixed $2 < p < \infty$ and $1 < q < p'$,*

$$\begin{aligned} &\left\| \left(\sum_{j \in \mathbb{Z}} \int_1^2 \left| \sum_{k \in \mathbb{Z}} S_{j+k} g_{j,k,t} \right|^2 dt \right)^{1/2} \right\|_p^q \\ &\leq C \sum_{k \in \mathbb{Z}} \left(\int_1^2 \left\| \left(\sum_{j \in \mathbb{Z}} |g_{j,k,t}|^2 \right)^{1/2} \right\|_p^2 dt \right)^{q/2}. \end{aligned} \tag{2.3}$$

The following lemma is our main ingredient in the proof of our main results.

Lemma 2.3 *Let $\{\tau_{k,t} : k \in \mathbb{Z}, t \in \mathbb{R}^+\}$ be a family of uniformly bounded Borel measures on \mathbb{R}^n . Let $\{a_k\}_{k \in \mathbb{Z}}$ be a sequence of real numbers with satisfying $\inf_{k \in \mathbb{Z}} a_{k+1}/a_k = a > 1$. Let $M \in \mathbb{N} \setminus \{0\}$ and $L : \mathbb{R}^n \rightarrow \mathbb{R}^M$ be a linear transformation. Suppose that*

$$(i) \quad |\widehat{\tau_{k,t}}(\xi)| \leq C \min\{1, a_k |L(\xi)|\}; \tag{2.4}$$

$$(ii) \quad |\widehat{\tau_{k,t}}(\xi)| \leq C (\log |a_k L(\xi)|)^{-\beta} \quad \text{for some } \beta > 0, \text{ if } a_k |L(\xi)| > 1; \tag{2.5}$$

$$(iii) \quad \left\| \sup_{k \in \mathbb{Z}} \sup_{t > 0} |\tau_{k,t}| * f \right\|_q \leq C \|f\|_q \tag{2.6}$$

for all $1 < q < \infty$. Then for $p \in (1 + 1/(2\beta), 1 + 2\beta)$ and $\beta > 1/2$, there exists a constant $C(a) > 0$ such that

$$\left\| \left(\int_1^2 \sum_{k \in \mathbb{Z}} |\tau_{k,t} * f|^2 dt \right)^{1/2} \right\|_p \leq C(a) \|f\|_p.$$

Proof Let S_k be as in (2.1). Then we can write

$$\begin{aligned} G(f)(x) &:= \left(\int_1^2 \sum_{k \in \mathbb{Z}} |\tau_{k,t} * f(x)|^2 dt \right)^{1/2} \\ &= \left(\int_1^2 \sum_{k \in \mathbb{Z}} \left| \tau_{k,t} * \left(\sum_{j \in \mathbb{Z}} S_{j+k} S_{j+k} f \right)(x) \right|^2 dt \right)^{1/2} \\ &= \left(\sum_{k \in \mathbb{Z}} \int_1^2 \left| \sum_{j \in \mathbb{Z}} S_{j+k} (\tau_{k,t} * S_{j+k} f)(x) \right|^2 dt \right)^{1/2}. \end{aligned} \tag{2.7}$$

Case 1. $1 + 1/(2\beta) < p < 2$. It follows from (2.2) and (2.7) that

$$\|G(f)\|_p^q \leq C \sum_{j \in \mathbb{Z}} \left\| \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\tau_{k,t} * S_{j+k} f|^2 dt \right)^{1/2} \right\|_p^q, \quad \forall 1 < q < p. \tag{2.8}$$

For each fixed $k \in \mathbb{Z}$, we set

$$I_{kj}(x) := \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\tau_{k,t} * S_{j+k} f(x)|^2 dt \right)^{1/2}.$$

Invoking Lemma 2.1 and the Littlewood-Paley theory imply

$$\|I_{kj}\|_p \leq C \left\| \left(\sum_{k \in \mathbb{Z}} |S_{j+k} f|^2 \right)^{1/2} \right\|_p \leq C \|f\|_p, \quad \forall 1 < p < \infty. \tag{2.9}$$

On the other hand, by Plancherel's theorem and (2.4)-(2.5), we have

$$\begin{aligned} \|I_{kj}\|_2^2 &= \int_1^2 \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |\lambda_{j+k}(|L(\xi)|)|^2 |\widehat{\tau_{k,t}}(\xi)|^2 d\xi dt \\ &\leq C \int_1^2 \sum_{k \in \mathbb{Z}} \int_{\{a_{j+k-1}^{-1} \leq |L(\xi)| \leq a_{j+k-1}^{-1}\}} |\hat{f}(\xi)|^2 |\widehat{\tau_{k,t}}(\xi)|^2 d\xi dt \\ &\leq CB_j^2 \|f\|_2^2, \end{aligned}$$

where $B_j = a^{-j+1} \chi_{\{j \geq -1\}} + (|j+1| \log a)^{-\beta} \chi_{\{j < -1\}}$. That is,

$$\|I_{kj}\|_2 \leq CB_j \|f\|_2. \tag{2.10}$$

Interpolating between (2.9) and (2.10), there exists $\epsilon \in (2/(2\beta + 1), 1)$ such that

$$\|I_{kj}\|_p \leq CB_j^\epsilon \|f\|_p, \quad \forall 1 + 1/(2\beta) < p < 2.$$

For fixed $p \in (1 + 1/(2\beta), 2)$ and $\beta > 1/2$, we can choose $q \in (1, p)$ such that $q\epsilon\beta > 1$. Thus

$$\sum_{j \in \mathbb{Z}} \|I_{kj}\|_p^q \leq C \left(\sum_{j \geq -1} a^{-q\epsilon(j-1)} + \sum_{j < -1} (|j+1| \log a)^{-q\epsilon\beta} \right) \|f\|_p^q \leq C(a) \|f\|_p^q,$$

which, together with (2.8), implies

$$\|G(f)\|_p \leq C(a)\|f\|_p, \quad \text{for } 1 + 1/(2\beta) < p < 2. \tag{2.11}$$

Case 2. $2 < p < 1 + 2\beta$. By (2.3) and (2.7), we have for $2 < p < \infty$ and $1 < q < p'$,

$$\|G(f)\|_p^q \leq C \sum_{j \in \mathbb{Z}} \left(\int_1^2 \left\| \left(\sum_{k \in \mathbb{Z}} |\tau_{k,t} * S_{j+k}f|^2 \right)^{1/2} \right\|_p^2 dt \right)^{q/2}. \tag{2.12}$$

Let

$$J_{j,t}f(x) := \left(\sum_{k \in \mathbb{Z}} |\tau_{k,t} * S_{j+k}f(x)|^2 \right)^{1/2}.$$

By (2.6), [23, p.544, Lemma] and the Littlewood-Paley theory, we have, for $k \in \mathbb{Z}$ and $t \in [1, 2]$,

$$\|J_{j,t}f\|_{p_0} \leq C \left\| \left(\sum_{k \in \mathbb{Z}} |S_{j+k}f|^2 \right)^{1/2} \right\|_{p_0} \leq C\|f\|_{p_0}, \quad \forall 1 < p_0 < \infty. \tag{2.13}$$

On the other hand, by the same arguments as in (2.10), we have

$$\|J_{j,t}f\|_2 \leq CB_j\|f\|_2, \tag{2.14}$$

where B_j is as in (2.10). On interpolation between (2.13) and (2.14), for fixed $p \in (2, 1 + 2\beta)$ and $\beta > 1/2$, we can choose $q \in (1, p')$ and $\delta \in (2/(2\beta + 1), 1)$ such that $q\delta\beta > 1$ and

$$\|J_{j,t}f\|_p \leq CB_j^\delta\|f\|_p, \quad \text{for } 2 < p < 1 + 2\beta.$$

This, combined with (2.12), implies

$$\|G(f)\|_p^q \leq C \left(\sum_{j \geq -1} a^{-q\delta(j-1)} + \sum_{j < -1} (|j+1| \log a)^{-q\delta\beta} \right) \|f\|_p^q \leq C(a)\|f\|_p^q,$$

which, together with (2.11), completes the proof of Lemma 2.3. □

Lemma 2.4 ([13, Lemma 2.2]) *Suppose $\Phi(t) = t^{\alpha_1} + \mu_2 t^{\alpha_2} + \dots + \mu_n t^{\alpha_n}$ and $\varphi \in \mathfrak{F}$, where μ_2, \dots, μ_n are real parameters, and $\alpha_1, \dots, \alpha_n$ are distinct positive (not necessarily integer) exponents. Then for any $r > 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$,*

$$\left| \int_{r/2}^r \exp(i\lambda\Phi(\varphi(t))) \frac{dt}{t} \right| \leq C(\varphi) |\lambda\varphi(r)^{\alpha_1}|^{-\epsilon},$$

where $\epsilon = \min\{1/\alpha_1, 1/n\}$ and $C(\varphi)$ does not depend on μ_2, \dots, μ_n .

Lemma 2.5 ([24, Lemma 2.2]) *Let $\mathcal{P}(t) = (P_1(t), \dots, P_d(t))$ with P_j being real polynomials defined on \mathbb{R}^+ . Suppose that $\varphi \in \mathfrak{F}$. Then the operator $M_{\mathcal{P},\varphi}$ defined by*

$$M_{\mathcal{P},\varphi}(f)(x) = \sup_{r>0} \int_r^{2r} |f(x - \mathcal{P}(\varphi(t)))| \frac{dt}{t}$$

is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. The bound is independent of the coefficients of P_j for all $1 \leq j \leq d$ and f but depends on φ .

Lemma 2.6 ([25]) *Let $\Phi : S^1 \rightarrow \mathbb{R}^d$, $\Phi = (\Phi_1, \dots, \Phi_d)$ be real analytic on S^1 . Suppose that $\{\Phi_1, \dots, \Phi_d\}$ is linearly independent set. If $\Omega \in \mathcal{F}_\beta(S^1)$ for some $\beta > 1$, then*

$$\sup_{\xi \in S^{d-1}} \int_{S^1} |\Omega(y')| \left(\log^+ \frac{1}{|\xi \cdot \Phi(y')|} \right)^\beta d\sigma(y') < \infty.$$

3 Proofs of main theorems

Proof of Theorem 1.1 Let $\mathcal{N} = \max_{1 \leq j \leq n} \deg(P_j)$. For $1 \leq l \leq n$, let $P_l(t) = \sum_{i=1}^{\mathcal{N}} a_{i,l} t^i$. For $1 \leq s \leq \mathcal{N}$, and $1 \leq l \leq n$, let $P_l^{(s)}(t) = \sum_{i=1}^s a_{i,l} t^i$ and $P^{(s)}(t) = (P_1^{(s)}(t), \dots, P_n^{(s)}(t))$. Set $P^{(0)}(t) = 0$ and

$$\Phi_s(y) = (P_1^{(s)}(\varphi(\rho(y)))y'_1, \dots, P_n^{(s)}(\varphi(\rho(y)))y'_n).$$

Then we can write

$$\begin{aligned} \Phi_s(y) \cdot \xi &= \sum_{l=1}^n \xi_l y'_l P_l^{(s)}(\varphi(\rho(y))) \\ &= \sum_{l=1}^n \sum_{i=1}^s \xi_l y'_l a_{i,l} \varphi(\rho(y))^i = \sum_{i=1}^s (L_i(\xi) \cdot y') \varphi(\rho(y))^i, \end{aligned} \tag{3.1}$$

where $L_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the linear transformation given by

$$L_i(\xi) = (a_{i,1}\xi_1, \dots, a_{i,n}\xi_n).$$

For each $j \in \mathbb{Z}$, $t \in \mathbb{R}^+$ and $1 \leq s \leq \mathcal{N}$, we define the measures $\{\sigma_{j,t}^s\}$ and $\{|\sigma_{j,t}^s|\}$ by

$$\begin{aligned} \widehat{\sigma_{j,t}^s}(\xi) &= \frac{1}{2^j t} \int_{2^{j-1}t < \rho(y) \leq 2^j t} \exp(-2\pi i \Phi_s(y) \cdot \xi) \frac{\Omega(y)}{\rho(y)^{\alpha-1}} dy, \\ |\widehat{\sigma_{j,t}^s}|(\xi) &= \frac{1}{2^j t} \int_{2^{j-1}t < \rho(y) \leq 2^j t} \exp(-2\pi i \Phi_s(y) \cdot \xi) \frac{|\Omega(y)|}{\rho(y)^{\alpha-1}} dy. \end{aligned}$$

We get from (3.1)

$$\begin{aligned} &|\widehat{\sigma_{j,t}^s}(\xi) - \widehat{\sigma_{j,t}^{s-1}}(\xi)| \\ &\leq \frac{1}{2^j t} \int_{2^{j-1}t < \rho(y) \leq 2^j t} |\exp(-2\pi i \Phi_s(y) \cdot \xi) - \exp(-2\pi i \Phi_{s-1}(y) \cdot \xi)| \frac{|\Omega(y)|}{\rho(y)^{\alpha-1}} dy \\ &\leq C(\varphi(2^j t))^s |L_s(\xi)|. \end{aligned} \tag{3.2}$$

On the other hand, by a change of variable, we have

$$\begin{aligned} |\widehat{\sigma}_{j,t}^s(\xi)| &= \frac{1}{2^j t} \left| \int_{2^{j-1}t}^{2^j t} \int_{S^{n-1}} \exp\left(-2\pi i \sum_{k=1}^s L_k(\xi) \cdot \theta \varphi(r)^k\right) \Omega(\theta) J(\theta) d\sigma(\theta) dr \right| \\ &\leq C \int_{S^{n-1}} |\Omega(\theta)| |I_{j,t,s,\xi}(\theta)| d\sigma(\theta), \end{aligned} \tag{3.3}$$

where

$$I_{j,t,s,\xi}(\theta) := \frac{1}{2^j t} \int_{2^{j-1}t}^{2^j t} \exp\left(-2\pi i \sum_{k=1}^s L_k(\xi) \cdot \theta \varphi(r)^k\right) dr.$$

By Lemma 2.4, we have

$$|I_{j,t,s,\xi}(\theta)| \leq C |\varphi(2^j t)^s L_s(\xi) \cdot \theta|^{-1/s}.$$

Combining the trivial inequality $|I_{j,t,s,\xi}(\theta)| \leq C$ with the fact that $t/(\log t)^\beta$ is increasing in (e^β, ∞) , we have

$$|I_{j,t,s,\xi}(\theta)| \leq C \frac{(\log e^{\beta s} |\eta \cdot \theta|^{-1})^\beta}{(\log |\varphi(2^j t)^s L_s(\xi)|)^\beta}, \quad \text{if } |\varphi(2^j t)^s L_s(\xi)| > 1, \tag{3.4}$$

where $\eta = L_s(\xi)/|L_s(\xi)|$. This, together (3.3) with the fact that $\Omega \in \mathcal{F}_\beta(S^{n-1})$, implies

$$|\widehat{\sigma}_{j,t}^s(\xi)| \leq C (\log |\varphi(2^j t)^s L_s(\xi)|)^{-\beta}, \quad \text{if } |\varphi(2^j t)^s L_s(\xi)| > 1. \tag{3.5}$$

Now we can choose a function $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(t) \equiv 1$ for $|t| \leq 1/2$ and $\psi(t) \equiv 0$ for $|t| > 1$. For $1 \leq s \leq \mathcal{N}$, $j \in \mathbb{Z}$ and $t \in \mathbb{R}^+$, we define the measures $\{\tau_{j,t}^s\}$ by

$$\begin{aligned} \widehat{\tau}_{j,t}^s(\xi) &= \widehat{\sigma}_{j,t}^s(\xi) \prod_{k=s+1}^{\mathcal{N}} \psi(|\varphi(2^j t)^k L_k(\xi)|) \\ &\quad - \widehat{\sigma}_{j,t}^{s-1}(\xi) \prod_{k=s}^{\mathcal{N}} \psi(|\varphi(2^j t)^k L_k(\xi)|). \end{aligned}$$

Here we use the convention $\prod_{j \in \emptyset} a_j = 1$. It is easy to see that

$$\sigma_{j,t}^{\mathcal{N}} = \sum_{s=1}^{\mathcal{N}} \tau_{j,t}^s. \tag{3.6}$$

It follows from (3.2), (3.5), and the trivial estimate $|\widehat{\sigma}_{j,t}^s(\xi)| \leq C$ that, for $1 \leq s \leq \mathcal{N}$,

$$|\widehat{\tau}_{j,t}^s(\xi)| \leq C(\varphi) \min\{1, \varphi(2^j t)^s |L_s(\xi)|\}, \tag{3.7}$$

$$|\widehat{\tau}_{j,t}^s(\xi)| \leq C(\varphi) (\log |\varphi(2^j t)^s L_s(\xi)|)^{-\beta}, \quad \text{if } |\varphi(2^j t)^s L_s(\xi)| > 1. \tag{3.8}$$

By the definition of $\sigma_{j,t}^s$ and (3.6), we can write

$$\begin{aligned}
 \mathcal{M}_\Omega^\Phi(f)(x) &= \left(\int_0^\infty \left| \sum_{j=-\infty}^0 2^j \sigma_{j,t}^{\mathcal{N}} * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \\
 &\leq \sum_{j=-\infty}^0 2^j \left(\int_0^\infty |\sigma_{j,t}^{\mathcal{N}} * f(x)|^2 \frac{dt}{t} \right)^{1/2} \\
 &\leq 2 \left(\sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} |\sigma_{0,t}^{\mathcal{N}} * f(x)|^2 \frac{dt}{t} \right)^{1/2} \\
 &\leq 2 \left(\sum_{j \in \mathbb{Z}} \int_1^2 |\sigma_{j,t}^{\mathcal{N}} * f(x)|^2 \frac{dt}{t} \right)^{1/2} \\
 &\leq 2 \sum_{s=1}^{\mathcal{N}} \left(\int_1^2 \sum_{j \in \mathbb{Z}} |\tau_{j,t}^s * f(x)|^2 \frac{dt}{t} \right)^{1/2}. \tag{3.9}
 \end{aligned}$$

On the other hand, by a change of variable we have

$$\begin{aligned}
 \|\sigma_{j,t}^s * f(x)\| &\leq \frac{1}{2^j t} \int_{2^{j-1}t < \rho(y) \leq 2^j t} |f(x - \Phi_s(y))| \frac{|\Omega(y)|}{\rho(y)^{\alpha-1}} dy \\
 &\leq C \int_{S^{n-1}} |\Omega(y')| \left(\frac{1}{2^j t} \int_{2^{j-1}t}^{2^j t} |f(x - \Phi_s(A_r y'))| dr \right) d\sigma(y') \\
 &\leq C \int_{S^{n-1}} |\Omega(y')| M_{\mathcal{P},\varphi}(f)(x) d\sigma(y'),
 \end{aligned}$$

where $M_{\mathcal{P},\varphi}$ is as in Lemma 2.5 and $\mathcal{P}(t) = (P_1^{(s)}(t)y'_1, \dots, P_n^{(s)}(t)y'_n)$. By Lemma 2.5 and Minkowski's inequality, we have

$$\left\| \sup_{j \in \mathbb{Z}} \sup_{t > 0} \|\sigma_{j,t}^s * f\| \right\|_p \leq C \|f\|_p.$$

This inequality, together with the definition of $\tau_{j,t}^s$, yields

$$\left\| \sup_{j \in \mathbb{Z}} \sup_{t > 0} \|\tau_{j,t}^s * f\| \right\|_p \leq C \|f\|_p. \tag{3.10}$$

Then Theorem 1.1 follows from (3.7)-(3.10) and Lemma 2.3. □

Proof of Theorem 1.3 Let Φ, P_N, φ, ϕ be as in Theorem 1.3. For $1 \leq s \leq N$, we set $P_s(t) = \sum_{m=1}^s a_m t^m$. Define the measures $\{\sigma_{j,t}^s\}$ and $\{|\sigma_{j,t}^s|\}$ by

$$\widehat{\sigma_{j,t}^s}(\xi) = \frac{1}{2^j t} \int_{2^{j-1}t < \rho(y) \leq 2^j t} \exp(-2\pi i P_s(\varphi(\rho(y)))\phi(y') \cdot \xi) \frac{\Omega(y)}{\rho(y)^{\alpha-1}} dy; \tag{3.11}$$

$$\widehat{|\sigma_{j,t}^s|}(\xi) = \frac{1}{2^j t} \int_{2^{j-1}t < \rho(y) \leq 2^j t} \exp(-2\pi i P_s(\varphi(\rho(y)))\phi(y') \cdot \xi) \frac{|\Omega(y)|}{\rho(y)^{\alpha-1}} dy. \tag{3.12}$$

Following the notation in [20], let $\{\phi_{i_1}, \dots, \phi_{i_l}\}$ be a maximal linearly independent subset of $\{\phi_1, \dots, \phi_d\}$, where $1 \leq l \leq d, 1 \leq i_r \leq d$ and $r = 1, \dots, l$. Thus, for $j \notin \{i_1, \dots, i_l\}$, there exist

$a^{(j)} = (a_{j,1}, \dots, a_{j,l}) \in \mathbb{R}^l$ such that

$$\phi_j(y') = a^{(j)} \cdot (\phi_{i_1}(y'), \dots, \phi_{i_l}(y')) = \sum_{k=1}^l a_{j,k} \phi_{i_k}(y').$$

This implies that there exists a linear transformation $L : \mathbb{R}^d \rightarrow \mathbb{R}^l$ such that

$$\phi(y') \cdot \xi = L(\xi) \cdot \tilde{\phi}(y'), \quad \xi \in \mathbb{R}^d,$$

where $\tilde{\phi}(y') = (\phi_{i_1}(y'), \dots, \phi_{i_l}(y'))$. Thus

$$|\widehat{\sigma}_{j,t}^s(\xi)| \leq C \int_{S^1} |\Omega(z')| |I_{j,t,s,\xi}(z')| d\sigma(z'),$$

where

$$I_{j,t,s,\xi}(z') := \frac{1}{2^j t} \int_{2^{j-1}t}^{2^j t} \exp[-2\pi i P_s(\varphi(u))(L(\xi) \cdot \tilde{\phi}(z'))] du.$$

By Lemma 2.4, we have

$$|I_{j,t,s,\xi}(z')| \leq C |\varphi(2^j t)^s a_s L(\xi) \cdot \tilde{\phi}(z')|^{-1/s}. \tag{3.13}$$

Since $t/(\log t)^\beta$ is increasing in (e^β, ∞) for any $\beta > 0$, and $|\tilde{\phi}(z')| \leq B$ with $B > 1$ for any $z' \in S^{n-1}$. We can deduce from (3.13) and the trivial estimate $|I_{j,t,s,\xi}(z')| \leq C$ that

$$|I_{j,t,s,\xi}(z')| \leq C \frac{(\log(Be^{\beta s} |\zeta \cdot \tilde{\phi}(z')|^{-1}))^\beta}{(\log |\varphi(2^j t)^s a_s L(\xi)|)^\beta}, \quad \text{if } |\varphi(2^j t)^s a_s L(\xi)| > 1, \tag{3.14}$$

where $\zeta = L(\xi)/|L(\xi)|$. Invoking Lemma 2.6 and (3.14), we obtain, for $\beta > 1$,

$$|\widehat{\sigma}_{j,t}^s(\xi)| \leq C (\log |\varphi(2^j t)^s a_s L(\xi)|)^{-\beta}, \quad \text{if } |\varphi(2^j t)^s a_s L(\xi)| > 1. \tag{3.15}$$

On the other hand, we have

$$\begin{aligned} |\widehat{\sigma}_{j,t}^s(\xi) - \widehat{\sigma}_{j,t}^{s-1}(\xi)| &\leq \frac{1}{2^j t} \int_{2^{j-1}t < \rho(y) \leq 2^j t} |\exp(-2\pi i (a_s \varphi(\rho(y))^s |L(\xi)| \tilde{\phi}(y))) - 1| \frac{|\Omega(y)|}{\rho(y)^{\alpha-1}} dy \\ &\leq C |\varphi(2^j t)^s a_s L(\xi)| \int_{S^1} |\Omega(z')| |\tilde{\phi}(z')| d\sigma(z') \\ &\leq C |\varphi(2^j t)^s a_s L(\xi)| \sup_{z' \in S^1} |\tilde{\phi}(z')| \\ &\leq C |\varphi(2^j t)^s a_s L(\xi)|. \end{aligned} \tag{3.16}$$

Notice that

$$\begin{aligned} \|\sigma_{j,t}^s * f(x)\| &\leq \frac{1}{2^j t} \int_{2^{j-1}t < \rho(y) \leq 2^j t} |f(x - P_s(\varphi(\rho(y)))\phi(y))| \frac{|\Omega(y)|}{\rho(y)^{\alpha-1}} dy \\ &\leq C \int_{S^1} |\Omega(y')| \left(\int_0^{2^j t} |f(x - P_s(\varphi(u))\phi(y'))| \frac{du}{u} \right) d\sigma(y'). \end{aligned}$$

This combining with Lemma 2.5 and Minkowski's inequality, implies

$$\left\| \sup_{j \in \mathbb{Z}} \sup_{t > 0} |\sigma_{j,t}^s| * f \right\|_p \leq C \|f\|_p. \tag{3.17}$$

Then the rest of the proof of Theorem 1.3 follows from an argument which is similar to those in the proof of Theorem 1.1 and (3.15)-(3.17). We omit the details. \square

Proof of Theorem 1.4 Let $n \geq 2$, $\mathcal{P} = (P_1, \dots, P_d)$, where $P_j : \mathbb{R}^n \rightarrow \mathbb{R}$ is a polynomial for $1 \leq j \leq d$. Let

$$M = \max \{ \deg(P_1), \dots, \deg(P_d) \},$$

and

$$P_i(y) = \sum_{|\gamma| \leq M} a_{i\gamma} y^\gamma \quad \text{for } i = 1, \dots, d.$$

For $1 \leq s \leq M$, we let

$$\mathcal{P}^{(s)} = (P_{1,s}, \dots, P_{d,s}),$$

where

$$P_{i,s}(y) = \sum_{|\gamma| \leq s} a_{i\gamma} y^\gamma.$$

Set $\mathcal{P}^{(0)} = 0$ and $\Phi_s(y) = \mathcal{P}^{(s)}(\varphi(\rho(y)))$.

For each $j \in \mathbb{Z}$, $t \in \mathbb{R}^+$ and $0 \leq s \leq M$, we define the measures $\{\sigma_{j,t}^s\}$ and $\{|\sigma_{j,t}^s|\}$ by

$$\begin{aligned} \widehat{\sigma_{j,t}^s}(\xi) &= \frac{1}{2^j t} \int_{2^{j-1}t < \rho(y) \leq 2^j t} \exp(-2\pi i \Phi_s(y) \cdot \xi) \frac{\Omega(y)}{\rho(y)^{\alpha-1}} dy; \\ |\widehat{\sigma_{j,t}^s}|(\xi) &= \frac{1}{2^j t} \int_{2^{j-1}t < \rho(y) \leq 2^j t} \exp(-2\pi i \Phi_s(y) \cdot \xi) \frac{|\Omega(y)|}{\rho(y)^{\alpha-1}} dy. \end{aligned}$$

For $1 \leq s \leq M$, let l_s denote the number of multi-indices $\gamma = (\gamma_1, \dots, \gamma_n)$ satisfying $|\gamma| = s$, and define the linear transformation $L_s : \mathbb{R}^d \rightarrow \mathbb{R}^{l_s}$ by

$$L_s(\xi) = \left((L_s(\xi))_\gamma \right)_{|\gamma|=s} = \left(\sum_{i=1}^d a_{i\gamma} \xi_i \right)_{|\gamma|=s}.$$

By the change of variables, we have

$$\begin{aligned} |\widehat{\sigma_{j,t}^s}(\xi)| &= \left| \int_{S^{n-1}} \frac{1}{2^j t} \int_{2^{j-1}t}^{2^j t} \exp(-2\pi i \xi \cdot \mathcal{P}^{(s)}(\varphi(u)\theta)) du \Omega(\theta) J(\theta) d\sigma(\theta) \right| \\ &\leq C \int_{S^{n-1}} |\Omega(\theta)| \left| \frac{1}{2^j t} \int_{2^{j-1}t}^{2^j t} \exp(-2\pi i \xi \cdot \mathcal{P}^{(s)}(\varphi(u)\theta)) du \right| d\sigma(\theta) \end{aligned}$$

$$\begin{aligned} &\leq C \int_{S^{n-1}} |\Omega(\theta)| \left| \frac{1}{2^j t} \int_{2^{j-1}t}^{2^j t} \exp\left(-2\pi i \sum_{|\gamma| \leq s} \sum_{i=1}^d \xi_j a_{j\gamma} \theta^\gamma \varphi(u)^{|\gamma|}\right) du \right| d\sigma(\theta) \\ &\leq C \int_{S^{n-1}} |\Omega(\theta)| |J_{j,t,s,\xi}(\theta)| d\sigma(\theta), \end{aligned}$$

where

$$J_{j,t,s,\xi}(\theta) := \frac{1}{2^j t} \int_{2^{j-1}t}^{2^j t} \exp\left(-2\pi i \sum_{|\gamma|=s} (L_s(\xi))^\gamma \theta^\gamma \varphi(u)^s + \text{lower powers in } u\right) du.$$

Let

$$Q_{s,\xi}(\theta) := |L_s(\xi)|^{-1} \sum_{|\gamma|=s} (L_s(\xi))^\gamma \theta^\gamma.$$

By Lemma 2.4, we have

$$|J_{j,t,s,\xi}(\theta)| \leq C(\varphi(2^j t)^s |L_s(\xi)| |Q_{s,\xi}(\theta)|)^{-1/s}.$$

By this inequality, together with the trivial estimate $|J_{j,t,s,\xi}(\theta)| \leq C$, we get

$$|\widehat{\sigma}_{j,t,s,\theta}(\xi)| \leq C \frac{(\log(e^{\beta s} L_s^{1/2} |Q_{s,\xi}(\theta)|^{-1}))^\beta}{(\log |\varphi(2^j t)^s L_s(\xi)|)^\beta}, \quad \text{if } |\varphi(2^j t)^s L_s(\xi)| > 1.$$

Since $\Omega \in \bigcap_{s=1}^\infty \mathcal{F}(n, s, \beta)$, $Q_{s,\xi} \in V(n, s)$ and $\|Q_{s,\xi}\| = 1$, we immediately obtain

$$|\widehat{\sigma}_{j,t}^s(\xi)| \leq C(\log |\varphi(2^j t)^s L_s(\xi)|)^{-\beta}, \quad \text{if } |\varphi(2^j t)^s L_s(\xi)| > 1. \tag{3.18}$$

On the other hand, we have

$$\begin{aligned} &|\widehat{\sigma}_{j,t}^s(\xi) - \widehat{\sigma}_{j,t}^{s-1}(\xi)| \\ &\leq \frac{1}{2^j t} \int_{2^{j-1}t < \rho(y) \leq 2^j t} \left| \exp\left(-2\pi i \sum_{j=1}^d \sum_{|\iota|=s} \xi_j a_{j\iota} \varphi(\rho(y))^s (y)^\iota\right) - 1 \right| \frac{|\Omega(y)|}{\rho(y)^{\alpha-1}} dy \\ &\leq C |\varphi(2^j t)^s L_s(\xi)|. \end{aligned} \tag{3.19}$$

In addition, using Lemma 2.5, one can easily check that

$$\left\| \sup_{j \in \mathbb{Z}} \sup_{t > 0} |\sigma_{j,t}^s| * f \right\|_p \leq C \|f\|_p. \tag{3.20}$$

Then the rest proof of Theorem 1.4 follows from similar arguments to the proof of Theorem 1.1 and (3.18)-(3.20). Details will be omitted. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors worked jointly in drafting and approving the final manuscript.

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