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Notes on the paper entitled 'Generalizations of the logarithmic Hardy inequality in critical Sobolev-Lorentz spaces'

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The purpose of this note is to clarify the novelty of the paper entitled 'Generalizations of the logarithmic Hardy inequality in critical Sobolev-Lorentz spaces' which was published in the J. Inequal. Appl. 2013:381, 2013 [1]. After this paper was published, the authors were informed of the references [2–5], and [6], the results of which partly overlap with those of [1]. In the paper [1], the authors established the Hardy inequality of the logarithmic type in the critical Sobolev-Lorentz spaces $H^{\frac{n}{p}}_{p,q}(\mathbb{R}^n)$; see Section 2 in [1] for the precise definition of $H^{\frac{n}{p}}_{p,q}(\mathbb{R}^n)$. The main theorem in [1] is stated as follows.

Theorem A [1, Theorem 1.1] *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$ and $1 < \alpha, \beta < \infty$. Then the inequality*

$$\left(\int_{\{|x|<\frac{1}{2}\}} \frac{|u(x)|^\alpha}{|\log |x||^\beta} \frac{dx}{|x|^n} \right)^{\frac{1}{\alpha}} \le C \|u\|_{H^{\frac{n}{p}}_{p,q}} \tag{1}$$

holds for all $u \in H^{\frac{n}{p}}_{p,q}(\mathbb{R}^n)$ if and only if one of the following conditions (i), (ii), *and* (iii) *is fulfilled*:

$$\begin{cases} \text{(i)} & 1+\alpha-\beta < 0; \\ \text{(ii)} & 1+\alpha-\beta \ge 0 \quad \text{and} \quad q < \frac{\alpha}{1+\alpha-\beta}; \\ \text{(iii)} & 1+\alpha-\beta > 0, \qquad q = \frac{\alpha}{1+\alpha-\beta} \quad \text{and} \quad \alpha \ge \beta. \end{cases} \tag{2}$$

However, the inequality (1) was established under the condition (iii) in [7] and [6]. More precisely, the particular case of $\alpha = \beta = p = q$ was considered in [7], where the authors also obtained similar inequalities in the critical Besov spaces. In [6], the authors considered Bessel potential spaces with a logarithmic smoothness on Lorentz-Zygmund spaces which extend the critical Sobolev-Lorentz spaces $H^{\frac{n}{p}}_{p,q}(\mathbb{R}^n)$ for instance. Indeed, the authors in [6] proved logarithmic Hardy inequalities corresponding to (1) in critical Sobolev-Lorentz-Zygmund spaces with a logarithmic smoothness in [6, Theorem 5.1]. This theorem includes the inequality (1) under the condition (iii) as special cases.

Based on the historical remarks above, the novelty in the paper [1] is to have given other conditions (i) and (ii) with which the inequality (1) holds and to have shown that the conditions (i), (ii), and (iii) are also necessary for (1). Especially, the marginal case $q = \infty$ was

also considered in [1], where the norm $\|u\|_{H^{\frac{n}{p}}_{p,\infty}}$ becomes smallest in the sense of Sobolev-Lorentz spaces, and Theorem A shows that the condition (i) is necessary and sufficient for (1) to hold.

We also add other references [2–4], and [5] in this note, which are closely related to the results in [1] and which were not cited in [1].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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