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Inequalities for dual quermassintegrals of the radial pth mean bodies

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Abstract

Gardner and Zhang defined the notion of radial pth mean body ($p > -1$) in the Euclidean space $\mathbb{R}^n$. In this paper, we obtain inequalities for dual quermassintegrals of the radial pth mean bodies. Further, we establish dual quermassintegrals forms of the Zhang projection inequality and the Rogers-Shephard inequality, respectively. Finally, Shephard's problem concerning the radial pth mean bodies is shown when $p > 0$.

MSC: 52A40; 52A20

Keywords: radial pth mean body; dual quermassintegrals; Zhang projection inequality; Rogers-Shephard inequality

1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in the Euclidean space $\mathbb{R}^n$ for the set of convex bodies containing the origin in their interiors in $\mathbb{R}^n$ by $\mathcal{K}^n_o$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, denote by $V(K)$ the n-dimensional volume of body K for the standard unit ball B in $\mathbb{R}^n$, define $\omega_n = V(B)$.

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K,\cdot)$, is defined by (see [1, 2])

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\} \tag{1.1}$$

for all $u \in S^{n-1}$. If ρ_K is positive and continuous, K will be called a star body (about the origin). Let $\mathcal{S}^n_o$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

The notion of radial pth mean body was given by Gardner and Zhang (see [3]). For $K \in \mathcal{K}^n$, the radial pth mean body R_pK of K is defined for nonzero $p > -1$ by

$$\rho_{R_pK}(u) = \left(\frac{1}{V(K)}\int_K \rho_K^p(x,u)\,dx\right)^{\frac{1}{p}} \tag{1.2}$$

for each $u \in S^{n-1}$; define R_0K by

$$\rho_{R_0K}(u) = \exp\left(\frac{1}{V(K)}\int_K \log\rho_K(x,u)\,dx\right)$$

for each $u \in S^{n-1}$.

In [3], Gardner and Zhang showed the following.

Theorem 1.A *If $K \in \mathcal{K}^n$, $-1 < p < q$, then*

$$DK \subseteq c_{n,q}R_qK \subseteq c_{n,p}R_pK \subseteq nV(K)\Pi^*K, \tag{1.3}$$

in each inclusion equality holds if and only if K is a simplex. Here

$$c_{n,p} = \big(nB(p+1,n)\big)^{-\frac{1}{p}} \tag{1.4}$$

for nonzero $p > -1$, $c_{n,0} = \lim_{p\to 0}(nB(p+1,n))^{-\frac{1}{p}}$, and DK and Π^*K denote the difference body and the polar of projection body, respectively.

From Theorem 1.A, Gardner and Zhang [3] again proved the Zhang projection inequality (also see [4]) and the Rogers-Shephard inequality (also see [5]).

Theorem 1.B (Zhang projection inequality) *If $K \in \mathcal{K}^n$, then*

$$V\big(\Pi^*K\big)V(K)^{n-1} \geq \frac{1}{n^n}\binom{2n}{n}, \tag{1.5}$$

with equality if and only if K is a simplex.

Theorem 1.C (Rogers-Shephard inequality) *If $K \in \mathcal{K}^n$, then*

$$V(DK) \leq \binom{2n}{n}V(K), \tag{1.6}$$

with equality if and only if K is a simplex.

In this paper, we continuously research the radial pth mean body. First, we establish inequalities for dual quermassintegrals of the radial pth mean body R_pK as follows.

Theorem 1.1 *If $K \in \mathcal{K}^n$, $p > 0$, real $i \neq n$, then there exists $x_0 \in K$ such that for $i < n-p$ or $i > n$,*

$$\widetilde{W}_i(R_pK) \leq \widetilde{W}_i(K - x_0); \tag{1.7}$$

for $n-p < i < n$,

$$\widetilde{W}_i(R_pK) \geq \widetilde{W}_i(K - x_0). \tag{1.8}$$

In every inequality, equality holds if and only if $R_pK = K - x_0$. For $i = n-p$, (1.7) (or (1.8)) is identic. Here, $\widetilde{W}_i(K)$ denotes the dual quermassintegrals of K which are given by (see [6])

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}\,dS(u). \tag{1.9}$$

Obviously, let $i = 0$ *in* (1.9), *then*

$$\widetilde{W}_0(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n \, dS(u) = V(K). \tag{1.10}$$

Let $i = 0$ in Theorem 1.1 and notice that $V(K - x_0) = V(K)$, we easily get the following.

Corollary 1.1 *If* $K \in \mathcal{K}^n$, $p > 0$, *then for* $p < n$,

$$V(R_p K) \leq V(K);$$

for $p > n$,

$$V(R_p K) \geq V(K).$$

All with equality if and only if $R_p K = K$. *For* $p = n$, *above inequalities are identic.*

Note that Corollary 1.1 can be found in [7].

As an application of Theorem 1.1, we obtain the following dual quermassintegrals form of the Zhang projection inequality.

Theorem 1.2 *If* $K \in \mathcal{K}^n$, $p > 0$, *real* $i \neq n$, *then there exists* $x_0 \in K$ *such that for* $n - p \leq i < n$,

$$\widetilde{W}_i(\Pi^* K) \geq \left[\frac{c_{n,p}}{nV(K)}\right]^{n-i} \widetilde{W}_i(K - x_0), \tag{1.11}$$

with equality for $i = n - p$ *if and only if* K *is a simplex, for* $n - p < i < n$ *if and only if* K *is a simplex and* $R_p K = K - x_0$.

Note that the case of $p = n - i$ in (1.11) can be found in [8].

If p is a positive integer in Theorem 1.2, then by (1.4) we get that

$$(c_{n,p})^{n-i} = \binom{n+p}{n}^{\frac{n-i}{p}}. \tag{1.12}$$

Hence, we have the following.

Corollary 1.2 *If* $K \in \mathcal{K}^n$, p *is a positive integer,* i *is any real, if* $n - p \leq i < n$, *then there exists* $x_0 \in K$ *such that*

$$\widetilde{W}_i(\Pi^* K) \geq \left[\frac{1}{nV(K)}\right]^{n-i} \binom{n+p}{n}^{\frac{n-i}{p}} \widetilde{W}_i(K - x_0),$$

with equality for $i = n - p$ *if and only if* K *is a simplex, for* $n - p < i < n$ *if and only if* K *is a simplex and* $R_p K = K - x_0$.

Let $i = 0$ in Corollary 1.2, and together with (1.12) and (1.10), we have the following.

Corollary 1.3 *If* $K \in \mathcal{K}^n$, $p \geq n$ *and* p *is an integer, then*

$$V(\Pi^* K) V(K)^{n-1} \geq \frac{1}{n^n}\binom{n+p}{n}^{\frac{n}{p}}, \tag{1.13}$$

with equality for $p = n$ if and only if K is a simplex, for $p > n$ if and only if K is a simplex and there exists $x_0 \in K$ such that $R_pK = K - x_0$.

Compared to (1.13) and the Zhang projection inequality (1.5), inequality (1.13) may be regarded as a general Zhang projection inequality.

As another application of Theorem 1.1, we obtain the following dual quermassintegrals form of the Rogers-Shephard inequality.

Theorem 1.3 *If $K \in \mathcal{K}^n$, $p > 0$ and real $i \neq n$, if $i \leq n - p$ or $i > n$, then there exists $x_0 \in K$ such that*

$$\widetilde{W}_i(DK) \leq (c_{n,p})^{n-i}\widetilde{W}_i(K - x_0), \tag{1.14}$$

with equality for $i = n - p$ if and only if K is a simplex, for $i < n - p$ or $i > n$ if and only if K is a simplex and $R_pK = K - x_0$.

Similarly, if p is a positive integer in Theorem 1.3, then by (1.12) we obtain the following.

Corollary 1.4 *If $K \in \mathcal{K}^n$, p is a positive integer, i is any real, if $i \leq n - p$ or $i > n$, then there exists $x_0 \in K$ such that*

$$\widetilde{W}_i(DK) \leq \binom{n+p}{n}^{\frac{n-i}{p}} \widetilde{W}_i(K - x_0),$$

with equality for $i = n - p$ if and only if K is a simplex, for $i < n - p$ or $i > n$ if and only if K is a simplex and $R_pK = K - x_0$.

Taking $i = 0$ in Corollary 1.4, and using (1.12) and (1.10), we get the following.

Corollary 1.5 *If $K \in \mathcal{K}^n$, p is a positive integer and $p \leq n$, then*

$$V(DK) \leq \binom{n+p}{n}^{\frac{n}{p}} V(K), \tag{1.15}$$

with equality for $p = n$ if and only if K is a simplex, for $p < n$ if and only if K is a simplex and there exists $x_0 \in K$ such that $R_pK = K - x_0$.

Compared to (1.15) and the Rogers-Shephard inequality (1.6), inequality (1.15) may be regarded as a general Rogers-Shephard inequality.

In addition, we also give the Shephard-type problem for the radial pth mean bodies in Section 4.

2 Preliminaries

2.1 Support function, difference body and projection body

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot)$, is defined by (see [1, 2])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

If $K \in \mathcal{K}_o^n$, the polar body of K, K^*, is defined by (see [1, 2])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}.$$

If $K \in \mathcal{K}^n$, the difference body, $DK = K + (-K)$, of K is defined by (see [1])

$$h(DK,u) = h(K,u) + h(K,-u)$$

for all $u \in S^{n-1}$.

For $K \in \mathcal{K}^n$, the projection body of K, ΠK, is a centered convex body whose support function is given by (see [1])

$$h(\Pi K,u) = V_{n-1}(K|u^\perp)$$

for all $u \in S^{n-1}$, where V_{n-1} denotes $(n-1)$-dimensional volume, and $K|u^\perp$ denotes the image of orthogonal projection of K onto the codimensional 1 subspace orthogonal to u.

2.2 Extended radial function

If K is compact star-shaped with respect to $x \in \mathbb{R}^n$, its radial function $\rho_K(x,\cdot)$ with respect to x is defined, for all $u \in S^{n-1}$ such that the line through x parallel to u intersects K, by (see [3])

$$\rho_K(x,u) = \max\{\lambda \geq 0 : x + \lambda u \in K\}. \tag{2.1}$$

From (1.1) and (2.1), we easily know that

$$\rho_K(x,u) = \rho_{K-x}(u) \tag{2.2}$$

for all $u \in S^{n-1}$. We call $\rho_K(x,\cdot)$ the extended radial function of K with respect to x. If x is the origin o, then $\rho_K(x,u) = \rho_K(u)$ for all $u \in S^{n-1}$.

From (2.2) and (1.2), obviously,

$$\rho_{R_pK}(u) = \left(\frac{1}{V(K)}\int_K \rho_{K-x}^p(u)\,dx\right)^{\frac{1}{p}}. \tag{2.3}$$

2.3 L_p-Dual mixed quermassintegrals

If $K, L \in \mathcal{S}_o^n$, $p > 0$, $\lambda, \mu \geq 0$ (not both zero), the L_p-radial combination, $\lambda \cdot K \tilde{+}_p \mu \cdot L \in \mathcal{S}_o^n$, of K and L is defined by (see [9, 10])

$$\rho(\lambda \cdot K \tilde{+}_p \mu \cdot L, \cdot)^p = \lambda\rho(K,\cdot)^p + \mu\rho(L,\cdot)^p. \tag{2.4}$$

Associated with (2.4) and (1.9), we define a class of L_p-dual mixed quermassintegrals as follows: For $K, L \in \mathcal{S}_o^n$, $p > 0$, $\varepsilon > 0$ and real $i \neq n$, the L_p-dual mixed quermassintegrals, $\widetilde{W}_{p,i}(K,L)$, of K and L are defined by

$$\frac{n-i}{p}\widetilde{W}_{p,i}(K,L) = \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K \tilde{+}_p \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon}. \tag{2.5}$$

Let $i = 0$ in definition (2.5), and together with (1.10), we write that $\widetilde{W}_{p,0}(K,L) = \widetilde{V}_p(K,L)$, then

$$\frac{n}{p}\widetilde{V}_p(K,L) = \lim_{\varepsilon\to 0^+}\frac{V(K \tilde{+}_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Here $\widetilde{V}_p(K,L)$ denotes a type of L_p-dual mixed volume of K and L which is defined in [9, 11] (for $p \geq 1$ also see [12]).

From definition (2.5), the integral representation of L_p-dual mixed quermassintegrals can be established as follows.

Theorem 2.1 *If $K, L \in \mathcal{S}_o^n$, $p > 0$, and real $i \neq n$, then*

$$\widetilde{W}_{p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-p-i}(u)\rho_L^p(u)\,dS(u). \tag{2.6}$$

Proof From (2.4) and (2.5), for $i \neq n$, we have that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K \hat{+}_p \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon}\\
&\quad= \lim_{\varepsilon\to 0^+}\frac{1}{n}\int_{S^{n-1}}\frac{\rho(K \hat{+}_p \varepsilon \cdot L, u)^{n-i} - \rho(K,u)^{n-i}}{\varepsilon}\,dS(u)\\
&\quad= \lim_{\varepsilon\to 0^+}\frac{1}{n}\int_{S^{n-1}}\frac{[\rho(K,u)^p + \varepsilon\rho(L,u)^p]^{\frac{n-i}{p}} - \rho(K,u)^{n-i}}{\varepsilon}\,dS(u).
\end{aligned}$$

By Hospital's rule we see that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\frac{[\rho(K,\cdot)^p + \varepsilon\rho(L,\cdot)^p]^{\frac{n-i}{p}} - \rho(K,\cdot)^{n-i}}{\varepsilon}\\
&\quad= \lim_{\varepsilon\to 0^+}\rho_K^{n-i}\frac{[1 + \varepsilon(\rho_K/\rho_L)^p]^{\frac{n-i}{p}} - 1}{\varepsilon}\\
&\quad= \frac{n-i}{p}\rho_K^{n-p-i}\rho_L^p,
\end{aligned}$$

thus we get formula (2.6) by definition (2.5). □

From (2.6), we easily know that

$$\widetilde{W}_{p,i}(K,K) = \frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i}\,dS(u) = \widetilde{W}_i(K), \tag{2.7}$$

$$\widetilde{W}_{p,n-p}(K,L) = \widetilde{W}_{n-p}(L). \tag{2.8}$$

The Minkowski inequality for the L_p-dual mixed quermassintegrals is given as follows.

Theorem 2.2 *Let $K, L \in \mathcal{S}_o^n$, $p > 0$, and real $i \neq n$, then for $i < n - p$,*

$$\widetilde{W}_{p,i}(K,L) \leq \widetilde{W}_i(K)^{(n-p-i)/(n-i)}\widetilde{W}_i(L)^{p/(n-i)}; \tag{2.9}$$

for $n-p<i<n$ or $i>n$,

$$\widetilde{W}_{p,i}(K,L) \geq \widetilde{W}_i(K)^{(n-p-i)/(n-i)}\widetilde{W}_i(L)^{p/(n-i)}. \tag{2.10}$$

In every inequality, equality holds if and only if K and L are dilates. For $i=n-p$, (2.9) (*or* (2.10)) *is identic.*

Proof For $i<n-p$, from (2.6) and together with the Hölder inequality (see [13]), we have that

$$\begin{aligned}
\widetilde{W}_{p,i}(K,L) &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-p-i}(u)\rho_L^p(u)\,dS(u) \\
&\leq \left[\frac{1}{n}\int_{S^{n-1}} \left[\rho_K^{n-p-i}(u)\right]^{\frac{n-i}{n-p-i}}\,dS(u)\right]^{\frac{n-p-i}{n-i}} \left[\frac{1}{n}\int_{S^{n-1}} \left[\rho_L^p(u)\right]^{\frac{n-i}{p}}\,dS(u)\right]^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\,dS(u)\right]^{\frac{n-p-i}{n-i}} \left[\frac{1}{n}\int_{S^{n-1}} \rho_L^{n-i}(u)\,dS(u)\right]^{\frac{p}{n-i}} \\
&= \widetilde{W}_i(K)^{\frac{n-p-i}{n-i}}\widetilde{W}_i(L)^{\frac{p}{n-i}},
\end{aligned}$$

which gives inequality (2.9) when $i<n-p$. According to the condition that equality holds for the Hölder inequality, we know that the equality holds in inequality (2.9) if and only if K and L are dilates.

Similarly, we can prove for $n-p<i<n$ or $i>n$, inequality (2.10) is true.

For $i=n-p$, by (2.8) and (2.3) then

$$\widetilde{W}_{p,i}(K,L)^{n-i} = \widetilde{W}_{p,n-p}(K,L)^p = \widetilde{W}_{n-p}(L)^p$$

and

$$\widetilde{W}_i(K)^{n-p-i}\widetilde{W}_i(L)^p = \widetilde{W}_{n-p}(K)^{n-p-i}\widetilde{W}_{n-p}(L)^p = \widetilde{W}_{n-p}(L)^p,$$

thus (2.9) (or (2.10)) is identic when $i=n-p$. □

3 Proofs of the theorems

The proofs of the theorems require the following lemma.

Lemma 3.1 *If $K\in\mathcal{K}^n$, $p>0$, and real $i\neq n$, then for any $Q\in\mathcal{S}_o^n$,*

$$\widetilde{W}_{p,i}(Q,R_pK) = \frac{1}{V(K)}\int_K \widetilde{W}_{p,i}(Q,K-x)\,dx. \tag{3.1}$$

Proof Using (2.6) and (2.3), then for any $Q\in\mathcal{S}_o^n$, we have that

$$\begin{aligned}
\widetilde{W}_{p,i}(Q,R_pK) &= \frac{1}{n}\int_{S^{n-1}} \rho(Q,u)^{n-p-i}\rho(R_pK,u)^p\,dS(u) \\
&= \frac{1}{nV(K)}\int_{S^{n-1}}\int_K \rho(Q,u)^{n-p-i}\rho(K-x,u)^p\,dx\,dS(u) \\
&= \frac{1}{V(K)}\int_K \widetilde{W}_{p,i}(Q,K-x)\,dx.
\end{aligned}$$

□

Proof of Theorem 1.1 For $i < n - p$, let $Q = R_pK$ in (3.1), this together with (2.6), (2.7) and (2.9) gives

$$\begin{aligned}\widetilde{W}_i(R_pK) &= \frac{1}{V(K)}\int_K \widetilde{W}_{p,i}(R_pK, K - x)\,dx \\ &\le \frac{1}{V(K)}\int_K \widetilde{W}_i(R_pK)^{\frac{n-p-i}{n-i}}\,\widetilde{W}_i(K-x)^{\frac{p}{n-i}}\,dx \\ &= \frac{1}{V(K)}\widetilde{W}_i(R_pK)^{\frac{n-p-i}{n-i}}\int_K \widetilde{W}_i(K-x)^{\frac{p}{n-i}}\,dx,\end{aligned}$$

i.e.,

$$\widetilde{W}_i(R_pK)^{\frac{p}{n-i}} \le \frac{1}{V(K)}\int_K \widetilde{W}_i(K-x)^{\frac{p}{n-i}}\,dx.$$

Therefore, according to the integral mean value theorem, there exists $x_0 \in K$ such that

$$\widetilde{W}_i(R_pK)^{\frac{p}{n-i}} \le \frac{1}{V(K)}\widetilde{W}_i(K-x_0)^{\frac{p}{n-i}}\int_K dx = \widetilde{W}_i(K-x_0)^{\frac{p}{n-i}}.$$

Since $p > 0$ and $i < n - p$, thus we get inequality (1.7). According to the condition that equality holds in inequality (2.9), we see that with equality in (1.7) if and only if R_pK and $K - x_0$ are dilates. This combined with (1.7), we know that equality holds in (1.7) if and only if $R_pK = K - x_0$.

Similarly, for $n - p < i < n$ or $i > n$, from inequality (2.10) and equality (3.1), then

$$\widetilde{W}_i(R_pK)^{\frac{p}{n-i}} \ge \widetilde{W}_i(K-x_0)^{\frac{p}{n-i}}.$$

Hence, we have that for $i > n$ and $p > 0$,

$$\widetilde{W}_i(R_pK) \le \widetilde{W}_i(K-x_0);$$

for $n - p < i < n$ and $p > 0$,

$$\widetilde{W}_i(R_pK) \ge \widetilde{W}_i(K-x_0).$$

From this, we get inequality (1.7) and inequality (1.8), respectively, and equality holds in the above inequalities if and only if $R_pK = K - x_0$.

For $i = n - p$, by (2.8) and (3.1) we see that (1.7) (or (1.8)) is identic. □

Proof of Theorem 1.2 From (1.3), we have that $c_{n,p}R_pK \subseteq nV(K)\Pi^*K$ for $p > -1$, then

$$(c_{n,p})^{n-i}\widetilde{W}_i(R_pK) \le \bigl(nV(K)\bigr)^{n-i}\widetilde{W}_i\bigl(\Pi^*K\bigr), \tag{3.2}$$

with equality if and only if K is a simplex. Hence, together with (1.8), then for $n - p < i < n$ and $p > 0$, we obtain that

$$\widetilde{W}_i\bigl(\Pi^*K\bigr) \ge \left[\frac{c_{n,p}}{nV(K)}\right]^{n-i}\widetilde{W}_i(K-x_0),$$

which is desired (1.11).

Associated with the cases of equality holding in (3.2) and (1.8), we see that equality holds in (1.11) for $i = n - p$ if and only if K is a simplex, for $n - p < i < n$ if and only if K is a simplex and $R_pK = K - x_0$. □

Proof of Theorem 1.3 From (1.3), we know that $DK \subseteq c_{n,p}R_pK$ for $p > -1$, thus

$$\widetilde{W}_i(DK) \leq (c_{n,p})^{n-i}\widetilde{W}_i(R_pK), \tag{3.3}$$

with equality if and only if K is a simplex. Hence, together with (1.7), then for $p > 0$, $i < n - p$ or $i > n$, we get that

$$\widetilde{W}_i(DK) \leq (c_{n,p})^{n-i}\widetilde{W}_i(K - x_0),$$

this is just (1.14).

Combining with the cases of equality holding in (3.3) and (1.7), we see that equality holds in (1.14) for $i = n - p$ if and only if K is a simplex, for $i < n - p$ or $i > n$ if and only if K is a simplex and $R_pK = K - x_0$. □

4 Shephard-type problem

In this section, we research the Shephard-type problem for the radial pth mean bodies. Recall that Zhou and Wang in [7] gave the Shephard-type problem for the radial pth mean bodies as follows.

Theorem 4.A *Let $K, L \in \mathcal{K}^n$, $p > 0$, if $R_pK \subseteq R_pL$, then*

$$V(K) \leq V(L),$$

with equality if and only if $R_pK = R_pL$ and K is a translation of L.

Here, we obtain a stronger result for the Shephard-type problem of the radial pth mean bodies. Our result is the following theorem.

Theorem 4.1 *Let $K, L \in \mathcal{K}^n$, $p > 0$, if $R_pK \subseteq R_pL$, then there exist $x_0 \in K$ and $y_0 \in L$ such that*

$$K - x_0 \subseteq L - y_0, \tag{4.1}$$

with equality if and only if $R_pK = R_pL$ and $K - x_0 = L - y_0$.

Proof Since $R_pK \subseteq R_pL$ for $p > 0$, thus $\rho^p_{R_pK}(u) \leq \rho^p_{R_pL}(u)$ for all $u \in S^{n-1}$, *i.e.*,

$$\frac{1}{V(K)}\int_K \rho^p_{K-x}(u)\,dx \leq \frac{1}{V(L)}\int_L \rho^p_{L-y}(u)\,dy.$$

Therefore, by the integral mean value theorem, there exist $x_0 \in K$ and $y_0 \in L$ such that

$$\rho^p_{K-x_0}(u)\frac{1}{V(K)}\int_K dx \leq \rho^p_{L-y_0}(u)\frac{1}{V(L)}\int_L dy,$$

thus

$$\rho^p_{K-x_0}(u) \le \rho^p_{L-y_0}(u)$$

for all $u \in S^{n-1}$. This yields (4.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Gardner, RJ: Geometric Tomography, 2nd edn. Cambridge University Press, Cambridge (2006)
2. Schneider, R: Convex Bodies: The Brunn-Minkowski Theory, 2nd edn. Cambridge University Press, Cambridge (2014)
3. Gardner, RJ, Zhang, GY: Affine inequalities and radial mean bodies. Am. J. Math. **120**, 505-528 (1998)
4. Zhang, GY: Restricted chord projection and affine inequalities. Geom. Dedic. **39**, 213-222 (1991)
5. Rogers, CA, Shephard, GC: The difference body of a convex body. Arch. Math. **8**, 220-233 (1957)
6. Lutwak, E: Dual mixed volumes. Pac. J. Math. **58**, 531-538 (1975)
7. Zhou, YP, Wang, WD: Inequalities for the radial p-th mean bodies. Wuhan Univ. J. Nat. Sci. **17**(4), 1-5 (2012)
8. Wang, WD: On extensions of the Zhang's projection inequality. Adv. Appl. Math. Sci. **2**, 199-207 (2010)
9. Haberl, C: L_p Intersection bodies. Adv. Math. **217**(6), 2599-2624 (2008)
10. Haberl, C, Ludwig, M: A characterization of L_p intersection bodies. Int. Math. Res. Not. **2006**, Article ID 10548 (2006)
11. Yuan, J, Cheung, W-S: L_p-Intersection bodies. J. Math. Anal. Appl. **339**(2), 1431-1439 (2008)
12. Yu, WY, Wu, DH, Leng, GS: Quasi L_p-intersection bodies. Acta Math. Sin. **23**(11), 1937-1948 (2007)
13. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Cambridge University Press, Cambridge (1959)

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