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Inequalities and asymptotic expansions associated with the Wallis sequence

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Abstract
We present the asymptotic expansions of functions involving the ratio of gamma functions and provide formulas for determining the coefficients of the asymptotic expansions. As consequences, we obtain the asymptotic expansions of the Wallis sequence. Also, we establish inequalities for the Wallis sequence.
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1 Introduction

The Wallis sequence to which the title refers is

$$W_n = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}, \quad n \in \mathbb{N} := \{1,2,3,\ldots\}. \tag{1.1}$$

Wallis (1616-1703) discovered that

$$\prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\cdots = \frac{\pi}{2} \tag{1.2}$$

(see [1, p.68]). Based on Wallis' infinite product (1.2), the first infinite continued fraction of π was given by Brouncker (1620-1684):

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \ddots}}}}. \tag{1.3}$$

Euler's analysis of Wallis' proof led him to formulas for the gamma and beta functions. Stirling (1692-1770) used (1.2) to determine the constant factor in his asymptotic formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \quad n \to \infty.$$

Several elementary proofs of (1.2) can be found (see, for example, [2–4]). An interesting geometric construction produces (1.2) [5]. Many formulas exist for the representation

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of π, and a collection of these formulas is listed in [6, 7]. For more on the history of π see [1, 8–10].

Some inequalities and asymptotic formulas associated with the Wallis sequence W_n can be found (see, for example, [11–23]). In [13], Hirschhorn proved that for $n \in \mathbb{N}$,

$$\frac{\pi}{2}\left(1-\frac{1}{4n+\frac{7}{3}}\right) < W_n < \frac{\pi}{2}\left(1-\frac{1}{4n+\frac{8}{3}}\right). \tag{1.4}$$

Also in [13], Hirschhorn pointed out that if the c_j are given by

$$\tanh\left(\frac{x}{4}\right) = \sum_{j=0}^{\infty} c_j \frac{x^{2j+1}}{(2j)!}, \tag{1.5}$$

then, as $n \to \infty$,

$$W_n \sim \frac{\pi}{2}\left(1+\frac{1}{2n}\right)^{-1} \prod_{j\geq 0} \exp\left(\frac{c_j}{n^{2j+1}}\right) = \frac{\pi}{2}\left(1+\frac{1}{2n}\right)^{-1} \exp\left(\sum_{j=0}^{\infty} \frac{c_j}{n^{2j+1}}\right). \tag{1.6}$$

Remark 1 It is well known (see [24, p.85]) that

$$\tanh z = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} z^{2k-1}, \quad |z| < \frac{\pi}{2},$$

where B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) are the Bernoulli numbers. We then obtain

$$\tanh\left(\frac{x}{4}\right) = \sum_{j=0}^{\infty} \frac{(2^{2j+2}-1)B_{2j+2}}{2^{2j+1}(2j+1)(j+1)} \frac{x^{2j+1}}{(2j)!}, \quad |x| < 2\pi. \tag{1.7}$$

Thus we have

$$c_j = \frac{(2^{2j+2}-1)B_{2j+2}}{2^{2j+1}(2j+1)(j+1)}, \quad j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{1.8}$$

Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The first aim of this paper is to determine the coefficients $c_j(\ell, r)$ (for $j \in \mathbb{N}$) such that

$$W(x) \sim \frac{\pi}{2}\left(1+\sum_{j=1}^{\infty} \frac{c_j(\ell,r)}{x^j}\right)^{x^{\ell}/r} \quad \text{as } x \to \infty,$$

where the function $W(x)$ is defined by

$$W(x) = \frac{\pi}{2}\left(1+\frac{1}{2x}\right)^{-1} \frac{1}{x}\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2. \tag{1.9}$$

Clearly, $W_n = W(n)$. The second aim of this paper is to establish inequalities for the Wallis sequence W_n.

2 A useful lemma

The classical Euler's gamma function is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is called psi (or digamma) function, and $\psi^{(k)}(x)$ $(k \in \mathbb{N})$ are called polygamma functions.

The following lemma is required in our present investigation.

Lemma 1 ([25, Corollary 2.1]) *Let $m, n \in \mathbb{N}$. Then for $x > 0$,*

$$\begin{aligned}
&\sum_{j=1}^{2m} \left(1 - \frac{1}{2^{2j}}\right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}} \\
&\quad < (-1)^n \left(\psi^{(n-1)}(x+1) - \psi^{(n-1)}\left(x + \frac{1}{2}\right)\right) + \frac{(n-1)!}{2x^n} \\
&\quad < \sum_{j=1}^{2m-1} \left(1 - \frac{1}{2^{2j}}\right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}},
\end{aligned} \tag{2.1}$$

where B_n are the Bernoulli numbers.

It follows from (2.1) that, for $x > 0$,

$$L(x) < \psi(x+1) - \psi\left(x + \frac{1}{2}\right) < U(x), \tag{2.2}$$

where

$$\begin{aligned}
L(x) &= \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2{,}048x^8} \\
&\quad - \frac{31}{2{,}048x^{10}} + \frac{691}{16{,}384x^{12}} - \frac{5{,}461}{32{,}768x^{14}}
\end{aligned}$$

and

$$\begin{aligned}
U(x) &= \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2{,}048x^8} \\
&\quad - \frac{31}{2{,}048x^{10}} + \frac{691}{16{,}384x^{12}} - \frac{5{,}461}{32{,}768x^{14}} + \frac{929{,}569}{1{,}048{,}576x^{16}}.
\end{aligned}$$

In Section 4, the proofs of Theorems 3 and 4 make use of inequality (2.2).

3 Asymptotic expansions

The logarithm of gamma function has asymptotic expansion (see [26, p.32]):

$$\ln\Gamma(x+t) \sim \left(x + t - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{n=1}^\infty \frac{(-1)^{n+1} B_{n+1}(t)}{n(n+1)} \frac{1}{x^n} \tag{3.1}$$

as $x \to \infty$, where $B_n(t)$ denotes the Bernoulli polynomials defined by the following generating function:

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}. \tag{3.2}$$

Note that the Bernoulli numbers B_n (for $n \in \mathbb{N}_0$) are defined by (3.2) for $t = 0$.

From (3.1), we obtain, as $x \to \infty$,

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim x \exp\left(\frac{1}{t-s} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}(B_{j+1}(t) - B_{j+1}(s))}{j(j+1)} \frac{1}{x^j}\right). \tag{3.3}$$

Setting $(s,t) = (\frac{1}{2}, 1)$ and noting that

$$B_n(0) = (-1)^n B_n(1) = B_n \quad \text{and} \quad B_n\left(\frac{1}{2}\right) = \left(2^{1-n} - 1\right) B_n \quad \text{for } n \in \mathbb{N}_0$$

(see [24, p.805]), we obtain from (3.3), as $x \to \infty$,

$$\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2 \sim x \exp\left(\sum_{j=1}^{\infty} \frac{2(1-(-1)^{j+1}(2^{-j}-1))B_{j+1}}{j(j+1)} \frac{1}{x^j}\right), \tag{3.4}$$

or

$$\frac{1}{x}\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2 \sim \exp\left(\sum_{j=0}^{\infty} \frac{(2-2^{-2j-1})B_{2j+2}}{(2j+1)(j+1)} \frac{1}{x^{2j+1}}\right). \tag{3.5}$$

We see from (1.6) and (1.9) that

$$\frac{1}{n}\left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 \sim \exp\left(\sum_{j=0}^{\infty} \frac{c_j}{n^{2j+1}}\right), \quad n \to \infty, \tag{3.6}$$

with the coefficients c_j given by (1.5). From (3.5) and (3.6), we retrieve (1.8).

By using the Maclaurin expansion of $\ln(1+x)$,

$$\ln(1+x) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} x^j \quad \text{for } -1 < x \le 1,$$

we obtain

$$\left(1 + \frac{1}{2x}\right)^{-1} \sim \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^j}{j2^j} \frac{1}{x^j}\right) \quad \text{as } x \to \infty. \tag{3.7}$$

Applying (3.4) and (3.7) yields

$$W(x) \sim \frac{\pi}{2} \exp\left(\sum_{j=1}^{\infty} \frac{b_j}{x^j}\right) \quad \text{as } x \to \infty, \tag{3.8}$$

with the coefficients b_j (for $j \in \mathbb{N}$) given by

$$b_j = (-1)^{j-1}\left(-\frac{1}{j2^j} + \frac{2((-1)^{j+1} - (2^{-j} - 1))B_{j+1}}{j(j+1)}\right). \tag{3.9}$$

From (3.8), we obtain the following asymptotic expansion for the Wallis sequence W_n:

$$\begin{aligned} W_n \sim \frac{\pi}{2}\exp\biggl(&-\frac{1}{4n} + \frac{1}{8n^2} - \frac{5}{96n^3} + \frac{1}{64n^4} - \frac{1}{320n^5} + \frac{1}{384n^6} - \frac{25}{7{,}168n^7} \\ &+ \frac{1}{2{,}048n^8} + \frac{29}{9{,}216n^9} + \frac{1}{10{,}240n^{10}} - \frac{695}{90{,}112n^{11}} + \cdots\biggr), \quad n \to \infty. \end{aligned} \tag{3.10}$$

Using $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$, from (3.10) we deduce that

$$\begin{aligned} W_n \sim \frac{\pi}{2}\biggl(&1 - \frac{1}{4n} + \frac{5}{32n^2} - \frac{11}{128n^3} + \frac{83}{2{,}048n^4} - \frac{143}{8{,}192n^5} + \frac{625}{65{,}536n^6} - \frac{1{,}843}{262{,}144n^7} \\ &+ \frac{24{,}323}{8{,}388{,}608n^8} + \frac{61{,}477}{33{,}554{,}432n^9} - \frac{14{,}165}{268{,}435{,}456n^{10}} \\ &- \frac{8{,}084{,}893}{1{,}073{,}741{,}824n^{11}} + \cdots\biggr). \end{aligned} \tag{3.11}$$

Even though as many coefficients as we please on the right-hand side of (3.11) can be obtained by using Mathematica, here we aim at giving a formula for determining these coefficients. In fact, Theorem 1 below presents a general asymptotic expansion for $W(x)$ which includes (3.11) as its special case.

Theorem 1 *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. Then the function $W(x)$, as defined in* (1.9), *has the following asymptotic expansion*:

$$W(x) \sim \frac{\pi}{2}\left(1 + \sum_{j=1}^{\infty} \frac{c_j(\ell, r)}{x^j}\right)^{x^\ell / r} \quad \text{as } x \to \infty \tag{3.12}$$

with the coefficients $c_j(\ell, r)$ (for $j \in \mathbb{N}$) given by

$$c_j(\ell, r) = \sum \frac{r^{k_1 + k_2 + \cdots + k_j}}{k_1! k_2! \cdots k_j!} b_1^{k_1} b_2^{k_2} \cdots b_j^{k_j}, \tag{3.13}$$

where b_j are given in (3.9), *summed over all non-negative integers k_j satisfying the equation*

$$(1 + \ell)k_1 + (2 + \ell)k_2 + \cdots + (j + \ell)k_j = j.$$

Proof In view of (1.6), we can let

$$\left(\frac{2}{\pi} W(x)\right)^{r/x^\ell} = 1 + \sum_{j=1}^{m} \frac{c_j(\ell, r)}{x^j} + O\bigl(x^{-m-1}\bigr) \quad \text{as } x \to \infty, \tag{3.14}$$

where $c_1(\ell,r),\dots,c_m(\ell,r)$ are real numbers to be determined. Write (3.8) as

$$\ln\left(\frac{2}{\pi}W(x)\right)=\sum_{k=1}^{m}\frac{b_k}{x^k}+R_m(x),$$

where $R_m(x)=O(x^{-m-1})$. Further, we have

$$\begin{aligned}\left(\frac{2}{\pi}W(x)\right)^{r/x^\ell}&=e^{rR_m(x)/x^\ell}e^{\sum_{k=1}^{m}\frac{rb_k}{x^{k+\ell}}}\\&=e^{rR_m(x)/x^\ell}\prod_{k=1}^{m}\left[1+\left(\frac{rb_k}{x^{k+\ell}}\right)+\frac{1}{2!}\left(\frac{rb_k}{x^{k+\ell}}\right)^2+\cdots\right]\\&=e^{rR_m(x)/x^\ell}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\cdots k_m!}\\&\quad\times\left(\frac{rb_1}{x^{1+\ell}}\right)^{k_1}\left(\frac{rb_2}{x^{2+\ell}}\right)^{k_2}\cdots\left(\frac{rb_m}{x^{m+\ell}}\right)^{k_m}\\&=e^{rR_m(x)/x^\ell}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{r^{k_1+k_2+\cdots+k_m}}{k_1!k_2!\cdots k_m!}b_1^{k_1}b_2^{k_2}\cdots b_m^{k_m}\\&\quad\times\frac{1}{x^{(1+\ell)k_1+(2+\ell)k_2+\cdots+(m+\ell)k_m}}.\end{aligned}\tag{3.15}$$

Equating the coefficients by the equal powers of x in (3.14) and (3.15), we see that

$$c_j(\ell,r)=\sum_{(1+\ell)k_1+(2+\ell)k_2+\cdots+(j+\ell)k_j=j}\frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}b_1^{k_1}b_2^{k_2}\cdots b_j^{k_j}.$$

The proof of Theorem 1 is complete. □

Theorem 1 gives an explicit formula for determining the coefficients of the asymptotic expansion (3.12). Theorem 2 below provides a recurrence relation for determining the coefficients of the asymptotic expansion (3.12).

Theorem 2 *Let $r\neq 0$ be a given real number and $\ell\geq 0$ be a given integer. Then the function $W(x)$, as defined in* (1.9), *has the following asymptotic expansion*:

$$W(x)\sim\frac{\pi}{2}\left(\sum_{j=0}^{\infty}\frac{c_j(\ell,r)}{x^j}\right)^{x^\ell/r}\quad as\ x\to\infty\tag{3.16}$$

with the coefficients $c_j(\ell,r)$ (for $j\in\mathbb{N}_0$) given by the recurrence relation:

$$c_0(\ell,r)=1\quad and\quad c_j(\ell,r)=\frac{r}{j}\sum_{k=1}^{j-\ell}b_k(k+\ell)c_{j-\ell-k}(\ell,r)\quad for\ j\in\mathbb{N},\tag{3.17}$$

where b_j (for $j\in\mathbb{N}$) are given in (3.9).

Proof Taking the logarithm of (3.8)

$$\ln\left(\frac{2}{\pi}W(x)\right) \sim \sum_{j=1}^{\infty} b_j x^{-j} \quad \text{as } x \to \infty.$$

Write (3.14) as

$$\frac{r}{x^{\ell}} \ln\left(\frac{2}{\pi}W(x)\right) \sim \ln\left(\sum_{j=0}^{\infty} c_j(\ell, r) x^{-j}\right) \quad \text{as } x \to \infty.$$

It follows that

$$r\sum_{k=1}^{\infty} b_k x^{-k-\ell} \sim \ln\left(\sum_{j=0}^{\infty} c_j(\ell, r) x^{-j}\right) \quad \text{as } x \to \infty.$$

Differentiating each side with respect to x yields

$$r\left(\sum_{j=0}^{\infty} c_j(\ell, r) x^{-j}\right)\left(\sum_{k=1}^{\infty} b_k(k+\ell) x^{-k-\ell-1}\right) \sim \sum_{j=1}^{\infty} c_j(\ell, r) j x^{-j-1}.$$

Hence,

$$jc_j(\ell, r) = r\sum_{k=1}^{j-\ell} b_k(k+\ell) c_{j-\ell-k}(\ell, r) \quad \text{for } j \in \mathbb{N}$$

and (3.17) follows. The proof of Theorem 2 is complete. □

4 Inequalities

In this section, we establish inequalities for the Wallis sequence W_n.

Theorem 3 *For all* $n \in \mathbb{N}$,

$$\frac{\pi}{2}\alpha(n) < W_n < \frac{\pi}{2}\beta(n), \tag{4.1}$$

where

$$\alpha(n) = 1 - \cfrac{\frac{1}{4}}{n + \frac{5}{8} + \cfrac{\frac{3}{64}}{n+\frac{1}{2} + \cfrac{\frac{15}{64}}{n+\frac{1}{2} + \cfrac{\frac{35}{64}}{n+\frac{1}{2} + \cfrac{\frac{63}{64}}{n+\frac{1}{2}}}}}}$$

and

$$\beta(n) = 1 - \cfrac{\frac{1}{4}}{n + \frac{5}{8} + \cfrac{\frac{3}{64}}{n+\frac{1}{2} + \cfrac{\frac{15}{64}}{n+\frac{1}{2} + \cfrac{\frac{35}{64}}{n+\frac{1}{2} + \cfrac{\frac{63}{64}}{n+\frac{1}{2} + \cfrac{\frac{99}{64}}{n+\frac{1}{2}}}}}}}.$$

That is,

$$\alpha(n) = \frac{32{,}768n^5 + 77{,}824n^4 + 133{,}120n^3 + 116{,}672n^2 + 55{,}416n + 10{,}395}{32{,}768n^5 + 86{,}016n^4 + 149{,}504n^3 + 143{,}424n^2 + 73{,}976n + 16{,}413}$$

and

$$\beta(n) = \frac{262{,}144n^6 + 753{,}664n^5 + 1{,}781{,}760n^4 + 2{,}226{,}176n^3 + 1{,}778{,}048n^2 + 765{,}768n + 135{,}135}{262{,}144n^6 + 819{,}200n^5 + 1{,}945{,}600n^4 + 2{,}607{,}104n^3 + 2{,}185{,}600n^2 + 1{,}043{,}384n + 211{,}479}.$$

Proof In view of the fact that

$$W_n = \frac{\pi}{2} \cdot \frac{1}{n+\frac{1}{2}} \left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2,$$

the inequality (4.1) is equivalent to

$$\frac{1}{2}\ln\alpha(n) < \ln\Gamma(n+1) - \ln\Gamma\left(n+\frac{1}{2}\right) - \frac{1}{2}\ln\left(n+\frac{1}{2}\right) < \frac{1}{2}\ln\beta(n).$$

To obtain the left-hand inequality, define $f(x)$ for $x \geq 1$ by

$$f(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln\alpha(x).$$

Using Stirling's formula, we find that

$$\lim_{x\to\infty} f(x) = 0.$$

We now show that $f(x)$ is strictly decreasing for $x \geq 3$, and $f(1) > f(2) > f(3)$, so $f(n) > 0$ for $n \geq 1$. By using the second inequality in (2.2), we have

$$\begin{aligned} f'(x) &= \psi(x+1) - \psi\left(x+\frac{1}{2}\right) - \frac{1}{2x+1} - \frac{1}{2}\frac{\alpha'(x)}{\alpha(x)} \\ &< U(x) - \frac{1}{2x+1} - \frac{1}{2}\frac{\alpha'(x)}{\alpha(x)} = U(x) - \frac{1}{2x+1} - \frac{1}{2}\frac{P_8(x)}{P_{10}(x)}, \end{aligned}$$

where $P_k(x)$ is a polynomial of degree k with non-negative integer coefficients. In what follows, $P_k(x)$ has the same understanding.

On simplification, using MAPLE, we find that

$$f'(x) < -\frac{N(x)}{P_{27}(x)},$$

where $N(x)$ is a polynomial of degree 15 with integer coefficients (some positive, some negative). It can be shown further that

$$N(x) = (x-3)P_{14}(x) + 1{,}752{,}962{,}197{,}350{,}057{,}763{,}671,$$

so

$$N(x) > 0$$

for $x \geq 3$ and so

$$f'(x) < 0$$

for $x \geq 3$. Direct computation yields

$$f(1) = 1.83 \times 10^{-6}, \qquad f(2) = 1.92 \times 10^{-8}, \qquad f(3) = 7.06 \times 10^{-10}.$$

Consequently, the sequence $(f(n))_{n\in\mathbb{N}}$ is strictly decreasing. This leads to

$$f(n) > \lim_{n\to\infty} f(n) = 0, \quad n \in \mathbb{N},$$

which means that the first inequality in (4.1) is valid for $n \in \mathbb{N}$.

To obtain the right-hand inequality, define $g(x)$ for $x \geq 1$ by

$$g(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln\beta(x).$$

Using Stirling's formula, we find that

$$\lim_{x\to\infty} g(x) = 0.$$

Differentiating $g(x)$ and applying the first inequality in (2.2), we obtain

$$\begin{aligned} g'(x) &= \psi(x+1) - \psi\left(x+\frac{1}{2}\right) - \frac{1}{2x+1} - \frac{1}{2}\frac{\beta'(x)}{\beta(x)} \\ &> L(x) - \frac{1}{2x+1} - \frac{1}{2}\frac{\beta'(x)}{\beta(x)} = L(x) - \frac{1}{2x+1} - \frac{1}{2}\frac{P_{10}(x)}{P_{12}(x)}. \end{aligned}$$

On simplification, using MAPLE, we find that

$$g'(x) > \frac{1}{32{,}768}\frac{M(x)}{P_{26}(x)},$$

where $M(x)$ is a polynomial of degree 13 with integer coefficients (some positive, some negative). It can be shown further that

$$M(x) = (x-9)P_{12}(x) + 83{,}067{,}781{,}256{,}008{,}661{,}351{,}549{,}403,$$

so

$$M(x) > 0$$

for $x \geq 9$ and so

$$g'(x) > 0$$

for $x \geq 9$. Direct computation yields

$$g(1) = -7.03 \times 10^{-7}, \qquad g(2) = -4.217 \times 10^{-9}, \qquad g(3) = -9.61 \times 10^{-11},$$
$$g(4) = -4.88 \times 10^{-12}, \qquad g(5) = -4.22 \times 10^{-13}, \qquad g(6) = -5.3 \times 10^{-14},$$
$$g(7) = -8.78 \times 10^{-15}, \qquad g(8) = -1.8 \times 10^{-15}, \qquad g(9) = -4.37 \times 10^{-16}.$$

Consequently, the sequence $(g(n))_{n\in\mathbb{N}}$ is strictly increasing. This leads to

$$g(n) < \lim_{n\to\infty} g(n) = 0, \quad n \in \mathbb{N},$$

which means that the second inequality in (4.1) is valid for $n \in \mathbb{N}$. The proof of Theorem 3 is complete. □

We propose the following.

Conjecture 1 *Let* $a_k = \frac{(2k-1)(2k+1)}{64}$, $k \in \mathbb{N}$. *The Wallis sequence* W_n *has the following continued fraction representation*:

$$W_n = \frac{\pi}{2}\left(1 - \cfrac{\frac{1}{4}}{n + \frac{5}{8} + \cfrac{a_1}{n+\frac{1}{2} + \cfrac{a_2}{n+\frac{1}{2} + \cfrac{a_3}{n+\frac{1}{2} + \ddots}}}}\right).$$

Theorem 4 *The following inequalities hold*:

$$\frac{\pi}{2}\left(1 - \frac{1}{4n+\frac{5}{2}}\right)^{\lambda(n)} < W_n < \frac{\pi}{2}\left(1 - \frac{1}{4n+\frac{5}{2}}\right)^{\mu(n)}, \tag{4.2}$$

where

$$\lambda(n) = 1 - \frac{3}{64n^2} + \frac{3}{64n^3} - \frac{23}{1{,}024n^4}$$

and

$$\mu(n) = 1 - \frac{3}{64n^2} + \frac{3}{64n^3} - \frac{23}{1{,}024n^4} - \frac{1}{512n^5}.$$

The first inequality holds for $n \geq 5$, *while the second inequality is valid for all* $n \geq 1$.

Proof Inequality (4.2) can be written as

$$\left(1 - \frac{1}{4n+\frac{5}{2}}\right)^{\lambda(n)/2} < \frac{1}{\sqrt{n+\frac{1}{2}}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} < \left(1 - \frac{1}{4n+\frac{5}{2}}\right)^{\mu(n)/2}. \tag{4.3}$$

The lower bound in (4.3) is obtained by considering the function $F(x)$ defined by

$$F(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln\left(x+\frac{1}{2}\right) - \frac{\lambda(x)}{2}\ln\left(1 - \frac{1}{4x+\frac{5}{2}}\right).$$

Using Stirling's formula, we find that

$$\lim_{x\to\infty} F(x) = 0.$$

Differentiating $F(x)$ and applying the second inequality in (2.2), we obtain

$$\begin{aligned} F'(x) &= \psi(x+1) - \psi\left(x+\frac{1}{2}\right) - V(x) - \frac{24x^2-36x+23}{512x^5}\ln\left(\frac{8x+3}{8x+5}\right) \\ &< U(x) - V(x) - \frac{24x^2-36x+23}{512x^5}\ln\left(\frac{8x+3}{8x+5}\right), \end{aligned}$$

with

$$V(x) = \frac{8{,}192x^6 + 10{,}240x^5 + 2{,}944x^4 - 96x^3 + 48x^2 + 2x - 23}{128(8x+3)(8x+5)x^4(2x+1)}.$$

We claim that $F'(x) < 0$ for $x \geq 7$. It suffices to show that

$$G(x) := \frac{512x^5(U(x) - V(x))}{24x^2-36x+23} - \ln\left(\frac{8x+3}{8x+5}\right) < 0 \quad \text{for } x \geq 7.$$

Differentiation yields

$$G'(x) = \frac{R(x)}{2{,}048x^{12}(8x+3)^2(8x+5)^2(2x+1)^2(24x^2-36x+23)^2},$$

where $R(x)$ is a polynomial of degree 17 with integer coefficients (some positive, some negative). It can be shown further that

$$R(x) = (x-7)P_{16}(x) + 89{,}314{,}236{,}262{,}237{,}854{,}773{,}083,$$

so

$$R(x) > 0$$

for $x \geq 7$ and so

$$G'(x) > 0$$

for $x \geq 7$, and we have

$$G(x) < \lim_{x\to\infty} G(x) = 0, \quad x \geq 7.$$

This proves the claim.

Hence, $F(x)$ is strictly decreasing for $x \geq 7$. Direct computation yields

$$F(5) = 2.399 \times 10^{-9}, \qquad F(6) = 1.494 \times 10^{-9}, \qquad F(7) = 7.947 \times 10^{-10}.$$

Consequently, the sequence $(F(n))$ is strictly decreasing for $n \geq 5$. This leads to

$$F(n) > \lim_{n\to\infty} F(n) = 0, \quad n \geq 5,$$

which means that the first inequality in (4.2) is valid for $n \geq 5$.

The upper bound in (4.3) is obtained by considering the function $H(x)$ defined by

$$H(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln\left(x+\frac{1}{2}\right) - \frac{\mu(x)}{2}\ln\left(1-\frac{1}{4x+\frac{5}{2}}\right).$$

Using Stirling's formula, we find that

$$\lim_{x\to\infty} H(x) = 0.$$

Differentiating $H(x)$ and applying the first inequality in (2.2), we obtain

$$\begin{aligned} H'(x) &= \psi(x+1) - \psi\left(x+\frac{1}{2}\right) - J(x) - \frac{48x^3-72x^2+46x+5}{1{,}024x^6}\ln\left(\frac{8x+3}{8x+5}\right) \\ &> L(x) - J(x) - \frac{48x^3-72x^2+46x+5}{1{,}024x^6}\ln\left(\frac{8x+3}{8x+5}\right), \end{aligned}$$

with

$$J(x) = \frac{8{,}192x^7+10{,}240x^6+2{,}944x^5-96x^4+48x^3+2x^2-27x-2}{128(8x+3)(8x+5)x^5(2x+1)}.$$

We claim that the function $H'(x) > 0$ for $x \geq 2$. It suffices to show that

$$I(x) := \frac{1{,}024x^6(L(x)-J(x))}{48x^3-72x^2+46x+5} - \ln\left(\frac{8x+3}{8x+5}\right) > 0 \quad \text{for } x \geq 2.$$

Differentiation yields

$$I'(x) = -\frac{S(x)}{16x^9(8x+3)^2(8x+5)^2(2x+1)^2(48x^3-72x^2+46x+5)^2},$$

where $S(x)$ is a polynomial of degree 15 with integer coefficients (some positive, some negative). It can be shown further that

$$S(x) = (x-2)P_{14}(x) + 5{,}637{,}072{,}199{,}918,$$

so

$$S(x) > 0$$

for $x \geq 2$ and so

$$I'(x) < 0$$

for $x \geq 2$, and we have

$$I(x) > \lim_{x\to\infty} I(x) = 0, \quad x \geq 2.$$

This proves the claim.

Hence, $H(x)$ is strictly increasing for $x \geq 2$. Direct computation yields

$$H(1) = -0.000462508\ldots, \qquad H(2) = -0.000005843\ldots.$$

Consequently, the sequence $(H(n))$ is strictly increasing for $n \geq 1$. This leads to

$$H(n) < \lim_{n\to\infty} H(n) = 0, \quad n \geq 1,$$

which means that the second inequality in (4.2) is valid for $n \geq 1$. The proof of Theorem 4 is complete. □

In fact, it is proved that

$$W_n = \frac{\pi}{2}\left(1 - \frac{1}{4n + \frac{5}{2}}\right)^{1-\frac{3}{64n^2}+\frac{3}{64n^3}-\frac{23}{1{,}024n^4}+O(n^{-5})}, \quad n \to \infty. \tag{4.4}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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