

RESEARCH

Open Access

# A note on the strong law of large numbers for Markov chains indexed by irregular trees

Wei-cai Peng\*

\*Correspondence:  
weicaipeng@126.com  
Department of Mathematics,  
Chaohu University, Chaohu, 238000,  
P.R. China

## Abstract

In this paper, a kind of an infinite irregular tree is introduced. The strong law of large numbers and the Shannon-McMillan theorem for Markov chains indexed by an infinite irregular tree are established. The outcomes generalize some known results on regular trees and uniformly bounded degree trees.

**Keywords:** Markov chain; tree; strong law of large numbers; AEP

## 1 Introduction

By a tree  $T$  we mean an infinite, locally finite, connected graph with a distinguished vertex  $o$  called the *root* and without loops or cycles. We only consider trees without leaves. That is, the degree (the number of neighboring vertices) of each vertex (except  $o$ ) is required to be at least 2.

Let  $T$  be an infinite tree with root  $o$ , the set of all vertices with distance  $n$  from the root is called the  $n$ th generation (or  $n$ th level) of  $T$ . We denote by  $T^{(n)}$  the union of the first  $n$  generations of  $T$ , by  $T_{(m)}^{(n)}$  the union from the  $m$ th to  $n$ th generations of  $T$ , by  $L_n$  the subgraph of  $T$  containing the vertices in the  $n$ th generation. For each vertex  $t$ , there is a unique path from  $o$  to  $t$ , and  $|t|$  for the number of edges on this path. We denote the first predecessor of  $t$  by  $1_t$ , the second predecessor of  $t$  by  $2_t$ , and by  $n_t$  the  $n$ th predecessor of  $t$ . We also call  $t$  one of  $1_t$ 's sons. For any two vertices  $s$  and  $t$ , denote by  $s \leq t$ , if  $s$  is on the unique path from the root  $o$  to  $t$ , denote by  $s \wedge t$  the vertex farthest from  $o$  satisfying  $s \wedge t \leq s$  and  $s \wedge t \leq t$ .  $X^A = \{X_t, t \in A\}$  and denote by  $|A|$  the number of vertices of  $A$ .

If each vertex on a tree  $T$  has  $m + 1$  neighboring vertices, we call it a Bethe tree  $T_{B,m}$ ; if the root has  $m$  neighbors and the other vertices have  $m + 1$  neighbors on a tree  $T$ , we call it a Cayley tree  $T_{C,m}$ . Both the Bethe tree and the Cayley tree are called regular (or homogeneous) trees. If the degrees of all vertices on a tree  $T$  are uniformly bounded, then we call  $T$  a uniformly bounded degree tree (see [1] and [2]).

**Definition 1** (see [3]) Let  $T$  be a locally finite, infinite tree,  $S$  be a finite state-space,  $\{X_t, t \in T\}$  be a collection of  $S$ -valued random variables defined on the probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . Let

$$\{p(x), x \in S\} \tag{1}$$

be a distribution on  $S$ , and

$$(P(y|x)), \quad x, y \in S \tag{2}$$

be a stochastic matrix on  $S^2$ . If for any vertex  $t$ ,

$$\begin{aligned} & \mathbf{P}(X_t = y | X_{1_t} = x \text{ and } X_s \text{ for } t \wedge s \leq 1_t) \\ &= \mathbf{P}(X_t = y | X_{1_t} = x) = P(y|x) \quad \forall x, y \in S \end{aligned} \tag{3}$$

and

$$\mathbf{P}(X_0 = x) = p(x) \quad \forall x \in S,$$

then  $\{X_t, t \in T\}$  will be called  $S$ -valued Markov chains indexed by an infinite tree  $T$  with initial distribution (1) and transition matrix (2), or called  $T$ -indexed Markov chains with state-space  $S$ .

Benjamini and Peres [3] gave the notion of tree-indexed Markov chains and studied the recurrence and ray-recurrence for them. Berger and Ye [4] studied the existence of entropy rate for some stationary random fields on a homogeneous tree. Ye and Berger [5], by using Pemantle’s result [6] and a combinatorial approach, studied the Shannon-McMillan theorem with convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree. Yang and Liu [7] studied the strong law of large numbers and Shannon-McMillan theorems for Markov chains field on the Cayley tree. Yang [8] studied some strong limit theorems for homogeneous Markov chains indexed by a homogeneous tree and the strong law of large numbers and the asymptotic equipartition property (AEP) for finite homogeneous Markov chains indexed by a homogeneous tree. Yang and Ye [9] studied strong theorems for countable nonhomogeneous Markov chains indexed by a homogeneous tree and the strong law of large numbers and the AEP for finite nonhomogeneous Markov chains indexed by a homogeneous tree. Bao and Ye [10] studied the strong law of large numbers and asymptotic equipartition property for nonsymmetric Markov chain fields on Cayley trees. Takacs [1] studied the strong law of large numbers for the univariate functions of finite Markov chains indexed by an infinite tree with uniformly bounded degree. Huang and Yang [2] studied the strong law of large numbers for Markov chains indexed by uniformly bounded degree trees.

However, the degrees of the vertices in the tree models are uniformly bounded. What if the degrees of the vertices are not uniformly bounded? In this paper, we drop the uniformly bounded restriction. We mainly study the strong law of large numbers and AEP with a.e. convergence for finite Markov chains indexed by trees under the following assumption.

For any integer  $N \geq 0$ , let  $d^0(t) := 1$  and denote

$$d^N(t) := |\sigma \in T : N_\sigma = t| \tag{4}$$

by the amount of  $t$ ’s  $N$ th descendants. Denote

$$O(n) = \left\{ c_n : 0 < \limsup_{n \rightarrow \infty} \frac{c_n}{n} \leq c, c \text{ is a constant} \right\}.$$

We assume that for enough large  $n$  and any given integer  $N \geq 0$ ,

$$\max\{d^N(t) : t \in T^{(n)}\} \leq O\left(\ln \frac{|T^{(n+N)}|}{|T^{(n)}|}\right). \tag{5}$$

The following examples are used to explain assumption (5).

**Example 1** Both the Bethe tree  $T_{B,m}$  and the Cayley tree  $T_{C,m}$  satisfy assumption (5). Actually,  $\max\{d^N(t) : t \in T^{(n-N)}\}$  is a constant  $m^N$ , and  $\ln(|T^{(n)}|/|T^{(n-N)}|) = N \ln m$ .

**Example 2** A uniformly bounded degree tree satisfies assumption (5). In fact, if the tree  $T$  is a uniformly bounded degree tree, then  $\max\{d^N(t) : t \in T^{(n-N)}\}$  is no more than a constant  $a^N$ , and

$$\ln \frac{|T^{(n)}|}{|T^{(n-N)}|} \leq \ln \frac{|T^{(n-N)}| \times a^N}{|T^{(n-N)}|} = N \ln a$$

is also a constant.

**Example 3** Define the lower growth rate of the tree to be  $\text{gr } T = \liminf_{n \rightarrow \infty} |T^{(n)}|^{\frac{1}{n}}$  and the upper growth rate of the tree to be  $\text{Gr } T = \limsup_{n \rightarrow \infty} |T^{(n)}|^{\frac{1}{n}}$ .

If both the  $\text{gr } T$  and  $\text{Gr } T$  are finite, then

$$\ln \frac{|T^{(n+N)}|}{|T^{(n)}|} \leq \ln \frac{(\text{Gr } T)^{n+N}}{(\text{gr } T)^n} = n \ln \frac{\text{Gr } T}{\text{gr } T} + N \ln \text{Gr } T,$$

hence (5) implies that

$$\max\{d^N(t) : t \in T^{(n)}\} \leq O\left(\ln \frac{|T^{(n+N)}|}{|T^{(n)}|}\right) \leq O(n).$$

## 2 Some notations and lemmas

In the following, let  $\delta_k(\cdot)$  be a Kronecker  $\delta$ -function. For any given integer  $N \geq 0$ , denote

$$S_k^N(T^{(n)}) := \sum_{t \in T^{(n-N)}} \delta_k(X_t) d^N(t), \tag{6}$$

which can be considered as the number of  $k$ 's among the variables in  $T^{(n-N)}$ , weighted according to the number of  $N$ th descendants. By (6), we have

$$\sum_{k \in S} S_n^N(k) = |T^{(n-N)}| - 1. \tag{7}$$

Define

$$H_n(\omega) = \sum_{t \in T^{(n)} \setminus \{o\}} g_t(X_{1_t}, X_t) \tag{8}$$

and

$$G_n(\omega) = \sum_{t \in T^{(n)} \setminus \{o\}} E[g_t(X_{1_t}, X_t) | X_{1_t}]. \tag{9}$$

**Lemma 1** (see [2]) *Let  $T$  be an infinite tree with assumption (5) holds. Let  $(X_t)_{t \in T}$  be a  $T$ -indexed Markov chain with state-space  $S$  defined as before,  $\{g_t(x, y), t \in T\}$  be functions defined on  $S^2$ . Let  $L_o = \{o\}$ ,  $\mathcal{F}_n = \sigma(X^{T^{(n)}})$ ,*

$$t_n(\lambda, \omega) = \frac{e^{\lambda \sum_{t \in T^{(n)} \setminus \{o\}} g_t(X_{1_t}, X_t)}}{\prod_{t \in T^{(n)} \setminus \{o\}} E[e^{\lambda g_t(X_{1_t}, X_t)} | X_{1_t}]}, \tag{10}$$

where  $\lambda$  is a real number. Then  $\{t_n(\lambda, \omega), \mathcal{F}_n, n \geq 1\}$  is a nonnegative martingale.

**Lemma 2** (see [2]) *Under the assumption of Lemma 1, let  $\{a_n, n \geq 1\}$  be a sequence of nonnegative random variables,  $\alpha > 0$ . Set*

$$B = \left\{ \lim_{n \rightarrow \infty} a_n = \infty \right\} \tag{11}$$

and

$$D(\alpha) = \left\{ \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{t \in T^{(n)} \setminus \{o\}} E[g_t^2(X_{1_t}, X_t) e^{\alpha |g_t(X_{1_t}, X_t)|} | X_{1_t}] = M(\omega) < \infty \right\} \cap B. \tag{12}$$

Then

$$\lim_{n \rightarrow \infty} \frac{H_n(\omega) - G_n(\omega)}{a_n} = 0 \quad \text{a.e. on } D(\alpha). \tag{13}$$

### 3 Strong law of large numbers and Shannon-McMillan theorem

In this section, we study the strong law of large numbers and the Shannon-McMillan theorem for finite Markov chains indexed by an infinite tree with assumption (5) holds.

**Theorem 1** *Let  $T$  be an infinite tree with assumption (5) holds. Then under the assumption of Lemma 1, for all  $k \in S$  and  $N \geq 0$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \left\{ S_k^N(T^{(n)}) - \sum_{l \in S} S_l^{N+1}(T^{(n-1)}) P(k|l) \right\} = 0 \quad \text{a.e.} \tag{14}$$

*Proof* Let  $g_t(x, y) = d^N(t) \delta_k(y)$ ,  $a_n = |T^{(n+N)}|$ . Since

$$\begin{aligned} G_n(\omega) &= \sum_{t \in T^{(n)} \setminus \{o\}} E[g_t(X_{1_t}, X_t) | X_{1_t}] \\ &= \sum_{t \in T^{(n)} \setminus \{o\}} d^N(t) \sum_{x_t \in S} \delta_k(x_t) P(x_t | X_{1_t}) \\ &= \sum_{t \in T^{(n)} \setminus \{o\}} d^N(t) P(k | X_{1_t}) \\ &= \sum_{l \in S} \sum_{t \in T^{(n)} \setminus \{o\}} \delta_l(X_{1_t}) d^N(t) P(k|l) \\ &= \sum_{l \in S} \sum_{t \in T^{(n-1)}} \delta_l(X_t) d^{N+1}(t) P(k|l) \\ &= \sum_{l \in S} S_l^{N+1}(T^{(n-1)}) P(k|l) \end{aligned} \tag{15}$$

and

$$H_n(\omega) = \sum_{t \in T^{(n)} \setminus \{o\}} g_t(X_{1_t}, X_t) = \sum_{t \in T^{(n)} \setminus \{o\}} d^N(t) \delta_k(X_t) = S_k^N(T^{(n)}) - \delta_k(X_o) d^N(o). \tag{16}$$

By Lemma 1, we know that  $\{t_n(\lambda, \omega), \mathcal{F}_n, n \geq 1\}$  is a nonnegative martingale. According to the Doob martingale convergence theorem, we have

$$\lim_n t_n(\lambda, \omega) = t(\lambda, \omega) < \infty \quad \text{a.e.} \tag{17}$$

We have by (17)

$$\limsup_{n \rightarrow \infty} \frac{\ln t_n(\lambda, \omega)}{|T^{(n+N)}|} \leq 0 \quad \text{a.e. } \omega \in B. \tag{18}$$

By (10), (16) and (18), we get

$$\limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \left\{ \lambda H_n(\omega) - \sum_{t \in T^{(n)} \setminus \{o\}} \ln[E[e^{\lambda g_t(X_{1_t}, X_t)} | X_{1_t}]] \right\} \leq 0 \quad \text{a.e. } \omega \in B. \tag{19}$$

Let  $\lambda > 0$ . Dividing two sides of (19) by  $\lambda$ , we have

$$\limsup_{n \rightarrow \infty} \frac{1}{a_n} \left\{ H_n(\omega) - \sum_{t \in T^{(n)} \setminus \{o\}} \frac{\ln[E[e^{\lambda g_t(X_{1_t}, X_t)} | X_{1_t}]]}{\lambda} \right\} \leq 0 \quad \text{a.e. } \omega \in B. \tag{20}$$

The case  $\{d^N(t) : t \in T^{(n)}\}$  is uniformly bounded was considered in [2], we only consider the case  $\{d^N(t) : t \in T^{(n)}\}$  is not uniformly bounded. By (18) and inequalities  $\ln x \leq x - 1$  ( $x > 0$ ),  $0 \leq e^x - 1 - x \leq 2^{-1}x^2 e^{|x|}$ , as  $0 < \lambda \leq \alpha$ , we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \left[ H_n(\omega) - \sum_{t \in T^{(n)} \setminus \{o\}} E[g_t(X_{1_t}, X_t) | X_{1_t}] \right] \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T^{(n)}} \left\{ \frac{\ln[E[e^{\lambda g_t(X_{1_t}, X_t)} | X_{1_t}]]}{\lambda} - E[g_t(X_{1_t}, X_t) | X_{1_t}] \right\} \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T^{(n)}} \left\{ \frac{E[e^{\lambda g_t(X_{1_t}, X_t)} | X_{1_t}] - 1}{\lambda} - E[g_t(X_{1_t}, X_t) | X_{1_t}] \right\} \\ & \leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T^{(n)}} E[g_t^2(X_{1_t}, X_t) e^{\lambda |g_t(X_{1_t}, X_t)|} | X_{1_t}] \\ & = \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T_{(1)}^{(n)}} E[(d^N(t) \delta_k(X_t))^2 e^{\lambda |d^N(t) \delta_k(X_t)|} | X_{1_t}] \\ & \leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T_{(1)}^{(n)}} [(d^N(t))^2 e^{\lambda d^N(t)} P(k | X_{1_t})] \\ & \leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T_{(1)}^{(n)}} [(d^N(t))^2 e^{\lambda d^N(t)}] \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T_{(1)}^{(n)}} e^{2\lambda d^N(t)} \quad (\text{for enough large } d^N(t)) \\
 &\leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{|T^{(n)}| - 1}{|T^{(n+N)}|} \max\{e^{2\lambda d^N(t)}, t \in T_{(1)}^{(n)}\} \\
 &\leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{|T^{(n)}| - 1}{|T^{(n+N)}|} (e^{\max\{d^N(t), t \in T_{(1)}^{(n)}\}})^{2\lambda}. \tag{21}
 \end{aligned}$$

By (5), there exists a constant  $\beta > 0$  such that

$$\max\{d^N(t), t \in T_{(1)}^{(n)}\} \leq \beta \ln \frac{|T^{(n+N)}|}{|T^{(n)}|},$$

hence,

$$(e^{\max\{d^N(t), t \in T_{(1)}^{(n)}\}})^{2\lambda} < \left(\frac{|T^{(n+N)}|}{|T^{(n)}|}\right)^{2\lambda\beta}. \tag{22}$$

Set  $0 < \lambda < \frac{1}{2\beta}$ , by (21) and (22) we have

$$\limsup_{n \rightarrow \infty} \frac{H_n(\omega) - G_n(\omega)}{|T^{(n+N)}|} \leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{|T^{(n)}| - 1}{|T^{(n+N)}|} \times \left(\frac{|T^{(n+N)}|}{|T^{(n)}|}\right)^{2\lambda\beta}. \tag{23}$$

Let  $\lambda \rightarrow 0^+$  in (23), by (15) and (16) we have

$$\lim_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \left\{ S_k^N(T^{(n)}) - \sum_{l \in S} S_l^{N+1}(T^{(n-1)})P(k|l) \right\} \leq 0 \quad \text{a.e.} \tag{24}$$

Let  $-\frac{1}{2\beta} \leq \lambda \rightarrow 0^-$ . By (19), we similarly get

$$\lim_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \left\{ S_k^N(T^{(n)}) - \sum_{l \in S} S_l^{N+1}(T^{(n-1)})P(k|l) \right\} \geq 0 \quad \text{a.e.} \tag{25}$$

Combining (24) and (25), we obtain (14) directly. □

Let  $T$  be a tree,  $(X_t)_{t \in T}$  be a stochastic process indexed by the tree  $T$  with state-space  $S$ . Denote

$$P(x^{T^{(n)}}) = P(X^{T^{(n)}} = x^{T^{(n)}}).$$

Let

$$f_n(\omega) = -\frac{1}{|T^{(n)}|} \ln P(X^{T^{(n)}}), \tag{26}$$

$f_n(\omega)$  will be called the entropy density of  $X^{T^{(n)}}$ . If  $(X_t)_{t \in T}$  is a  $T$ -indexed Markov chain with state-space  $S$  defined by Definition 1, we have by (5)

$$f_n(\omega) = -\frac{1}{|T^{(n)}|} \left[ \ln P(X_0) + \sum_{t \in T^{(n)} \setminus \{o\}} \ln P(X_t, X_{I_t}) \right]. \tag{27}$$

The convergence of  $f_n(\omega)$  to a constant in a sense ( $L_1$  convergence, convergence in probability, a.e. convergence) is called the Shannon-McMillan theorem or the entropy theorem or the AEP in information theory.

**Theorem 2** *Let  $T$  be an infinite tree with assumption (5) holds. Let  $k \in S$ , and  $P$  be an ergodic stochastic matrix. Denote the unique stationary distribution of  $P$  by  $\pi$ . Let  $(X_t)_{t \in T}$  be a  $T$ -indexed Markov chain with state-space  $S$  generated by  $P$ . Then, for given integer  $N \geq 0$ ,*

$$\lim_{n \rightarrow \infty} \frac{S_k^N(T^{(n)})}{|T^{(n+N)}|} = \pi(k) \quad a.e. \tag{28}$$

Let  $S_{l,k}(T^{(n)}) := |\{t \in T^{(n)} : (X_t, X_t) = (l, k)\}|$ , then

$$\lim_{n \rightarrow \infty} \frac{S_{l,k}(T^{(n)})}{|T^{(n)}|} = \pi(l)P(k|l) \quad a.e. \tag{29}$$

Let  $f_n(\omega)$  be defined as (27), then

$$\lim_{n \rightarrow \infty} f_n(\omega) = - \sum_{l \in S} \sum_{k \in S} \pi(l)P(k|l) \ln P(k|l) \quad a.e. \tag{30}$$

*Proof* The proofs of (28) and (29) are similar to those of Huang and Yang ([2], Theorem 2 and Corollary 3). Letting  $g_t(x, y) = -\ln P(y|x)$  in Lemma 1, then

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(\omega) &= \lim_{n \rightarrow \infty} \frac{H_n(\omega)}{|T^{(n)}|} = - \lim_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \setminus \{o\}} \ln P(X_{1_t}, X_t) \\ &= - \lim_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \sum_{l \in S} \sum_{k \in S} \delta_l(X_{1_t}) \delta_k(X_t) \ln P(k|l) \\ &= - \lim_{n \rightarrow \infty} \sum_{l \in S} \sum_{k \in S} \ln P(k|l) \frac{S_{l,k}(T^{(n)})}{|T^{(n)}|} \end{aligned}$$

by (29), (30) holds. □

**Competing interests**

The author declares that they have no competing interests.

**Acknowledgements**

This work is supported by the Foundation of Anhui Educational Committee (No. KJ2014A174).

Received: 9 February 2014 Accepted: 4 June 2014 Published: 23 June 2014

**References**

1. Takacs, C: Strong law of large numbers for branching Markov chains. *Markov Process. Relat. Fields* **8**, 107-116 (2001)
2. Huang, HL, Yang, WG: Strong law of large numbers for Markov chains indexed by an infinite tree with uniformly bounded degree. *Sci. China Ser. A* **51**(2), 195-202 (2008)
3. Benjamini, I, Peres, Y: Markov chains indexed by trees. *Ann. Probab.* **22**, 219-243 (1994)
4. Berger, T, Ye, Z: Entropic aspects of random fields on trees. *IEEE Trans. Inf. Theory* **36**, 1006-1018 (1990)
5. Ye, Z, Berger, T: Ergodic, regular and asymptotic equipartition property of random fields on trees. *J. Comb. Inf. Syst. Sci.* **21**, 157-184 (1996)
6. Pemantle, R: Automorphism invariant measure on trees. *Ann. Probab.* **20**, 1549-1566 (1992)

7. Yang, WG, Liu, W: Strong law of large numbers and Shannon-McMillan theorem for Markov chains field on Cayley tree. *Acta Math. Sci. Ser. B* **21**(4), 495-502 (2001)
8. Yang, WG: Some limit properties for Markov chains indexed by a homogeneous tree. *Stat. Probab. Lett.* **65**, 241-250 (2003)
9. Yang, WG, Ye, Z: The asymptotic equipartition property for nonhomogeneous Markov chains indexed by a homogeneous tree. *IEEE Trans. Inf. Theory* **53**(9), 3275-3280 (2007)
10. Bao, ZH, Ye, Z: Strong law of large numbers and asymptotic equipartition property for nonsymmetric Markov chain fields on Cayley trees. *Acta Math. Sci. Ser. B* **27**(4), 829-837 (2007)

doi:10.1186/1029-242X-2014-244

**Cite this article as:** Peng: A note on the strong law of large numbers for Markov chains indexed by irregular trees. *Journal of Inequalities and Applications* 2014 **2014**:244.

**Submit your manuscript to a SpringerOpen<sup>®</sup> journal and benefit from:**

- Convenient online submission
- Rigorous peer review
- Immediate publication on acceptance
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

---

Submit your next manuscript at ► [springeropen.com](http://springeropen.com)

---