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A note on the almost sure central limit theorem for the product of some partial sums

Yang Chen[1], Zhongquan Tan[2*] and Kaiyong Wang[1]

*Correspondence: tzq728@163.com
[2]College of Mathematics, Physics and Information Engineering, Jiaxing University, Jiaxing, 314001, P.R. China
Full list of author information is available at the end of the article

Abstract

Let (X_n) be a sequence of i.i.d., positive, square integrable random variables with $E(X_1) = \mu > 0$, $\mathrm{Var}(X_1) = \sigma^2$. Denote by $S_{n,k} = \sum_{i=1}^n X_i - X_k$ and by $\gamma = \sigma/\mu$ the coefficient of variation. Our goal is to show the unbounded, measurable functions g, which satisfy the almost sure central limit theorem, *i.e.*,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} g\left(\left(\frac{\prod_{k=1}^n S_{n,k}}{(n-1)^n \mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}}\right) = \int_0^\infty g(x)\,dF(x) \quad \text{a.s.,}$$

where $F(\cdot)$ is the distribution function of the random variable $e^{\mathcal{N}}$ and $\mathcal{N}$ is a standard normal random variable.

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1 Introduction

The almost sure central limit theorem (ASCLT) has been first introduced independently by Schatte [1] and Brosamler [2]. Since then, many studies have been done to prove the ASCLT in different situations, for example, in the case of function-typed almost sure central limit theorem (FASCLT) (see Berkes *et al.* [3], Ibragimov and Lifshits [4]). The purpose of this paper is to investigate the FASCLT for the product of some partial sums.

Let (X_n) be a sequence of i.i.d. random variables and define the partial sum $S_n = \sum_{k=1}^n X_k$ for $n \geq 1$. In a recent paper of Rempala and Wesolowski [5], it is showed under the assumption $E(X^2) < \infty$ and $X > 0$ that

$$\left(\frac{\prod_{k=1}^n S_k}{n!\mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}}, \tag{1}$$

where $\mathcal{N}$ is a standard normal random variable, $\mu = E(X)$ and $\gamma = \sigma/\mu$ with $\sigma^2 = \mathrm{var}(X)$. For further results in this field, we refer to Qi [6], Lu and Qi [7] and Rempala and Wesolowski [8].

Recently Gonchigdanzan and Rempala [9] obtained the almost sure limit theorem related to (1) as follows.

Theorem A *Let (X_n) be a sequence of i.i.d., positive random variables with $E(X_1) = \mu > 0$ and $\operatorname{Var}(X_1) = \sigma^2$. Denote by $\gamma = \sigma/\mu$ the coefficient of variation. Then, for any real x,*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}} \leq x\right) = G(x) \quad a.s., \tag{2}$$

where $G(x)$ is the distribution function of $e^{\sqrt{2}\mathcal{N}}$, $\mathcal{N}$ is a standard normal random variable. Some extensions on the above result can be found in Ye and Wu [10] *and the reference therein.*

A similar result on the product of partial sums was provided by Miao [11], which stated the following.

Theorem B *Let (X_n) be a sequence of i.i.d., positive, square integrable random variables with $E(X_1) = \mu > 0$ and $\operatorname{Var}(X_1) = \sigma^2$. Denote by $S_{n,k} = \sum_{i=1}^{n} X_i - X_k$ and $\gamma = \sigma/\mu$ the coefficient of variation. Then*

$$\left(\frac{\prod_{k=1}^{n} S_{n,k}}{(n-1)^n \mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}} \xrightarrow{d} e^{\mathcal{N}}, \tag{3}$$

and for any real x,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(\left(\frac{\prod_{k=1}^{n} S_{n,k}}{(n-1)^n \mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}} \leq x\right) = F(x) \quad a.s., \tag{4}$$

where $F(\cdot)$ is the distribution function of the random variable $e^{\mathcal{N}}$ and $\mathcal{N}$ is a standard normal random variable.

The purpose of this paper is to investigate the validity of (4) for some class of unbounded measurable functions g.

Throughout this article, (X_n) is a sequence of i.i.d. positive, square integrable random variables with $E(X_1) = \mu > 0$ and $\operatorname{Var}(X_1) = \sigma^2$. We denote by $S_{n,k} = \sum_{i=1}^{n} X_i - X_k$ and by $\gamma = \sigma/\mu$ the coefficient of variation. Furthermore, $\mathcal{N}$ is the standard normal random variable, Φ is the standard normal distribution function, ϕ is its density function and $a \ll b$ stands for $\limsup_{n\to\infty} |a_n/b_n| < \infty$.

2 Main result

We state our main result as follows.

Theorem 1 *Let $g(x)$ be a real-valued, almost everywhere continuous function on R such that $|g(e^x)\phi(x)| \leq c(1+|x|)^{-\alpha}$ with some $c > 0$ and $\alpha > 5$. Then, for any real x,*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} g\left(\left(\frac{\prod_{k=1}^{n} S_{n,k}}{(n-1)^n \mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}}\right) = \int_0^\infty g(x)\, dF(x) \quad a.s., \tag{5}$$

where $F(\cdot)$ is the distribution function of the random variable $e^{\mathcal{N}}$.

Let $f(x) = g(e^x)$. By a simple calculation, we can get the following result.

Remark 1 Let $f(x)$ be a real-valued, almost everywhere continuous function on R such that $|f(x)\phi(x)| \le c(1+|x|)^{-\alpha}$ with some $c > 0$ and $\alpha > 5$. Then (5) is equivalent to

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} f\left(\frac{1}{\gamma\sqrt{n}} \sum_{k=1}^{n} \log \frac{S_{n,k}}{(n-1)\mu} \right) = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx \quad \text{a.s.} \tag{6}$$

Remark 2 Lu *et al.* [12] proved the function-typed almost sure central limit theorem for a type of random function, which can include U-statistics, Von-Mises statistics, linear processes and some other types of statistics, but their results cannot imply Theorem 1.

3 Auxiliary results

In this section, we state and prove several auxiliary results, which will be useful in the proof of Theorem 1.

Let $\widetilde{S}_n = \sum_{i=1}^{n} \frac{X_i-\mu}{\sigma}$ and $U_i = \frac{1}{\gamma\sqrt{i}} \sum_{k=1}^{i} \log \frac{S_{i,k}}{(i-1)\mu}$. Observe that for $|x| < 1$ we have

$$\log(1+x) = x + \frac{\theta}{2}x^2,$$

where $\theta \in (-1, 0)$. Thus

$$\begin{aligned} U_i &= \frac{1}{\gamma\sqrt{i}} \sum_{k=1}^{i} \log \frac{S_{i,k}}{(i-1)\mu} \\ &= \frac{1}{\gamma\sqrt{i}} \sum_{k=1}^{i} \left(\frac{S_{i,k}}{(i-1)\mu} - 1 \right) + \frac{1}{\gamma\sqrt{i}} \sum_{k=1}^{i} \frac{\theta_k}{2} \left(\frac{S_{i,k}}{(i-1)\mu} - 1 \right)^2 \\ &= \frac{1}{\sqrt{i}} \sum_{k=1}^{i} \left(\frac{\sum_{j\ne k, j\le i}(X_j-\mu)}{(i-1)\sigma} \right) + \frac{1}{\gamma\sqrt{i}} \sum_{k=1}^{i} \frac{\theta_k}{2} \left(\frac{S_{i,k}}{(i-1)\mu} - 1 \right)^2 \\ &= \frac{1}{\sqrt{i}} \sum_{k=1}^{i} \frac{X_k-\mu}{\sigma} + \frac{1}{\gamma\sqrt{i}} \sum_{k=1}^{i} \frac{\theta_k}{2} \left(\frac{S_{i,k}}{(i-1)\mu} - 1 \right)^2 \\ &=: \frac{1}{\sqrt{i}} \widetilde{S}_i + R_i. \end{aligned} \tag{7}$$

By the law of iterated logarithm, we have for $k \to \infty$

$$\max_{1\le k\le i} \left| \frac{S_{i,k}}{(i-1)\mu} - 1 \right| = O\big((\log\log i/i)^{1/2} \big) \quad \text{a.s.}$$

Therefore,

$$|R_i| = \left| \frac{1}{\gamma\sqrt{i}} \sum_{k=1}^{i} \frac{\theta_k}{2} \left(\frac{S_{i,k}}{(i-1)\mu} - 1 \right)^2 \right| \ll \frac{1}{\sqrt{i}} \sum_{k=1}^{i} \left(\frac{S_{i,k}}{(i-1)\mu} - 1 \right)^2 \ll \frac{\log\log i}{i^{1/2}} \quad \text{a.s.} \tag{8}$$

Obviously,

$$\begin{aligned} E|R_i| &= E\left| \frac{1}{\gamma\sqrt{i}} \sum_{k=1}^{i} \frac{\theta_k}{2} \left(\frac{S_{i,k}}{(i-1)\mu} - 1 \right)^2 \right| \\ &\ll \frac{1}{\sqrt{i}} \sum_{k=1}^{i} E\left(\frac{S_{i,k}}{(i-1)\mu} - 1 \right)^2 \ll \frac{1}{\sqrt{i}} \sum_{k=1}^{i} \frac{1}{i-1} \ll \frac{1}{i^{1/2}}. \end{aligned} \tag{9}$$

Our proof mainly relies on decomposition (7). Properties (8) and (9) will be extensively used in the following parts of this section.

Lemma 1 *Let X and Y be random variables. We write $F(x) = P(X < x)$, $G(x) = P(X + Y < x)$. Then*

$$F(x-\varepsilon) - P\big(|Y| \geq \varepsilon\big) \leq G(x) \leq F(x+\varepsilon) + P\big(|Y| \geq \varepsilon\big)$$

for every $\varepsilon > 0$ and x.

Proof It is Lemma 1.3 of Petrov [13]. □

Lemma 2 *Let (X_n) be a sequence of i.i.d. random variables. Let $S_n = \sum_{k\leq n} X_k$, F^s denote the distribution function obtained from F by symmetrization, and choose $L > 0$ so large that $\int_{|x|\leq L} x^2\,dF^s \geq 1$. Then, for any $n \geq 1$, $\lambda > 0$,*

$$\sup_a P\left(a \leq \frac{S_n}{\sqrt{n}} \leq a + \lambda\right) \leq A\lambda$$

with some absolute constant A, provided $\lambda\sqrt{n} \geq L$.

Proof It can be obtained from Berkes *et al.* [3]. □

Lemma 3 *Assume that* (6) *is true for all indicator functions of intervals and for a fixed a.e. continuous function $f(x) = f_0(x)$. Then* (6) *is also true for all a.e. continuous functions f such that $|f(x)| \leq |f_0(x)|$, $x \in R$, and, moreover, the exceptional set of probability* 0 *can be chosen universally for all such f.*

Proof See Berkes *et al.* [3]. □

In view of Lemma 3 and Remark 1, in order to prove Theorem 1, it suffices to prove (6) for the case when $f(x)\phi(x) = (1 + |x|)^{-\alpha}$, $\alpha > 5$. Thus, in the following part, we put $f(x)\phi(x) = (1 + |x|)^{-\alpha}$, $\alpha > 5$ and

$$\xi_k = \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} f(U_i),$$

$$\xi_k^* = \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} f(U_i) I\left\{f(U_i) \leq \frac{k}{(\log k)^\beta}\right\},$$

where $1 < \beta < \frac{1}{2}(\alpha - 3)$.

Lemma 4 *Under the conditions of Theorem* 1, *we have $P(\xi_k \neq \xi_k^*$ i.o.$) = 0$.*

Proof Let f^{-1} denote an inverse function of f in some interval, and let α, β satisfy $1 < \beta < \frac{1}{2}(\alpha - 3)$. It is easy to check that

$$\left\{\xi_k \neq \xi_k^*\right\} \subseteq \left\{|U_i| \geq f^{-1}\big(k/(\log k)^\beta\big) \text{ for some } 2^k < i \leq 2^{k+1}\right\}$$

and

$$f\big((2\log k+(\alpha-2\beta)\log\log k)^{1/2}\big) = \frac{k}{(\log k)^{\beta}}\frac{\sqrt{2\pi}(\log k)^{\alpha/2}}{\{1+(2\log k+(\alpha-2\beta)\log\log k)^{1/2}\}^{\alpha}} \le \frac{k}{(\log k)^{\beta}}. \tag{10}$$

Note that the function f is even and strictly increasing for $x \ge x_0$. We have

$$f^{-1}\big(k/(\log k)^{\beta}\big) \ge \big(2\log k+(\alpha-2\beta)\log\log k\big)^{1/2}. \tag{11}$$

Observing that $2^k < i \le 2^{k+1}$ implies $k \ge \frac{1}{2}\log i$, in view of (8) we get

$$\begin{aligned} P\big(\xi_k \neq \xi_k^* \text{ i.o.}\big) &\le P\big(|U_i| \ge \big(2\log\log i+(\alpha-2\beta)\log\log\log i - O(1)\big)^{1/2} \text{ i.o.}\big) \\ &= P\bigg(\bigg|\frac{\widetilde{S}_i}{\sqrt{i}}+R_i\bigg| \ge \big(2\log\log i+(\alpha-2\beta)\log\log\log i - O(1)\big)^{1/2} \text{ i.o.}\bigg) \\ &\le P\bigg(\bigg|\frac{\widetilde{S}_i}{\sqrt{i}}\bigg| \ge \big(2\log\log i+(\alpha-2\beta)\log\log\log i - O(1)\big)^{1/2} \text{ i.o.}\bigg) \\ &= 0, \end{aligned}$$

where in the last step we use the assumption $\alpha - 2\beta > 3$ and a version of the Kolmogorov-Erdös-Feller-Petrovski test (see Feller [14], Theorem 2). This completes the proof of Lemma 4. □

Let $a_k = f^{-1}(k/(\log k)^{\beta})$ and let G_i and F_i denote, respectively, the distribution function of U_i and $\frac{\widetilde{S}_i}{\sqrt{i}}$. Set

$$\begin{aligned} \sigma_i^2 &= \int_{-\sqrt{i}}^{\sqrt{i}} x^2\,dF_i(x) - \bigg(\int_{-\sqrt{i}}^{\sqrt{i}} x\,dF_i(x)\bigg)^2, \\ \eta_i &= \sup_x \bigg|G_i(x) - \Phi\bigg(\frac{x}{\sigma_i}\bigg)\bigg|, \\ \varepsilon_i &= \sup_x \bigg|F_i(x) - \Phi\bigg(\frac{x}{\sigma_i}\bigg)\bigg|. \end{aligned}$$

Clearly, $\sigma_i \le 1$, $\lim_{i\to\infty}\sigma_i = 1$.

Lemma 5 *Under the conditions of Theorem* 1, *we have*

$$\sum_{k\le N} E\big(\xi_k^*\big)^2 \ll \frac{N^2}{(\log N)^{2\beta}}.$$

Proof Observe now that the relation

$$\bigg|\int_{-a}^{a}\psi(x)\,d\big(G_1(x)-G_2(x)\big)\bigg| \le \sup_{-a\le x\le a}\big|\psi(x)\big| \cdot \sup_{-a\le x\le a}\big|G_1(x)-G_2(x)\big| \tag{12}$$

is valid for any bounded, measurable functions ψ and distribution functions G_1, G_2. Let, as previously, $a_k = f^{-1}(k/(\log k)^\beta)$. Thus, for any $2^k < i \le 2^{k+1}$, we obtain that

$$\begin{aligned} Ef^2(U_i)I\left\{f(U_i) \le \frac{k}{(\log k)^\beta}\right\} &= \int_{|x|\le a_k} f^2(x)\,dG_i(x) \\ &\le \int_{|x|\le a_k} f^2(x)\,d\Phi\left(\frac{x}{\sigma_i}\right) + \eta_i \frac{k^2}{(\log k)^{2\beta}} \\ &\ll \int_{|x|\le a_k} f^2(x)\,d\Phi(x) + \eta_i \frac{k^2}{(\log k)^{2\beta}}, \end{aligned}$$

where in the last step, we have used the fact that $\sigma_i \le 1$, $\lim_{i\to\infty}\sigma_i = 1$. Hence, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} E\left(\xi_k^*\right)^2 &\ll E\left[\left(\sum_{i=2^k+1}^{2^{k+1}}\left(\frac{1}{i}\right)^2\right)^{1/2}\left(\sum_{i=2^k+1}^{2^{k+1}} f^2(U_i)I\left\{f(U_i)\le\frac{k}{(\log k)^\beta}\right\}\right)^{1/2}\right]^2 \\ &\ll \left(\sum_{i=2^k+1}^{2^{k+1}}\frac{1}{i^2}\right)\left(\sum_{i=2^k+1}^{2^{k+1}}\left(\int_{|x|\le a_k} f^2(x)\,d\Phi(x) + \eta_i\frac{k^2}{(\log k)^{2\beta}}\right)\right) \\ &\ll \frac{1}{2^k}\left(2^k\int_{|x|\le a_k} f^2(x)\,d\Phi(x) + \frac{k^2}{(\log k)^{2\beta}}\sum_{i=2^k+1}^{2^{k+1}}\eta_i\right) \\ &\ll \int_{|x|\le a_k}\frac{e^{x^2/2}}{(1+|x|)^{2\alpha}}\,dx + \frac{k^2}{(\log k)^{2\beta}}\sum_{i=2^k+1}^{2^{k+1}}\frac{\eta_i}{i}. \end{aligned} \tag{13}$$

Note that

$$\int_0^t \frac{e^{x^2/2}}{(1+|x|)^{2\alpha}}\,dx = \int_0^{t/2} + \int_{t/2}^t \ll te^{t^2/8} + \frac{1}{t^{2\alpha+1}}\int_{t/2}^t xe^{x^2/2}\,dx \ll \frac{e^{t^2/2}}{t^{2\alpha+1}},$$

and thus by (10) and (11), we have

$$\int_{|x|\le a_k}\frac{e^{x^2/2}}{(1+|x|)^{2\alpha}}\,dx \ll \frac{e^{a_k^2/2}}{a_k^{2\alpha+1}} \ll f(a_k)\frac{1}{a_k^{\alpha+1}} \ll \frac{k}{(\log k)^{\beta+(\alpha+1)/2}}. \tag{14}$$

Now we estimate η_i. By Lemma 1, we have that for some $\varepsilon > 0$,

$$\begin{aligned} \eta_i &= \sup_x\left|G_i(x) - \Phi\left(\frac{x}{\sigma_i}\right)\right| \\ &\le \sup_x\left|G_i(x) - F_i(x)\right| + \sup_x\left|F_i(x) - \Phi\left(\frac{x}{\sigma_i}\right)\right| \\ &= \sup_x\left|P(U_i \le x) - P\left(\frac{\widetilde{S}_i}{\sqrt{i}} \le x\right)\right| + \varepsilon_i \\ &= \sup_x\left|P\left(\left(\frac{\widetilde{S}_i}{\sqrt{i}} + R_i\right) \le x\right) - P\left(\frac{\widetilde{S}_i}{\sqrt{i}} \le x\right)\right| + \varepsilon_i \\ &\le P\left(|R_i| \ge \varepsilon\right) + \sup_x\left\{P\left(\frac{\widetilde{S}_i}{\sqrt{i}} \le x+\varepsilon\right) - P\left(\frac{\widetilde{S}_i}{\sqrt{i}} \le x\right)\right\} + \varepsilon_i. \end{aligned}$$

The Markov inequality and (9) imply that

$$P\big(|R_i| \ge \varepsilon\big) \le \frac{E|R_i|}{\varepsilon} \ll \frac{1}{i^{1/2}\varepsilon}.$$

In addition, Lemma 2 yields

$$\sup_x \left\{ P\left(\frac{\widetilde{S}_i}{\sqrt{i}} \le x + \varepsilon \right) - P\left(\frac{\widetilde{S}_i}{\sqrt{i}} \le x \right) \right\} \ll \varepsilon.$$

Setting $\varepsilon = i^{-1/3}$, we have

$$\eta_i \ll \frac{1}{i^{1/6}} + \frac{1}{i^{1/3}} + \varepsilon_i.$$

Using Theorem 1 of Friedman *et al.* [15], we get

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{i} < \infty.$$

Hence,

$$\sum_{i=1}^{\infty} \frac{\eta_i}{i} \ll \sum_{i=1}^{\infty} \frac{\frac{1}{i^{1/6}} + \varepsilon_i}{i} < \infty, \tag{15}$$

which, coupled with (13), (14) and the fact $\frac{1}{2}(\alpha+1) > \beta$, yields

$$\begin{aligned} \sum_{k \le N} E\big(\xi_k^*\big)^2 &\ll \sum_{k \le N} \frac{k}{(\log k)^{\beta + (\alpha+1)/2}} + \sum_{k \le N} \frac{k^2}{(\log k)^{2\beta}} \sum_{i=2^k+1}^{2^{k+1}} \frac{\eta_i}{i} \\ &\ll \frac{N^2}{(\log N)^{2\beta}}, \end{aligned}$$

which completes the proof. □

Lemma 6 *Let* $\xi_k^* = \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} f(U_i) I\{f(U_i) \le \frac{k}{(\log k)^\beta}\}$, $\xi_l^* = \sum_{i=2^l+1}^{2^{l+1}} \frac{1}{i} f(U_i) I\{f(U_i) \le \frac{l}{(\log l)^\beta}\}$. *Under the conditions of Theorem* 1, *we have for* $l \ge l_0$

$$\big|\mathrm{cov}\big(\xi_k^*, \xi_l^*\big)\big| \ll \frac{kl}{(\log k)^\beta (\log l)^\beta} 2^{-(l-k-1)/4}.$$

Proof We first show the following result, for any $1 \le i \le \frac{j}{2}$ and real x, y,

$$\big|P(U_i \le x, U_j \le y) - P(U_i \le x)P(U_j \le y)\big| \ll \left(\frac{i}{j}\right)^{1/4}. \tag{16}$$

Letting $\rho = \frac{i}{j}$, the Chebyshev inequality yields

$$P\left(\left| \frac{\widetilde{S}_i}{\sqrt{j}} \right| \ge \rho^{1/4} \right) \le \frac{1}{j} \rho^{-1/2} E|\widetilde{S}_i|^2 = \rho^{1/2}. \tag{17}$$

Using the Markov inequality and (9), we have

$$P\bigl(|R_j| \ge \rho^{1/4}\bigr) \le \frac{E|R_j|}{\rho^{1/4}} \ll \frac{1}{j^{1/2}\rho^{1/4}} = \frac{1}{j^{1/4}i^{1/4}} \le \rho^{1/4}. \tag{18}$$

It follows from Lemma 1, Lemma 2, (17), (18) and the positivity and independence of (X_n) that

$$\begin{aligned}
&P(U_i \le x, U_j \le y) \\
&\quad = P\biggl(U_i \le x, \frac{\widetilde{S}_j}{\sqrt{j}} + R_j \le y\biggr) \\
&\quad = P\biggl(U_i \le x, \frac{\widetilde{S}_i}{\sqrt{j}} + \sqrt{1-\rho}\frac{\widetilde{S}_j - \widetilde{S}_i}{\sqrt{j-i}} + R_j \le y\biggr) \\
&\quad \ge P\biggl(U_i \le x, \sqrt{1-\rho}\frac{\widetilde{S}_j - \widetilde{S}_i}{\sqrt{j-i}} \le y\biggr) \\
&\qquad - P\biggl(y - 2\rho^{1/4} \le \sqrt{1-\rho}\frac{\widetilde{S}_j - \widetilde{S}_i}{\sqrt{j-i}} \le y\biggr) - P\biggl(\biggl|\frac{\widetilde{S}_i}{\sqrt{j}}\biggr| \ge \rho^{1/4}\biggr) - P\bigl(|R_j| \ge \rho^{1/4}\bigr) \\
&\quad \ge P\biggl(U_i \le x, \sqrt{1-\rho}\frac{\widetilde{S}_j - \widetilde{S}_i}{\sqrt{j-i}} \le y\biggr) - \bigl(4A + O(1) + 1\bigr)\rho^{1/4} \\
&\quad = P(U_i \le x)P\biggl(\sqrt{1-\rho}\frac{\widetilde{S}_j - \widetilde{S}_i}{\sqrt{j-i}} \le y\biggr) - \bigl(4A + O(1) + 1\bigr)\rho^{1/4}.
\end{aligned} \tag{19}$$

We can obtain an analogous upper estimate for the first probability in (19) by the same way. Thus

$$P(U_i \le x, U_j \le y) = P(U_i \le x)P\biggl(\sqrt{1-\rho}\frac{\widetilde{S}_j - \widetilde{S}_i}{\sqrt{j-i}} \le y\biggr) - \theta\bigl(4A + O(1) + 1\bigr)\rho^{1/4},$$

where $|\theta| \le 1$. A similar argument yields

$$P(U_i \le x)P(U_j \le y) = P(U_i \le x)P\biggl(\sqrt{1-\rho}\frac{\widetilde{S}_j - \widetilde{S}_i}{\sqrt{j-i}} \le y\biggr) - \theta'\bigl(4A + O(1) + 1\bigr)\rho^{1/4},$$

where $|\theta'| \le 1$, and (16) follows. Letting $G_{i,j}(x,y)$ denote the joint distribution function of U_i and U_j, in view of (12), (16), we get for $l \ge l_0$

$$\begin{aligned}
&\biggl|\operatorname{cov}\biggl(f(U_i)I\biggl\{f(U_i) \le \frac{k}{(\log k)^\beta}\biggr\}, f(U_j)I\biggl\{f(U_j) \le \frac{l}{(\log l)^\beta}\biggr\}\biggr)\biggr| \\
&\quad = \biggl|\int_{|x|\le a_k}\int_{|y|\le a_l} f(x)f(y)\,d\bigl(G_{i,j}(x,y) - G_i(x)G_j(y)\bigr)\biggr| \\
&\quad \ll \frac{kl}{(\log k)^\beta(\log l)^\beta}2^{-(l-k-1)/4},
\end{aligned}$$

where the last relation follows from the facts that: f is strictly increasing for $x \geq x_0$, $f(a_i) = \frac{i}{(\log i)^\beta}$ and $2^k < i \leq 2^{k+1}$, $2^l < j \leq 2^{l+1}$. Thus

$$\left|\operatorname{cov}\left(\xi_k^*, \xi_l^*\right)\right| \ll \frac{kl}{(\log k)^\beta (\log l)^\beta} 2^{-(l-k-1)/4}. \qquad \square$$

Lemma 7 *Under the conditions of Theorem* 1, *letting* $\zeta_k = \xi_k^* - E\xi_k^*$, *we have*

$$E(\zeta_1 + \cdots + \zeta_N)^2 = O\left(\frac{N^2}{(\log N)^{2\beta-1}}\right), \quad N \to \infty.$$

Proof By Lemma 6, we have

$$\left|\sum_{\substack{1\leq k\leq l\leq N\\ l-k>40\log N}} E(\zeta_k\zeta_l)\right| \ll \frac{N^2}{(\log N)^{2\beta}} N^2 2^{-10\log N} = o(1).$$

On the other hand, letting $\|\cdot\|$ denote the L_2 norm, Lemma 5 and the Cauchy-Schwarz inequality imply

$$\begin{aligned}
\left|\sum_{\substack{1\leq k\leq l\leq N\\ l-k\leq 40\log N}} E(\zeta_k\zeta_l)\right| &\leq \sum_{\substack{1\leq k\leq l\leq N\\ l-k\leq 40\log N}} \|\zeta_k\|\|\zeta_l\| \\
&\leq \sum_{\substack{1\leq k\leq l\leq N\\ l-k\leq 40\log N}} \|\xi_k^*\|\|\xi_l^*\| \\
&= \sum_{0\leq j\leq 40\log N} \sum_{k=1}^{N-j} \|\xi_k^*\|\|\xi_{k+j}^*\| \\
&\leq \left(\sum_{k=1}^{N} \|\xi_k^*\|^2\right)^{1/2} \left(\sum_{l=1}^{N} \|\xi_l^*\|^2\right)^{1/2} 40\log N \\
&= O\left(\frac{N^2}{(\log N)^{2\beta-1}}\right),
\end{aligned}$$

and Lemma 7 is proved. $\square$

4 Proof of the main result

We only prove the property in (6), since, in view of Remark 1, it is sufficient for the proof of Theorem 1.

Proof of Theorem 1 By Lemma 7 we have

$$E\left(\frac{\zeta_1 + \cdots + \zeta_N}{N}\right)^2 = O\left((\log N)^{1-2\beta}\right),$$

and thus setting $N_k = [\exp(k^\lambda)]$ with $(2\beta - 1)^{-1} < \lambda < 1$, we get

$$\sum_{k=1}^{\infty} E\left(\frac{\zeta_1 + \cdots + \zeta_{N_k}}{N_k}\right)^2 < \infty,$$

and therefore

$$\lim_{k\to\infty} \frac{\zeta_1 + \cdots + \zeta_{N_k}}{N_k} = 0 \quad \text{a.s.} \tag{20}$$

Observe now that for $2^k < i \le 2^{k+1}$ we have

$$\begin{aligned} Ef(U_i)I\left\{f(U_i) \le \frac{k}{(\log k)^\beta}\right\} &= \int_{|x|\le a_k} f(x)\,dG_i(x) \\ &= \int_{|x|\le a_k} f(x)\,d\Phi\left(\frac{x}{\sigma_i}\right) + \int_{|x|\le a_k} f(x)\,d\left(G_i(x) - \Phi\left(\frac{x}{\sigma_i}\right)\right). \end{aligned}$$

Put $m = \int_{-\infty}^{\infty} f(x)\,d\Phi(x)$. Since $\sigma_i \le 1$, $\lim_{i\to\infty} \sigma_i = 1$ and $a_k \to \infty$ as $k \to \infty$, we have

$$\lim_{k\to\infty} \sup_{2^k < i \le 2^{k+1}} \left| \int_{|x|\le a_k} f(x)\,d\Phi\left(\frac{x}{\sigma_i}\right) - m \right| = 0,$$

and thus, using (12), we get

$$\left| Ef(U_i)I\left\{f(U_i) \le \frac{k}{(\log k)^\beta}\right\} - m \right| \le \frac{k\eta_i}{(\log k)^\beta} + o_k(1).$$

Thus we have

$$E\xi_k^* = m \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} + \vartheta_k \frac{k}{(\log k)^\beta} \sum_{i=2^k+1}^{2^{k+1}} \frac{\eta_i}{i} + o_k(1), \quad |\vartheta_k| \le 1.$$

Consequently, using the relation $\sum_{i\le L} 1/i = \log L + O(1)$ and (15), we conclude

$$\begin{aligned} \left| \frac{E(\xi_1^* + \cdots + \xi_N^*)}{\log 2^{N+1}} - m \right| &\ll \frac{1}{N} \sum_{k\le N} \frac{k}{(\log k)^\beta} \sum_{i=2^k+1}^{2^{k+1}} \frac{\eta_i}{i} + o_N(1) \\ &= O\big((\log N)^{-\beta}\big) + o_N(1) = o_N(1), \end{aligned}$$

and thus (20) gives

$$\lim_{k\to\infty} \frac{\xi_1^* + \cdots + \xi_{N_k}^*}{\log 2^{N_k+1}} = m \quad \text{a.s.}$$

By Lemma 4 this implies

$$\lim_{k\to\infty} \frac{\xi_1 + \cdots + \xi_{N_k}}{\log 2^{N_k+1}} = m \quad \text{a.s.} \tag{21}$$

The relation $\lambda < 1$ implies $\lim_{k\to\infty} N_{k+1}/N_k = 1$, and thus (21) and the positivity of ξ_k yield

$$\lim_{N\to\infty} \frac{\xi_1 + \cdots + \xi_N}{\log 2^{N+1}} = m \quad \text{a.s.,} \tag{22}$$

i.e., (6) holds for the subsequence $\{2^{N+1}\}$. Now, for each $N \geq 4$, there exists n, depending on N, such that $2^{n+1} \leq N \leq 2^{n+2}$. Then

$$\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{\log 2^{n+1}} \leq \frac{\sum_{i=1}^{N} \frac{1}{i} f(U_i)}{\log N} \frac{\log N}{\log 2^{n+1}} \leq \frac{\xi_1 + \xi_2 + \cdots + \xi_{n+2}}{\log 2^{n+2}} \frac{\log 2^{n+2}}{\log 2^{n+1}} \tag{23}$$

by the positivity of each term of (ξ_k). Noting that $(n+1)\log 2 \sim \log N \sim (n+2)\log 2$ as $N \to \infty$, we get (6) by (22) and (23). $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Physics, Suzhou University of Science and Technology, Suzhou, 215009, P.R. China. [2]College of Mathematics, Physics and Information Engineering, Jiaxing University, Jiaxing, 314001, P.R. China.

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