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On the sum of the two largest Laplacian eigenvalues of trees

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Abstract

For $S(T)$, the sum of the two largest Laplacian eigenvalues of a tree T, an upper bound is obtained. Moreover, among all trees with $n \geq 4$ vertices, the unique tree which attains the maximal value of $S(T)$ is determined.

MSC: 05C50

Keywords: upper bound; tree; Laplacian matrix

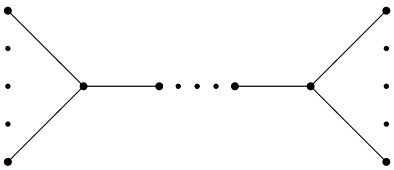
1 Introduction

Let $V(G)$ be the vertex set and $E(G)$ be the edge set of a graph G. The numbers of vertices and edges of G are denoted by $n(G)$ and $m(G)$, respectively. For a vertex $v \in V(G)$, let $N_G(v)$ be the set of vertices adjacent to v and $d_G(v) = |N_G(v)|$ be the degree of v. Particularly, denote by $\Delta(G)$ the maximum degree of G. The diameter of a connected graph G, denoted by $d(G)$, is the maximum distance among all pairs of vertices in G. Let $A(G)$ be the adjacency matrix of G and $D(G)$ be the diagonal matrix of vertex degrees. The matrix $D(G) - A(G)$ is called the *Laplacian matrix* of G and its eigenvalues are called the Laplacian eigenvalues of G. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the Laplacian eigenvalues of a graph G with n vertices. It is well known that $\mu_n = 0$ and $\sum_{i=1}^{n-1} \mu_i = 2m(G)$. In particular, μ_{n-1} is called the algebraic connectivity of G and it is denoted by $\alpha(G)$.

The Laplacian matrix is an important topic in the theory of graph spectra. Particularly, much literature has paid attention to μ_1, μ_2, μ_{n-1} or $\mu_1 - \mu_{n-1}$ for trees (see, for example, [1–6]). Let S_n be the star of order n, $S_{a,b}^k$ be the tree obtained from two stars S_{a+1}, S_{b+1} by joining a path of length k between their central vertices (see Figure 1). As is well known, among all trees of order n, S_n has the largest value of μ_1 (see [7]) and $S_{n-3,1}^1$ has the second largest value of μ_1 (see [6]). On the other hand, Guo [4] proved that these two trees also attain the first two smallest values of μ_2, respectively. This implies that μ_1, μ_2 cannot attain simultaneously the maximal (or minimal) value and even the relation between them seems like a seesaw. Therefore, it is interesting to investigate the value of $\mu_1 + \mu_2$. Moreover, Zhang [6] showed that the $S_{k-1,k-1}^1$, $S_{k-1,k-2}^2$, $S_{k-2,k-2}^3$ attain simultaneously the largest value of μ_2 among all trees with $2k$ vertices. Then Shao *et al.* [5] showed that $S_{k-1,k-1}^2$ attains the largest value of μ_2 among all trees with $2k+1$ vertices.

Another motivation to study the value of $\mu_1 + \mu_2$ came from a result of Haemers *et al.* [8], who showed that $\mu_1 + \mu_2 \leq m(G) + 3$ for any graph G. This result implies that Brouwer's

Figure 1 $S^k_{a,b}$: a tree of order $a+b+k+1$.



conjecture [9],

$$\mu_1+\mu_2+\cdots+\mu_k \le m(G)+\binom{k+1}{2},$$

is true for $k=2$. Considering a tree T, we have $\mu_1+\mu_2 \le n(T)+2$. Recently, Fritscher *et al.* [10] improved this bound by giving $\mu_1+\mu_2 < n(T)+2-\frac{2}{n(T)}$. This paper determines the extremal tree that attains the bound of $\mu_1+\mu_2$. Moreover, for general connected graphs, we also give a conjecture on the extremal graphs for $\mu_1+\mu_2$.

2 A sharp upper bound of $\mu_1+\mu_2$

Let $S_k(G)$ be the sum of the largest k Laplacian eigenvalues of a graph G. When $k=2$, we shall write $S(G)$ instead of $S_k(G)$ for simplicity. For graphs G and H, we denote by $G\cup H$ the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)$. The following lemmas come from an important result as regards a real symmetric matrix.

Lemma 2.1 ([8]) *Let $G_1, G_2, \ldots, G_r$ be some edge-disjoint graphs. Then $S_k(\bigcup_{i=1}^r G_i) \le \sum_{i=1}^r S_k(G_i)$ for any k.*

Lemma 2.2 ([8]) *For any graph G, $S(G) \le m(G)+3$.*

Lemma 2.3 *Let G be a connected graph, $d_i=d_G(v_i)$ and $m_i=\Sigma_{v_j\in N_G(v_i)} d_j/d_i$. Then*

- (i) [11] $\mu_1(G)\ge \Delta(G)+1$, *with equality if and only if* $\Delta(G)=n(G)-1$.
- (ii) [7] $\mu_1(G)\le n(G)$, *with equality if and only if the complement of G is disconnected.*
- (iii) [12] $\mu_1(G)\le \max\{d_i+m_i \mid v_i\in V(G)\}$.

Lemma 2.4 ([6]) *Let T be a tree of order n. If $T\ncong S_n$, then $\mu_1(T)\le \mu_1(S^1_{n-3,1})$, with equality if and only if $T\cong S^1_{n-3,1}$.*

Corollary 2.5 *Let T be a tree with n vertices and diameter $d\ge 3$. Then $\mu_1(T) < n-0.5$.*

Proof Note that any tree T has diameter $d\ge 3$ if $T\ncong S_n$. According to Lemma 2.4, $\mu_1(T)\le \mu_1(S^1_{n-3,1})$. Further, by Lemma 2.3,

$$\mu_1\bigl(S^1_{n-3,1}\bigr) \le \max\{d_i+m_i\} = n-2+\frac{n-1}{n-2} = n-1+\frac{1}{n-2} < n-0.5$$

for $n\ge 5$. For $n=4$, a straightforward calculation shows that $\mu_1(S^1_{1,1}) = 2+\sqrt{2} < 3.5$. $\square$

Lemma 2.6 ([2]) *Let T be a tree of order n and diameter $d\ge 3$. Then $\alpha(T)\ge \alpha(S^{d-2}_{\lceil\frac{n-d+1}{2}\rceil,\lfloor\frac{n-d+1}{2}\rfloor})$, with equality if and only if $T\cong S^{d-2}_{\lceil\frac{n-d+1}{2}\rceil,\lfloor\frac{n-d+1}{2}\rfloor}$.*

Lemma 2.7 ([11]) *Let G be a graph with a vertex u of degree one. Then* $\alpha(G) \leq \alpha(G-u)$.

Lemma 2.7 implies that the algebraic connectivity of a tree is not greater than that of its subtree.

Lemma 2.8 ([4]) *Let* T_n^k $(n \geq 2k+1)$ *be a tree obtained from a star* S_{n-k} *by replacing its k edges with k paths of length two, respectively. If* $k \geq 2$, *then* $\mu_2(T_n^k) = \frac{3+\sqrt{5}}{2}$.

The following lemma can be found in [13] and is known as the Interlacing Theorem of Laplacian eigenvalues.

Lemma 2.9 *Let G be a graph of order n and H be a graph obtained from G by deleting an edge. Then*

$$\mu_1(G) \geq \mu_1(H) \geq \cdots \geq \mu_n(G) \geq \mu_n(H) = 0.$$

Next we give the main theorem of this section. Its proof is divided into several sequent claims.

Theorem 2.10 *For any tree T with order* $n \geq 4$, $S(T) \leq S(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor})$. *The equality holds if and only if* $T \cong S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor}$.

Claim 2.11 *For any tree T with order* $n \geq 4$ *and diameter* $d \leq 3$, $S(T) < S(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor})$ *except that* $T \cong S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor}$.

Proof If $d(T) = 3$, then $T \cong S^1_{a,b}$ for some positive integers a, b with $a + b = n - 2$. It is well known that the Laplacian characteristic polynomial of $S^1_{a,b}$ is $\mu(\mu-1)^{n-4}f_{a,b}(\mu)$, where

$$f_{a,b}(\mu) = \mu^3 - (n+2)\mu^2 + (ab + 2n + 1)\mu - n. \tag{1}$$

Note that $S^1_{a,b}$ contains $S^1_{1,1}$ as a subtree. By Lemma 2.9, $\mu_2(S^1_{a,b}) \geq \mu_2(S^1_{1,1}) = 2$. Moreover, we know that for any tree T, $\alpha(T) \leq 1$, with equality if and only if T is a star. These imply that $\mu_1(S^1_{a,b})$, $\mu_2(S^1_{a,b})$, and $\alpha(S^1_{a,b})$ consist of the three roots of $f_{a,b}(\mu)$. As follows from (1), we have

$$\mu_1(S^1_{a,b}) + \mu_2(S^1_{a,b}) + \alpha(S^1_{a,b}) = n + 2. \tag{2}$$

By virtue of Lemma 2.6, we have $\alpha(S^1_{a,b}) > \alpha(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor})$ except that $(a,b) = (\lceil\frac{n-2}{2}\rceil, \lfloor\frac{n-2}{2}\rfloor)$. Equivalently, $S(S^1_{a,b}) < S(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor})$ except that $(a,b) = (\lceil\frac{n-2}{2}\rceil, \lfloor\frac{n-2}{2}\rfloor)$.

If $d(T) = 2$, then $T \cong S_n$. We first give a lower bound of $S(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor})$ for $n \geq 6$:

$$S(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor}) > n + 1.5. \tag{3}$$

Indeed, by (2) it suffices to show $\alpha(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor}) < 0.5$. Note that for $n \geq 6$, $S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor}$ contains $S^1_{2,2}$ as a subtree. By Lemma 2.7, $\alpha(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor}) \leq \alpha(S^1_{2,2}) = \frac{5-\sqrt{17}}{2} < 0.5$.

Note that $S(S_n) = n + 1$ for $n \geq 3$. According to (3), we have $S(S_n) < S(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor})$ for $n \geq 6$. As for $n \in \{4, 5\}$, a straightforward calculation shows that

$$S(S^1_{1,1}) \approx 5.4142, \qquad S(S^1_{2,1}) \approx 6.4811. \tag{4}$$

Also we have $S(S_n) < S(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor})$. ☐

Claim 2.12 *For any tree T with order n and diameter $d \geq 5$, $S(T) < S(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor})$.*

Proof Since $d(T) \geq 5$, then $n \geq 6$ and there is a path of length 5 in T. By inequality (3), it suffices to show $S(T) \leq n + 1.5$. First suppose that there is a path $v_0v_1 \cdots v_5$ in T such that either $\max\{d_T(v_0), d_T(v_5)\} \geq 2$ or $\max\{d_T(v_2), d_T(v_3)\} \geq 3$. Let T_1, T_2 be the two components of $T - v_2v_3$. Clearly, both T_1 and T_2 have at least two edges.

If μ_1, μ_2 of $T_1 \cup T_2$ attain at the same component, say T_1, then by Lemma 2.2,

$$S(T_1 \cup T_2) = S(T_1) \leq m(T_1) + 3 \leq m(T_1 \cup T_2) + 1. \tag{5}$$

Note that $S(v_2v_3) = S(S_2) = 2$. By Lemma 2.1,

$$S(T) \leq S(T_1 \cup T_2) + S(v_2v_3) \leq m(T_1 \cup T_2) + 3 = m(T) + 2 = n + 1. \tag{6}$$

Otherwise, $S(T_1 \cup T_2) = \mu_1(T_1) + \mu_1(T_2)$. Whether $\max\{d_T(v_0), d_T(v_5)\} \geq 2$ or $\max\{d_T(v_2), d_T(v_3)\} \geq 3$, we can observe that $\max\{d(T_1), d(T_2)\} \geq 3$. Say $d(T_2) \geq 3$, then by Corollary 2.5, $\mu_1(T_2) < n(T_2) - 0.5$. By Lemma 2.3(ii), $\mu_1(T_1) \leq n(T_1)$. Hence,

$$S(T) \leq S(T_1 \cup T_2) + S(v_2v_3) = n(T_1) + n(T_2) - 0.5 + 2 = n + 1.5.$$

Next, we may assume that each path $v_0v_1 \cdots v_5$ of length 5 in T has $d_T(v_0) = d_T(v_5) = 1$ and $d_T(v_2) = d_T(v_3) = 2$. This implies that $d(T) = 5$ and $T \cong S^3_{a,b}$ for some integers a, b with $a + b = n - 4$. If $a = b = 1$, then T is isomorphic to a path of order 6 and a straightforward calculation shows that $S(T) = 5 + \sqrt{3} < n + 1.5$, as claimed. Otherwise, assume without loss of generality that $a \geq 2$. Then $d_T(v_1) \geq 3$. Let T_3, T_4 be the two components of $T - v_1v_2$ with $v_0v_1 \in E(T_3)$. Then both T_3 and T_4 have at least two edges. If μ_1, μ_2 of $T_3 \cup T_4$ attain at the same component, say T_3, then by Lemmas 2.1 and 2.2,

$$S(T) \leq S(T_3 \cup T_4) + S(v_1v_2) = S(T_3) + 2 \leq m(T_3) + 5 \leq m(T) + 2 = n + 1.$$

Otherwise, $S(T_3 \cup T_4) = \mu_1(T_3) + \mu_1(T_4)$. Note that $\mu_1(T_3) \leq n(T_3)$. Since $d(T_4) = 3$, by Corollary 2.5, $\mu_1(T_4) < n(T_4) - 0.5$. So

$$S(T) \leq S(T_3 \cup T_4) + S(v_1v_2) \leq n(T_3) + n(T_4) - 0.5 + 2 = n + 1.5.$$ ☐

Claim 2.13 *For any tree T with order n and diameter 4, $S(T) < S(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor})$.*

Proof First suppose that T contains a path $v_0v_1 \cdots v_4$ such that $\max\{d_T(v_1), d_T(v_3)\} \geq 3$. Now $n \geq 6$ and it suffices to show $S(T) \leq n + 1.5$. Without loss of generality assume that

$d_T(v_1) \geq 3$. Let T_1, T_2 be the two components of $T - v_1v_2$ with $v_0v_1 \in E(T_1)$. Then both T_1 and T_2 have at least two edges.

If μ_1, μ_2 of $T_1 \cup T_2$ attain at the same component, say T_1, then similarly to inequalities (5) and (6), we can observe that $S(T) \leq n + 1$.

Now let $S(T_1 \cup T_2) = \mu_1(T_1) + \mu_1(T_2)$. If $d_T(v_2) \geq 3$, then $d(T_2) \geq 3$ and hence $\mu_1(T_2) < n(T_2) - 0.5$. So

$$S(T) \leq S(T_1 \cup T_2) + S(v_1v_2) < n(T_1) + n(T_2) - 0.5 + 2 = n + 1.5.$$

If $d_T(v_2) = 2$, then $T \cong S^2_{a,b}$ for some positive integers a, b with $3 \leq a + b = n - 3$. Moreover, since $d_T(v_1) \geq 3$, then $a \geq 2$. If $(a,b) \in \{(2,1),(3,1)\}$, a straightforward calculations show that $S(S^2_{a,b}) < n + 1.5$. Otherwise, $S^2_{a,b}$ contains either $S^2_{4,1}$ or $S^2_{2,2}$ as a subtree. Since

$$\mu_3\bigl(S^2_{4,1}\bigr) \approx 1.5068, \qquad \mu_3\bigl(S^2_{2,2}\bigr) \approx 1.5858,$$

it follows from Lemma 2.9 that $\mu_3(S^2_{a,b}) > 1.5$. Since $S^2_{a,b}$ is not a star, $\mu_{n-1}(S^2_{a,b}) < 1$. On the other hand, note that the matrix $1 \cdot I_n - [D(S^2_{a,b}) - A(S^2_{a,b})]$ has a identical rows and b different identical rows, so the multiplicity of eigenvalue 1 is at least $a + b - 2$ and else five eigenvalues are μ_1, μ_2, μ_3, μ_{n-1} and $\mu_n = 0$. Since $\sum_{i=1}^n \mu_i(S^2_{a,b}) = 2(n-1)$, we have

$$\sum_{i=1}^{3} \mu_i\bigl(S^2_{a,b}\bigr) + \mu_{n-1}\bigl(S^2_{a,b}\bigr) + \mu_n\bigl(S^2_{a,b}\bigr) = 2(n-1) - (a+b-2) = n + 3.$$

This implies that $S(S^2_{a,b}) < n + 3 - \mu_3(S^2_{a,b}) < n + 1.5$.

Next, it suffices to consider the case that each path $v_0v_1 \cdots v_4$ of T has $d_T(v_1) = d_T(v_3) = 2$. This implies that $T \cong T^k_n$ for some $k \geq 2$ and $n \geq 2k + 1$, since $d(T) = 4$. According to Lemma 2.8, $\mu_2(T^k_n) = \frac{3+\sqrt{5}}{2}$. Moreover, by Lemma 2.3,

$$\mu_1\bigl(T^k_n\bigr) \leq \max\{d_i + m_i\} = n - k - 1 + \frac{n-1}{n-k-1} \leq n - 2 + \frac{2}{n-3}.$$

Thus for $n \geq 6$,

$$S\bigl(T^k_n\bigr) \leq n - 2 + \frac{2}{n-3} + \frac{3+\sqrt{5}}{2} < n + 1.5 < S\bigl(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor}\bigr).$$

When $n = 5$, T^k_n is a path. Comparing with (4), $S(T^k_n) = 4 + \sqrt{5} < S(S^1_{2,1})$. This completes the proof. $\square$

Following from Claims 2.11-2.13, Theorem 2.10 holds and the unique tree with maximal $S(T)$ is $S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor}$. According to (2),

$$\mu\bigl(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor}\bigr) < n + 2 = m\bigl(S^1_{\lceil\frac{n-2}{2}\rceil,\lfloor\frac{n-2}{2}\rfloor}\bigr) + 3.$$

Theorem 2.14 *Let m, n be two positive integers with $n \leq m \leq 2n - 3$ and $G_{m,n}$ be a graph of order n and size m obtained from a given edge uv by joining $m - n + 1$ independent vertices with u and v, respectively, and another $2n - m - 3$ independent vertices with u. Then $S(G_{m,n}) = m + 3$.*

Proof Let $H_{s,t}$ be a graph obtained by joining a vertex to s vertices of a given complete graph of order $s + t$ and $H_{s,t}^c$ be its complement graph. Then $H_{s,t}^c$ is isomorphic to the union of S_{t+1} and s isolated vertices. Clearly, the Laplacian eigenvalues of $H_{s,t}^c$ consist of $t+1$, 1 with multiplicity $t-1$ and 0 with multiplicity $s+1$. Recall that for any graph G with n vertices, $\mu_i(G) = n - \mu_{n-i}(G^c)$ for $1 \leq i \leq n-1$ and $\mu_n(G) = 0$. So the Laplacian eigenvalues of $H_{s,t}$ consist of $s + t + 1$ with multiplicity s, $s + t$ with multiplicity $t - 1$, s and 0.

Now $G_{m,n}^c$ is isomorphic to the union of $H_{2n-m-3,m-n+1}$ and an isolated vertex. So the Laplacian eigenvalues of $G_{m,n}^c$ consist of $n - 1$ with multiplicity $2n - m - 3$, $n - 2$ with multiplicity $m - n$, $2n - m - 3$, and 0 with multiplicity 2. Therefore, the Laplacian eigenvalues of $G_{m,n}$ consist of n, $m - n + 3$, 2 with multiplicity $m - n$, 1 with multiplicity $2n - m - 3$ and 0. So $S(G_{m,n}) = n + (m - n + 3) = m + 3$. □

Recall that $\mu_1(G) \leq n(G)$ for any graph G. When $m(G) > 2n(G) - 3$, Haemers' bound is clearly not attainable. Theorem 2.14 implies that if $m(G) \leq 2n(G) - 3$, Haemers' bound is always sharp for connected graphs other than trees. Ending the paper, we present a conjecture on the uniqueness of the extremal graph.

Conjecture 2.15 *Among all connected graphs with n vertices and $n \leq m \leq 2n - 3$ edges, $G_{m,n}$ is the unique graph with maximal value of $\mu_1 + \mu_2$.*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

MG carried out the proofs of the main results in the manuscript. MQZ and YFW participated in the design of the study and drafted the manuscript. All authors read and approved the final manuscript.

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