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Maximal regularity properties of Agranovich-Vishik type abstract elliptic operators in the half-plane

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Abstract

In this work, Agranovich-Vishik type abstract elliptic operators in the half-plane are studied. We derive maximal regularity properties of these operators in UMD-valued Sobolev spaces. Our main aim is to prove existence and uniqueness theorems for the solution of abstract elliptic equation with regular boundary conditions on these function spaces. First, by applying the Fourier multiplier, we prove the separability properties of this differential operator in $\mathbb{R}^n$. By using the embedding theorem and the trace theorem, we obtain the main result.

Keywords: maximal regularity; elliptic operators; L^p Fourier multiplier; embedding in Sobolev spaces; trace in Sobolev spaces

1 Introduction

Boundary value problems (BVPs) for differential-operator equations (DOEs) in abstract spaces have been studied extensively by many researchers [1–11]. The maximal regularity properties for partial differential equations (PDEs), and, particularly, for elliptic equations have been studied in [2–4, 12–15]. The main objective of the present paper is to discuss the BVP for a general elliptic equation with complex parameter in a Banach space-valued Sobolev class. Regularity properties in parameter dependent elliptic equations were derived in [14] for a polynomial dependence, and in [3, 13, 14] for the case of linear dependence of a complex parameter. Here, the complex parameter is included polynomially in the principal part of the equation.

Consider, on $\mathbb{R}^n$, the following differential operator:

$$A(D,q) = \sum_{|\alpha|+\beta\leq l} a_{\alpha,\beta} q^{\beta} D^{\alpha}, \tag{1}$$

depending polynomially on a complex parameter q, of order l with constant complex coefficients where $\beta \in \mathbb{N}\cup\{0\}$.

We firstly consider the following equation in the whole space:

$$A(D,q)u = \sum_{|\alpha|+\beta\leq l} a_{\alpha,\beta} q^{\beta} D^{\alpha} u = f(x), \quad x = (x_1,\ldots,x_n) \in \mathbb{R}^n, \tag{2}$$

then we prove the separability properties of differential operator (1) in the Bochner space $L_p(\mathbb{R}^n;E)$ (*i.e.*, E-valued L_p spaces where E is a Banach space). In particular, the existence and uniqueness of maximal regular solution is derived. In addition, we derive the uniformly coercive estimation of the solution in the space $L_p(\mathbb{R}^n;E)$.

Let $\mathbb{R}^n_+ = \{x : x \in \mathbb{R}^n, x_n > 0\}$ and $\mathbb{R}^n_- = \{x : x \in \mathbb{R}^n, x_n < 0\}$. We denote $x = (x', x_n) = (x_1, \dots, x_n)$. Consider the following BVP:

$$A(D,q)u = \sum_{|\alpha|+\beta\leq l} a_{\alpha,\beta} q^\beta D^\alpha u = f(x), \quad x = (x_1,\dots,x_n) \in \mathbb{R}^n_+, \tag{3}$$

$$B_j(x',D,q)u = \sum_{|\alpha|+\beta\leq 2m} b_{\alpha\beta} q^\beta D^\alpha u|_{x_n=0} = g_j(x'), \quad j = 1,\dots,m. \tag{4}$$

g_j are trace functions defined on $\mathbb{R}^{n-1}$. The boundary operators B_j are subject to an algebraic condition which we call Condition II (see Section 3). Then we prove the isomorphism theorem (algebraic and topological) of problem (3)-(4) between E-valued Sobolev type spaces $W^{l,p}(\mathbb{R}^n_+;E)$ and $W^{l-2m,p}(\mathbb{R}^n_+;E) \times \prod_{j=1}^m B_{p,p}^{l-m_j-\frac{1}{p}}(\mathbb{R}^{n-1};E)$ $(j = 1,\dots,m)$. Since E is an arbitrary UMD space, the maximal regularity properties of various class of elliptic BVPs is obtained by choosing a different E. This condition, when $q = 1$, becomes the well-known condition of Shapiro-Lopatinskii [16, 17], which is often also called the ellipticity condition for problem (3)-(4). When $q = 1$, the BVP (3)-(4) is considered in a bounded domain with sufficiently smooth boundary, satisfying the complementing condition for all properly elliptic differential operators in [18, 19]. In addition, Agranovich-Vishik worked out problem (3)-(4) in a half-plane and domain with sufficiently smooth boundary [14]. Extensive references can be found in [5] (see also [17]).

While studying the elliptic operator depending on a parameter q, it will be convenient for us to use norms depending on the parameter. We put

$$|||u|||_{l,p} = \left(\|u\|^p_{l,p} + |q|^{pl}\|u\|^p_{0,p}\right)^{\frac{1}{p}}. \tag{5}$$

For any fixed q, the norms $\|u\|_{l,p}$ and $|||u|||_{l,p}$ are clearly equivalent.

2 Notation and background

The notation follows the usual standard. Let E be a Banach space and $L_p(\Omega;E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|u\|_{L^p(\Omega,E)} = \left(\int_\Omega \|u\|^p_E \, dx\right)^{\frac{1}{p}} < \infty. \tag{6}$$

The Banach space E is often called a *UMD space* if the Hilbert operator

$$(Hf)(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, dy$$

is bounded in $L^p(\Omega,E)$ for $0 < p < \infty$.

The term 'UMD' is an abbreviation for 'unconditional martingale differences'. UMD spaces include spaces such as L^p, l_p for $p \in (1,\infty)$.

Let l be an integer ≥ 1. The E-valued *Sobolev space*, $W^{l,p}(\Omega;E)$, of order l on Ω is defined by

$$W^{l,p}(\Omega;E) = \left\{ u \,\middle|\, u \in L^p(\Omega;E), D^\alpha u \in L^p(\Omega;E), |\alpha| \leq l, \right. \\ \left. \|u\|_{l,p} = \|u\|_{W^{l,p}(\Omega;E)} = \sum_{|\alpha|\leq l} \left\| D^\alpha u \right\|_{L^p(\Omega;E)} < \infty \right\}, \tag{7}$$

where

$$D^\alpha = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}, \quad \alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

We shall set

$$L^p(\Omega,E) = W^{0,p}(\Omega,E).$$

By $S = S(\mathbb{R}^n;E)$, we denote the *Schwartz space* of rapidly decreasing smooth functions. Let F denote the E-valued Fourier transform and let $1 < p < \infty$. A function $\psi \in C(\mathbb{R}^n)$ is called a *multiplier* from $L^p(\mathbb{R}^n;E_1)$ to $L^p(\mathbb{R}^n;E_2)$ if the mapping $f \to T_\psi f = F^{-1}\psi(\xi)Ff$ for $f \in S(\mathbb{R}^n;E_1)$ is well defined and provided that there is a constant C so that

$$\|T_\psi f\|_{L^p(\mathbb{R}^n;E_2)} \leq C\|f\|_{L^p(\mathbb{R}^n;E_1)}.$$

The norm of ψ in $M_p(E)$ is defined by

$$\|\psi\|_{M_p(E)} = \|T_\psi\|_{L^p(\mathbb{R}^n;E_1)\to L^p(\mathbb{R}^n;E_1)}.$$

Let m_i be positive integers, k_i be non-negative integers, s_i be positive numbers and $m_i > s_i - k_i > 0$, $i = 1,2,\ldots,n$, $s = (s_1,s_2,\ldots,s_n)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $0 < y_0 < \infty$. The E-valued *Besov spaces* $B^s_{p,q}(\Omega;E)$ are defined as

$$B^s_{p,q}(\Omega;E) = \left\{ f : f \in L_p(\Omega;E), \|f\|_{B^s_{p,q}(\Omega;E)} = \|f\|_{L_p(\Omega;E)} \right. \\ \left. + \sum_{i=1}^n \left(\int_0^{h_0} h^{-[(s_i-k_i)q+1]} \left\| \Delta_i^{m_i}(h,\Omega) D_i^{k_i} f \right\|^q_{L_p(\Omega;E)} \, dh \right)^{\frac{1}{q}} < \infty \right\}$$

for $1 \leq q < \infty$. For $q = \infty$, recall that

$$\|f\|_{B^s_{p,\infty}(\Omega;E)} = \sum_{i=1}^n \sup_{0<h<h_0} \frac{\|\Delta_i^{m_i}(h,\Omega)D_i^{k_i} f\|_{L_p(\Omega;E)}}{h^{s_i-k_i}}.$$

The definition of $B^s_{p,q}(\Omega;E)$ is independent of m_i and k_i.

Let Ω be a domain in $\mathbb{R}^n$. A linear operator T mapping $W^{m,p}(\Omega;E)$ into $W^{m,p}(\mathbb{R}^n;E)$ is called an *extension* for Ω provided that, for every $u \in W^{m,p}(\Omega;E)$, the equality $Tu(x) = u(x)$ holds a.e. in Ω, and for each m there is a constant K such that

$$\|Tu\|_{W^{m,p}(\mathbb{R}^n;E)} \leq K\|u\|_{W^{m,p}(\Omega;E)}. \tag{8}$$

Now, we shall give some theorems which will be used to prove the maximal regularity properties of elliptic differential equations in E-valued Sobolev spaces.

By using [6] and an interpolation of Banach spaces (see [3–8, 11, 16, 17]), we obtain the following.

Theorem 1 (Trace theorem) *Let E be a UMD space. Then the transformations*

$$u \to u_{x_n}^j(x',0), \quad j = 0,1,\ldots,m-1,$$

are bounded, linear, and surjective from $W^{m,p}(R_+^n;E)$ into $B_{p,p}^{m-j-\frac{1}{p}}(\mathbb{R}^{n-1};E)$.

Proof It is clear that

$$W^{m,p}(R_+^n;E) = W^{m,p}(R_+;W^{m,p}(\mathbb{R}^{n-1};E),L^p(\mathbb{R}^{n-1};E)).$$

Then by the virtue of the trace theorem of [6], the operator

$$u \to u_{x_n}^j(x',0)$$

will be linear, bounded, and surjective from $W^{m,p}(R_+^n;E)$ into $(W^{m,p}(\mathbb{R}^{n-1};E),L^p(\mathbb{R}^{n-1};E))_{\frac{j+\frac{1}{p}}{m},p}$.

It is well known that (see, for example, [16])

$$\left(W^{m,p}(\mathbb{R}^{n-1};E),L^p(\mathbb{R}^{n-1};E)\right)_{\frac{j+\frac{1}{p}}{m},p} = B_{p,p}^{m-j-\frac{1}{p}}(\mathbb{R}^{n-1};E).$$

This completes the proof. □

Theorem 2 (Extension theorem) *Let E be a UMD-space. Then there exists a bounded linear extension operator from $W^{m,p}(\mathbb{R}_+^n;E)$ to $W^{m,p}(\mathbb{R}^n;E)$.*

Proof By virtue of [5], the restriction to $\mathbb{R}_+^n$ of a function in $C_0^\infty(\mathbb{R}^n;E)$ is dense in $W^{m,p}(\mathbb{R}_+^n;E)$ for any m and p. So, we define the extension operator T only for such functions. Let $\psi \in C^\infty([0,\infty))$ be a real-valued function satisfying $\psi(t) = 1$ if $t \in [0,\frac{1}{2}]$, $\psi(t) = 0$ if $t \geq 1$. Let $u \in C_0^\infty(\mathbb{R}^n;E)$. Then we set

$$Tu(x) = \begin{cases} u(x), & \text{if } x \in \overline{\mathbb{R}}_+^n, \\ \sum_{k=0}^\infty a_k f(-2^k x_n)u(x',-2^k x_n), & \text{if } x \in \mathbb{R}_-^n, \end{cases}$$

where a_k is a sequence defined as follows:

$$\sum_{k=0}^\infty 2^{nk} a_k = (-1)^n, \quad n \geq 0 \tag{9}$$

and

$$\sum_{k=0}^\infty 2^{nk} |a_k| < \infty. \tag{10}$$

It is clear that Tu has compact support in $C^\infty(\overline{\mathbb{R}}_+^n;E)\cap C^\infty(\overline{\mathbb{R}}_-^n;E)$. If $x\in\mathbb{R}_-^n$, we have

$$D^\alpha Tu(x)=\sum_{k=0}^{\infty}a_k\sum_{j=0}^{\alpha_n}\binom{\alpha_n}{j}\left(-2^k\right)^{\alpha_n}f^{\alpha_n-j}\left(-2^kx_n\right)D_n^jD^{\alpha'}u\left(x',-2^kx_n\right),$$

which we write

$$D^\alpha Tu(x)=\sum_{k=0}^{\infty}\Psi_k(x). \tag{11}$$

Since $\Psi_k(x)=0$ when $-x_n>\frac{1}{2^{k-1}}$, it follows from (10) that the above series converges absolutely and uniformly as x_n tends to zero. Using (11), we get

$$\lim_{x_n\to0^-}D^\alpha Tu(x)=\lim_{x_n\to0^+}D^\alpha Tu(x)=D^\alpha Tu(0).$$

Hence, we say that $D^\alpha Tu\in C_0^\infty(\mathbb{R}^n;E)$. Moreover, if $|\alpha|\le m$,

$$\left\|\Psi_k(x)\right\|_E^p\le K_1^p|a_k|^p2^{kmp}\sum_{|\beta|\le m}\left\|D^\beta u\left(x',-2^kx_n\right)\right\|_E^p,$$

where K_1 depends on m, p, n, and f. Integrating over $\mathbb{R}_-^n$,

$$\left\{\int_{\mathbb{R}_-^n}\|\Psi_k\|_E^p\right\}^{\frac{1}{p}}\le K_1|a_k|2^{km}\left\{\left(\frac{1}{2^k}\right)\sum_{|\beta|\le m}\int_{\mathbb{R}_-^n}\left\|D^\beta u\left(x',-2^kx_n\right)\right\|_E^p\,dx\right\}^{\frac{1}{p}}.$$

Hence

$$\begin{aligned}\|\Psi_k\|_{W^{0,p}(\mathbb{R}_-^n;E)}&=K_1|a_k|2^{km}\left\{\sum_{|\beta|\le m}\int_{\mathbb{R}_+^n}\left\|D^\beta u(y)\right\|_E^p\,dy\right\}^{\frac{1}{p}}\\&=K_1|a_k|2^{km}\|u\|_{W^{m,p}(\mathbb{R}_+^n;E)}.\end{aligned} \tag{12}$$

It follows from (10)-(11) that

$$\left\|D^\alpha Tu\right\|_{L^p(\mathbb{R}_-^n;E)}\le K_1\|u\|_{W^{m,p}(\mathbb{R}_+^n;E)}\sum_{k=0}^{\infty}|a_k|2^{km}\le K_2\|u\|_{W^{m,p}(\mathbb{R}_+^n;E)}.$$

It is obvious that

$$\left\|D^\alpha Tu\right\|_{L^p(\mathbb{R}_+^n;E)}\le K_3\|u\|_{W^{m,p}(\mathbb{R}_+^n;E)}.$$

Combining these, we obtain

$$\|Tu\|_{W^{m,p}(\mathbb{R}^n;E)}\le K_4\|u\|_{W^{m,p}(\mathbb{R}_+^n;E)},$$

with $K=K(m,p,n)$. Thus, the proof is finished. □

By virtue of [8] we state the following theorems.

Theorem 3 (Embedding theorem) *Let E be a UMD space and* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| \le m$, $0 \le \mu \le \frac{|\alpha|}{m}$. *Then the embedding*

$$D^\alpha W^{m,p}(\mathbb{R}^n_+; E) \hookrightarrow L^p(\mathbb{R}^n; E) \tag{13}$$

is continuous, and for all $u \in W^{m,p}(\mathbb{R}^n_+; E)$ *and* $0 < h < h_0$, *the following estimate holds*:

$$\|D^\alpha u\|_{L^p(\mathbb{R}^n_+;E)} \le h^\mu \|u\|_{W^{m,p}(\mathbb{R}^n_+;E)} + h^{-(1-\mu)} \|u\|_{L^p(\mathbb{R}^n;E)}. \tag{14}$$

Theorem 4 (Fourier multiplier theorem) *Let E be a UMD space and* ψ *be a* C^{n+1} *function defined on* $\mathbb{R}^n$. *Assume there is a constant* c_ψ *such that for all multi-indices* α *satisfying* $|\alpha| < n+1$ *we have*

$$|x|^{|\alpha|} |D^\alpha \psi(x)| < c_\psi. \tag{15}$$

Then for $1 < p < \infty$ *the operator* T_ψ, *which is defined as* $F^{-1}\psi F$, *has an extension to* $L^P(\mathbb{R}^n; E)$ *which satisfies*

$$\|T_\psi f\|_{L^p} \le c_p \|f\|_{L^p}.$$

The constants c_p *depend only on* c_ψ, n, *and* p (*see* [20]).

3 Elliptic problem in $\mathbb{R}^n$

We shall consider the equation

$$A(D,q)u(x,q) = f(x,q) \tag{16}$$

in the whole space $\mathbb{R}^n$, where $A(D,q)$ is a differential operator with constant complex coefficients depending polynomially on a complex parameter q in such a way that, after replacing $D = (D_1, \ldots, D_n)$ by $\xi = (\xi_1, \ldots, \xi_n)$, we get a homogeneous polynomial $A(\xi, q)$ of degree s. Here, s is a non-negative integer. We symbolize it as

$$A(D,q) = \sum_{|\alpha|+\beta \le s} a_{\alpha\beta} q^\beta D^\alpha \tag{17}$$

and the symbol of the operator is as follows:

$$A(\xi,q) = \sum_{|\alpha|+\beta \le s} a_{\alpha\beta} q^\beta \xi^\alpha.$$

The parameter varies among the limits of a closed sector Q of the complex plane, with vertex at the origin of coordinates. That is,

$$Q = \{q : \alpha \le \arg q \le \beta\}.$$

We begin our analysis by proving Proposition 1 with the help of Theorem 3.

Proposition 1 *Let E be a UMD space. For $l \geq s$, the operator $A(D,q)$ is a bounded operator from $W^{l,p}(\mathbb{R}^n;E)$ to $W^{l-s,p}(\mathbb{R}^n;E)$. More precisely, we have the following estimate*:

$$\|Au\|_{W^{l-s,p}} + |q|^{l-s}\|Au\|_{L^p} \leq C_2'\left\{\left\|u(x,q)\right\|_{W^{l,p}} + |q|^l\|u\|_{L^p}\right\}, \tag{18}$$

where the constants C_1' and C_2' are independent of q and u.

Proof Let $u \in W^{l,p}(\mathbb{R}^n;E)$ be any function. By taking the norm of $A(D,q)u$, and using (5) and interpolation, we obtain

$$\begin{aligned}
|||Au|||_{l-s,p} &= \|Au\|_{l-s,p} + |q|^{l-s}\|Au\|_{0,p} \\
&= \sum_{|\gamma|\leq l-s}\left\|\sum_{|\alpha|+\beta\leq s} a_{\alpha\beta}q^\beta D^{\alpha+\gamma}u\right\|_{0,p} + |q|^{l-s}\left\|\sum_{|\alpha|+\beta\leq s} a_{\alpha\beta}q^\beta D^\alpha u\right\|_{0,p} \\
&= \sum_{|\gamma|\leq l-s}\left\|a_{s,0}D^{s+\gamma}u + \cdots + a_{0,s}q^s u\right\|_{0,p} \\
&\quad + |q|^{l-s}\left\|a_{s,0}D^s u + \cdots + a_{0,s}q^s u\right\|_{0,p} \\
&\leq \sum_{|\gamma|\leq l-s}\left\|a_{s,0}D^{s+\gamma}u\right\|_{0,p} + \cdots + \sum_{|\gamma|=l-s}\left\|a_{0,s}q^s u\right\|_{0,p} \\
&\quad + |q|^{l-s}\left\{\left\|a_{s,0}D^s u\right\|_{0,p} + \cdots + \left\|a_{0,s}q^s u\right\|_{0,p}\right\} \\
&\leq \sum_{|\gamma|\leq l}\left\|a_{s,0}D^\gamma u\right\|_{0,p} + \cdots + |q|^l\|a_{0,s}u\|_{0,p}.
\end{aligned}$$

Since the coefficients of the operator A are constants, by using Theorem 3, we obtain

$$|||Au|||_{l-s,p}(\mathbb{R}^n;E) \leq C\left\{\sum_{|\gamma|\leq l}\left\|D^\gamma u\right\|_{0,p} + \cdots + |q|^l\|u\|_{0,p}\right\} = C|||u|||_{l,p}(\mathbb{R}^n;E),$$

where C is chosen such that $C = \max\{a_{j,k} : j,k = 1,\ldots,m\}$. This completes the proof. □

We require the following conditions.

Condition I

(a) If $|\xi| + |q| \neq 0$, then $A(\xi,q) \neq 0$ for all $q \in Q$ and $\xi \in \mathbb{R}^n$ ($\xi \neq 0$).
(b) We suppose $A(\xi,q) \geq M(|\xi|^s + |q|^s)^{-1}$ holds for all $q \in Q$ and $\xi \in \mathbb{R}^n$ ($\xi \neq 0$), where M is a constant.

The main conclusion of this section is the following result.

Theorem 5 *Suppose that Condition* I *is satisfied, $l \geq s$, and $0 < p < \infty$. Then for $f \in W^{l-s,p}(\mathbb{R}^n;E)$ there exists one and only one solution $u \in W^{l,p}(\mathbb{R}^n;E)$ of problem* (16). *Moreover, the coercive uniform estimate holds*:

$$|||u|||_{W^{l,p}} \leq C_1|||f|||_{W^{l-s,p}} \tag{19}$$

for $|q| > q_0$ where the constant C_1 is independent of q and u.

Proof First of all, we will prove that there is a solution $u \in W^{l,p}(\mathbb{R}^n; E)$. Consider the equation

$$A(D,q)u(x,q) = \sum_{|\alpha|+\beta\leq s} a_{\alpha\beta} q^{\beta} D^{\alpha} u(x,q) = f(x,q). \tag{20}$$

Applying the Fourier transformation F to both sides of (20), we get

$$A(\xi,q)Fu(x,q) = Ff(x,q), \qquad A(\xi,q) = \sum_{|\alpha|+\beta=s} a_{\alpha\beta} q^{\beta} \xi^{\alpha}. \tag{21}$$

By Condition I, for a non-vanishing $q \in Q$, $\xi \in \mathbb{R}^n$ we have $A(\xi,q) \neq 0$. Hence, from (21) we obtain

$$\hat{u}(\xi,q) = A^{-1}(\xi,q)\hat{f}(\xi,q).$$

Now, we rewrite the inequality (19) as follows:

$$\sum_{|\alpha|\leq l} \|D^{\alpha}u\|_{L^p} + |q|^l \|u\|_{L^p} \leq C\left\{ \sum_{|\beta|\leq l-s} \|D^{\beta}f\|_{L^p} + |q|^{l-s}\|f\|_{L^p} \right\}.$$

Moreover, by using the Fourier transformation, we see that the above estimate is equivalent to

$$\sum_{|\alpha|\leq 1} \|F^{-1}(i\xi)^{\alpha}\hat{u}\|_{L_p} + |q|^l \|F^{-1}\hat{u}\|_{L_p} \leq C\left\{ \sum_{|\beta|\leq l-s} \|F^{-1}(i\xi)^{\beta}\hat{f}\|_{L_p} + |q|^{l-s}\|F^{-1}\hat{f}\|_{L_p} \right\}.$$

Replacing $\hat{u}$ with $A^{-1}(\xi,q)\hat{f}$ we obtain

$$\begin{aligned} &\sum_{|\alpha|\leq l} \|F^{-1}(i\xi)^{\alpha}A^{-1}(\xi,q)\hat{f}\|_{L^p} + |q|^l \|F^{-1}A^{-1}(\xi,q)\hat{f}\|_{L^p} \\ &\quad \leq C\left\{ \sum_{|\beta|\leq l-s} \|F^{-1}(i\xi)^{\beta}\hat{f}\|_{L^p} + |q|^{l-s}\|F^{-1}\hat{f}\|_{L^p} \right\}. \end{aligned}$$

We have to verify the following inequality to finish the proof:

$$\begin{aligned} &\sum_{|\alpha|\leq l} \|F^{-1}(i\xi)^{\alpha}A^{-1}(\xi,q)\hat{f}\|_{L^p} + |q|^l \|F^{-1}A^{-1}(\xi,q)\hat{f}\|_{L^p} \\ &\quad \leq C\left\| F^{-1}\left(\sum_{|\beta|\leq l-s} (i\xi)^{\beta} + q^{l-s} \right)\hat{f} \right\|_{L^p}. \end{aligned}$$

Let us rewrite the inequalities as follows:

$$\begin{aligned} &\left\| F^{-1} \frac{\sum_{|\alpha|\leq l}(i\xi)^{\alpha}A^{-1}(\xi,q)}{\sum_{|\beta|\leq l-s}(i\xi)^{\beta} + q^{l-s}} \left(\sum_{|\beta|\leq l-s} (i\xi)^{\beta} + q^{l-s} \right)\hat{f} \right\|_{L^p(\mathbb{R}^n;E)} \\ &\qquad + \left\| F^{-1} \frac{|q|^l A^{-1}(\xi,q)}{\sum_{|\beta|\leq l-s}(i\xi)^{\beta} + q^{l-s}} \left(\sum_{|\beta|\leq l-s} (i\xi)^{\beta} + q^{l-s} \right)\hat{f} \right\|_{L^p(\mathbb{R}^n;E)} \\ &\quad \leq C\left\| F^{-1}\left(\sum_{|\beta|\leq l-s} (i\xi)^{\beta} + q^{l-s} \right)\hat{f} \right\|_{L^p(\mathbb{R}^n;E)}. \end{aligned}$$

Define the following functions:

$$\psi_1(\xi,q) = |q|^l A^{-1}(\xi,q)\left(\sum_{|\beta|\leq l-s}(i\xi)^\beta + q^{l-s}\right)^{-1},$$

$$\psi_2(\xi,q) = \sum_{|\alpha|\leq l}(i\xi)^\alpha A^{-1}(\xi,q)\left(\sum_{|\beta|\leq l-s}(i\xi)^\beta + q^{l-s}\right)^{-1}.$$

In order to prove the above estimate, we have to show that the functions $\psi_1(\xi,q)$ and $\psi_2(\xi,q)$ are Fourier multipliers in $L^p(\mathbb{R}^n;E)$. By applying the multiplier theorem (see, for example, [3, 20]), we will show that $\psi_k \in C^{n+1}(\mathbb{R}^n;B(E))$, $k=1,2$, for $|\alpha|\leq n+1$. That is,

$$|\xi|^\alpha \left|D^\alpha \psi_i(\xi,q)\right| \leq C_{\psi_i}, \quad i=1,2. \tag{22}$$

For every $k\in\mathbb{N}$ there exists a constant $C_{n,k}$ such that

$$|\xi|^k \leq C_{n,k}\sum_{|\beta|=k}\left|\xi^\beta\right|, \tag{23}$$

where $\xi=(\xi_1,\ldots,\xi_n)$.

Let $|\alpha|=0$. Using the inequality (23) and Condition I(a) we find that

$$|q|^l \leq C_1\left|A(\xi,q)\right|\left(|\xi|^{l-s}+|q|^{l-s}\right), \tag{24}$$

where C_1 is a constant depending on ψ_1. Similarly, we can apply the same process to $|\psi_2(\xi,q)|\leq C_2$ to obtain

$$|\xi|^l \leq C_2\left|A(\xi,q)\right|\left(|\xi|^{l-s}+|q|^{l-s}\right), \tag{25}$$

where C_2 is also a constant depending on ψ_2. If we choose $C'=\max\{C_1,C_2\}$, we obtain the following inequality from (24)-(25):

$$\frac{|\xi|^{l-s}|q|^{l-s}}{(|\xi|^{l-s}+|q|^{l-s})^2}\left(|\xi|^s+|q|^s\right) \leq C'\left|A(\xi,q)\right|. \tag{26}$$

Because ξ and q are not zero at the same time, and $\frac{|\xi|^{l-s}|q|^{l-s}}{(|\xi|^{l-s}+|q|^{l-s})^2}$ is bounded as ξ and q tend to infinity, we can write

$$\left(|\xi|^s+|q|^s\right) \leq C\left|A(\xi,q)\right|, \tag{27}$$

where C is a constant that does not dependent on ξ or q. That is, we get $|\psi_1(x)|\leq C_1$ and $|\psi_2(x)|\leq C_2$. Let $\alpha_1=\alpha_2=\cdots=\alpha_{k-1}=\alpha_{k+1}=\cdots=\alpha_n=0$, and $\alpha_k=1$. Then, by using the boundedness of ψ_1, we obtain

$$\begin{aligned}&|\xi_k|\left|\frac{\partial}{\partial\xi_k}\psi_1(\xi,q)\right| \\ &\quad\leq |\xi_k||q|^l\left|\frac{\partial}{\partial\xi_k}\left(\sum_{\substack{|\alpha|\leq l\\ \beta\leq s}} c_{\alpha\beta}q^\beta\xi'^{\alpha'}\xi_k^{\alpha_k}\right)^{-1}\right|\end{aligned}$$

$$= |q|^l \left| \left(\alpha_k \sum_{\substack{|\alpha| \leq l \\ \beta \leq s}} c_{\alpha\beta} q^\beta \xi'^{\alpha'} \xi_k^{\alpha_k} \right) \left(\sum_{\substack{|\alpha| \leq l \\ \beta \leq s}} c_{\alpha\beta} q^\beta \xi'^{\alpha'} \xi_k^{\alpha_k} \right)^{-1} \right|$$

$$= \alpha_k |q|^l \leq C_1,$$

where $\xi' = (\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n)$ and $\alpha' = (\alpha_1, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_n)$.

In a similar way, we obtain the above estimate for all α with $|\alpha| \leq n + 1$. Then we have $|\xi|^\alpha |D^\alpha \psi_1(x)| \leq C_1$. Moreover, applying the same operations we obtain $|\xi|^\alpha |D^\alpha \psi_2(c)| \leq C_2$. That is, the functions ψ_1, ψ_2 are multipliers in $L^p(\mathbb{R}^n; E)$. Hence, we find that there is a solution $u \in W^{l,p}(\mathbb{R}^n; E)$ for $f \in W^{l-s,p}(\mathbb{R}^n; E)$ and the coercive estimate (19) holds.

Finally, to show that the solution is unique, we use the inequality (19). Suppose that there are two solutions u_1 and u_2 satisfying $Au_1 = f$ and $Au_2 = f$. If we subtract $Au_2 = f$ from $Au_1 = f$ we get

$$A(u_1 - u_2) = 0.$$

Hence, by the estimate (19) we obtain

$$|||u_1 - u_2|||_{W^{l,p}(\mathbb{R}^n;E)} \leq C_1 |||0|||_{W^{l-s,p}(\mathbb{R}^n;E)} \quad \Rightarrow \quad |||u_1 - u_2|||_{W^{l,p}(\mathbb{R}^n;E)} = 0,$$

which implies that $u_1 = u_2$. □

4 Elliptic problem in the half-space

In this section, we consider the following boundary value problem:

$$A(D,q)u \equiv \sum_{|\alpha|+\beta \leq 2m} a_{\alpha\beta} q^\beta D^\alpha u = f(x), \quad x \in \mathbb{R}_+^n, \tag{28}$$

$$B_j(x', D, q)u = \sum_{|\alpha|+\beta \leq 2m} b_{\alpha\beta} q^\beta D^\alpha u|_{x_n=0} = g_j(x'), \quad j = 1, \dots, m. \tag{29}$$

Here $x' = (x_1, \dots, x_{n-1})$; $A(D,q)$ and $B_j(D,q)$ denote the differential operator with constant complex coefficients depending on a complex parameter q. By replacing D by ξ we obtain homogeneous polynomials $A(\xi, q)$ and $B_j(\xi, q)$ in $(\xi_1, \dots, \xi_n, q)$ of degree $2m$ and m_j, respectively. The parameter q is the same as before.

The operators in (28) and (29) can be connected as follows:

$$N = \{A(D,q), B_1(D,q), B_2(D,q), \dots, B_m(D,q)\}. \tag{30}$$

First, by using the embedding theorem (Theorem 3) and the trace theorem (Theorem 1) in the space $W^{l,p}(\mathbb{R}_+^n; E)$, we obtain the following.

Proposition 2 *Let E be a UMD space and $l \geq 2m$ be an integer. Then N is a bounded linear operator from $W^{l,p}(\mathbb{R}_+^n; E)$ to $W^{l-2m,p}(\mathbb{R}_+^n; E) \times \prod_{j=1}^m B_{p,p}^{l-m_j-\frac{1}{p}}(\mathbb{R}^{n-1}; E)$ $(j = 1, \dots, m)$. Moreover, we have the estimate*

$$|||Au|||_{W^{l-2m,p}} + \sum_{j=1}^m |||g_j|||_{B_{p,p}^{l-m_j-\frac{1}{p}}} \leq C_1'' |||u|||_{W^{l,p}}, \tag{31}$$

where the constant C_1'' does not depend on q or u.

Let F' denote the Fourier transform with respect to $x' = (x_1, \dots, x_{n-1})$. By applying the Fourier transformation F' in problem (28)-(29) with respect to x' ($x' \to \xi'$), supposing that $f = 0$, and by replacing x_n with y, we get the problem on the half-line

$$A\left(\xi', -i\frac{d}{dy}, q\right)v(y) = 0, \quad y > 0, \tag{32}$$

$$B_j\left(\xi', -i\frac{d}{dy}, q\right)v(y)\bigg|_{y=0} = h_j, \quad j = 1, \dots, m, \tag{33}$$

depending on the parameters ξ' and q.

We denote by $\lambda_j^+(\xi', q)$ the roots of $A(\xi', \lambda, q)$ with positive imaginary part and we set

$$A_+(\xi', \lambda, q) = \prod_{j=1}^{m}\left(\lambda - \lambda_j^+(\xi', q)\right) = \sum_{k=0}^{m} c_k^+(\xi', q)\lambda^{m-k},$$

where the coefficients $c_k^+(\xi', q)$ are analytic functions of $\xi' \in \mathbb{R}^{n-1}$ and $\xi', q \neq 0$, and they are homogeneous of degree k. Moreover, for every rectifiable Jordan curve γ in the complex plane which encircles all the roots λ_j^+ (see [5]) we have

$$\frac{1}{2\pi i}\int_\gamma \frac{A_{m-j-1}^+(\xi', \lambda, q)}{A^+(\xi', \lambda, q)}\lambda^k \, d\lambda = \delta_{jk}, \quad 0 \le j \le m-1, 0 \le k \le m-1, \tag{34}$$

where $A_{m-j-1}^+(\xi', \lambda, q)$ is defined as follows:

$$A_{m-j-1}^+(\xi', \lambda, q) = \sum_{k=0}^{m-j-1} c_k^+(\xi', q)\lambda^{m-j-1-k}.$$

Condition II For $|\xi'| + |q| \neq 0$, $q \in Q$ and $\xi' \in \mathbb{R}^{n-1}$, the polynomials

$$B_1(\xi', \lambda, q), B_2(\xi', \lambda, q), \dots, B_m(\xi', \lambda, q) \tag{35}$$

in λ are linearly independent modulo

$$A_+(\xi', \lambda, q) = \prod_{j=1}^{m}\left(\lambda - \lambda_j^+(\xi', q)\right).$$

Let

$$B_j(\xi', \lambda, q) = Q_j A^+(\xi', \lambda, q) + B_j'(\xi', \lambda, q).$$

Condition II is equivalent to the fact that, if B_j' is given by

$$B_j'(\xi, q) = \sum_{k=0}^{m-1} b_{jk}'(\xi', q)\lambda^k,$$

the determinant of the matrix $\{b_{jk}'\}$ is not equal to zero for all $\xi' \in \mathbb{R}^{n-1}$ such that $|\xi'| + |q| \neq 0$, $q \in Q$ (see [5]).

For $q = 0$ this condition is the same as the Shapiro-Lopatinskii condition.

Proposition 3 *Assume that Condition* I *holds. Then Condition* II *is equivalent to the fact that problem* (32)-(33) *admits a solution belonging to* $S(\mathbb{R}_+;E)$, *for all* $h_j \in E$, $j = 1,\ldots,m-1$.

Proof Assume that Condition II is satisfied and let $h_i : \mathbb{R}^{n-1} \times \mathbb{C} \to E$, $i = 0,\ldots,m-1$, be m given E-valued functions and ξ' $(\neq 0) \in \mathbb{R}^n$. Then, since the determinant of the matrix $\{b'_{ij}\}_{i,j=0}^{m-1}$ does not vanish, the system

$$\sum_{j=0}^{m-1} b'_{ij}(\xi',q)v_j(\xi',q) = h_i, \quad i = 0,\ldots,m-1,$$

has unique E-valued solutions $\{v_j(\xi',q)\}_{j=0}^{m-1}$ which depend on ξ' and q. We set, in a similar way to [5],

$$u(\xi',x_n,q) = \frac{1}{2\pi i}\int_\gamma \sum_{j=0}^{m-1} v_j(\xi',q)\frac{M^+_{m-j-1}(\xi',\lambda,q)}{M^+(\xi',\lambda,q)}e^{ix_n\lambda}\,d\lambda,$$

where γ is a rectifiable Jordan curve which encircles the roots $\lambda_k^+(\xi',q)$ of $A(\xi',\lambda,q)$ (see [5], p.130). Here, the function $u(\xi',x_n,q)$ is also an E-valued function, and it satisfies (32); furthermore, in a similar way to [5], we find that it also satisfies the boundary conditions (33).

Hence, u is a unique solution of (32)-(33). □

Theorem 6 *Suppose that Conditions* I *and* II *are satisfied. Let l be an integer greater than* $2m$ *and* $1 < p < \infty$. *Assume E is a UMD-space. Then, with a non-vanishing* $q \in Q$, *for any functions* $f \in W^{l-2m,p}(\mathbb{R}^n_+;E)$ *and* $g_j \in B_{p,p}^{l-m_j-\frac{1}{p}}(\mathbb{R}^{n-1};E)$ *there is a unique solution* $u \in W^{l,p}(\mathbb{R}^n_+;E)$ *of problem* (28)-(29). *In addition, for* $|q| \geq q_0 > 0$ *the following uniformly coercive estimate holds*:

$$|||u|||_{W^{l,p}} \leq C\left\{|||f|||_{W^{l-2m,p}} + \sum_{j=1}^{m} |||g_j|||_{B_{p,p}^{l-m_j-\frac{1}{p}}}\right\}, \tag{36}$$

where the constant C does not depend on q or u.

Proof Let T be the extension operator from $W^{l,p}(\mathbb{R}^n_+;E)$ to $W^{l,p}(\mathbb{R}^n;E)$. By the extension theorem (Theorem 2), T is a bounded linear operator from $W^{l,p}(\mathbb{R}^n_+;E)$ to $W^{l,p}(\mathbb{R}^n;E)$. First, we consider the equation

$$A(D,q)\tilde{u} = Tf, \quad x \in \mathbb{R}^n. \tag{37}$$

By using the Fourier transformation, we find that (37) has a solution expressed as

$$\tilde{u} = F^{-1}A^{-1}(\xi,q)F(Tf). \tag{38}$$

The following estimate holds:

$$|||\tilde{u}|||_{W^{l,p}(\mathbb{R}^n;E)} \leq C_1|||Tf|||_{W^{l-2m,p}(\mathbb{R}^n;E)}. \tag{39}$$

By Theorem 5, this solution belongs to $W^{l,p}(\mathbb{R}^n;E)$; so its restriction $u_0 = \tilde{u}|_{\mathbb{R}^n_+}$ belongs to $W^{l,p}(\mathbb{R}^n_+;E)$.

Then the following estimate holds:

$$|||u_0|||_{W^{l,p}(\mathbb{R}^n_+;E)} \leq C_2 |||f|||_{W^{l-2m,p}(\mathbb{R}^n_+;E)}. \tag{40}$$

Now, we consider the problem

$$A(D,q)u_1 = 0, \quad x \in \mathbb{R}^n_+, \tag{41}$$

$$B_j(D,q)u_1|_{x_n=0} = g_j - g_{0j}, \tag{42}$$

where

$$g_{0j} = B_j(D,q)u_0|_{x_n=0}, \quad j = 1,\dots,m \tag{43}$$

such that, by virtue of the trace theorem (Theorem 1), $g_{0j} \in B_{p,p}^{l-m_j-\frac{1}{p}}(\mathbb{R}^{n-1};E)$.

First, we have to show that problem (41)-(42) has a unique solution $u_1 \in W^{l,p}(\mathbb{R}^n_+;E)$ and that the following estimate holds:

$$|||u_1|||_{W^{l,p}(\mathbb{R}^n_+;E)} \leq C_3 \sum_{j=1}^{m} |||g_j - g_{j0}|||_{B_{p,p}^{l-m_j-\frac{1}{p}}(\mathbb{R}^{n-1};E)} \tag{44}$$

with a constant C_3 independent of q or of the functions considered. Then it is clear that $u = u_0 + u_1$ is a solution of (36), and by Proposition 2 and (40) the following estimate is satisfied:

$$|||g_{j0}|||_{B_{p,p}^{l-m_j-\frac{1}{p}}} \leq C_4 |||u_0|||_{B_{p,p}^{l-m_j-\frac{1}{p}}} \leq C_4 C_2 |||f|||_{W^{l-2m,p}}. \tag{45}$$

Moreover, by using the trace theorem (Theorem 1) and estimate (45), we get

$$\begin{aligned}
|||u|||_{W^{l,p}} &\leq |||u_0|||_{W^{l,p}} + |||u_1|||_{W^{l,p}} \\
&\leq C_2 |||f|||_{W^{l-2m,p}} + C_3 \sum_{j=1}^{m} |||g_j|||_{B_{p,p}^{l-m_j-\frac{1}{p}}} + C_3 \sum_{j=1}^{m} |||g_{j0}|||_{B_{p,p}^{l-m_j-\frac{1}{p}}} \\
&\leq C_4 |||f|||_{W^{l-2m,p}} + C_3 \sum_{j=1}^{m} |||g_j|||_{B_{p,p}^{l-m_j-\frac{1}{p}}}
\end{aligned}$$

with constants C_2, C_4 independent of q and u.

Thus, it suffices to prove that the problem

$$A(D,q)u(x) = 0, \quad x \in \mathbb{R}^n_+, \tag{46}$$

$$B_j(D,q)u(x)|_{x_n=0} = g_j \quad (j = 1,\dots,m) \tag{47}$$

has a unique solution $u \in W^{l,p}(\mathbb{R}^n_+;E)$ and that estimate (36) holds.

Applying the Fourier transformation with respect to $x' = (x_1, \dots, x_m)$ to problem (46)-(47), we obtain

$$A(\xi', D_n, q)F'u = 0, \tag{48}$$

$$B_j(\xi', D_n, q)F'u|_{x_n=0} = F'g_j \quad (j = 1, \dots, m). \tag{49}$$

Now we apply Proposition 3 to solve problem (48)-(49). We find that problem (48)-(49) has a solution $u \in W^{l,p}(\mathbb{R}^n_+, E)$, and it is obvious that this solution is unique. This completes the proof of the theorem. □

5 Application

Let $E = l_q$, where

$$l_q = \left\{ u = \{u_k\}_{k=1}^{\infty} : \left(\sum_{k=1}^{\infty} |u_k|^q \right)^{\frac{1}{q}} = \|u\|_{l_q} < \infty \right\}. \tag{50}$$

Consider the BVP for a system of elliptic equations in $\mathbb{R}^n$

$$L_k(x, D)u = \sum_{|\alpha|+\beta \le s} a_{k\alpha\beta} q^\beta D^\alpha u = f_k, \quad k = 1, \dots, N, s, N \in \mathbb{N}, x \in \mathbb{R}^n, \tag{51}$$

where $a_{k\alpha\beta}$, $k = 1, \dots, N$, are complex coefficients.

Theorem 7 *Let Condition* I *hold. Then for* $f_k \in W^{l-s,p}(\mathbb{R}^n; l_q)$, $p, q \in (1, \infty)$ *there is a unique solution* $u \in W^{l,p}(\mathbb{R}^n; l_q)$ *of problem* (51) *and the following coercive estimate holds*:

$$|||u|||_{W^{l,p}(\mathbb{R}^n; l_q)} \le K_1 |||f|||_{W^{l-s,p}(\mathbb{R}^n; l_q)}.$$

Now, consider the BVP for system of elliptic equations in $\mathbb{R}^n_+$ as follows:

$$\begin{cases} L_k(x, D)u = \sum_{|\alpha|+\beta \le 2m} a_{k\alpha\beta} q^\beta D^\alpha u = f_k, & k = 1, \dots, N, N \in \mathbb{N}, \\ B_{kj} u|_{x_n=0} = \sum_{|\alpha|+\beta \le m_{kj}} b_{k\alpha\beta j} q^\beta D^\alpha u|_{x_n=0} = g_{kj}, & x \in \mathbb{R}^n_+, \end{cases} \tag{52}$$

where $m_{kj} \le 2m - 1$, and $a_{k\alpha\beta}$ and $b_{k\alpha\beta j}$ are complex coefficients.

From Theorem 6 we obtain the following.

Theorem 8 *Let Conditions* I *and* II *hold. Then for* $f_k \in W^{l-s,p}(\mathbb{R}^n_+; l_q)$, $p, q \in (1, \infty)$ *and* $g_{kj} \in B_{p,p}^{l-m_{kj}-\frac{1}{p}}(\mathbb{R}^{n-1}; l_q)$, *problem* (52) *has a unique solution* $u \in W^{l,p}(\mathbb{R}^n_+; l_q)$ *and the uniform coercive estimate*

$$|||u|||_{W^{l,p}(\mathbb{R}^n_+; l_q)} \le K_2 \left\{ |||f|||_{W^{l-2m,p}(\mathbb{R}^n_+; l_q)} + \sum_{j=1}^{m} |||b_k j|||_{B_{p,p}^{l-m_{kj}-\frac{1}{p}}(\mathbb{R}^{n-1}; l_q)} \right\}$$

holds.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The ideas and methods in this paper have been consulted by VBS for accuracy and appropriateness. The paper is written by AO. Calculations and proof have been conducted by AO. All authors read and approved the final manuscript.

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