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On stability of functional equations related to quadratic mappings in fuzzy Banach spaces

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Abstract
In this paper, we establish the generalized Hyers-Ulam stability problem of radical quadratic functional equations $f(\sqrt{x^2+y^2}) = f(x) + f(y)$ in fuzzy Banach spaces via the direct and fixed point methods.

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1 Introduction

The stability problem concerning the stability of group homomorphisms of functional equations was originally introduced by Ulam [1] in 1940. The famous Ulam stability problem was partially solved by Hyers [2] for a linear functional equation of Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias has had a lot of influence in the development of what we call the generalized Hyers-Ulam stability of functional equations. A generalization of Rassias' theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. Cădariu and Radu [6] applied the *fixed point method* to the investigation of the Jensen functional equation. They could present a short and simple proof (different from the *direct method* initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of the Jensen functional and the quadratic functional equations.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called a *quadratic functional equation*. Quadratic functional equations were used to characterize inner product spaces. In particular, every solution of the quadratic equation is said to be a *quadratic mapping*. The generalized Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof [7]. Recently, the stability problem of the radical quadratic functional equations in various spaces was proved in the papers [8–11].

In 1984, Katsaras [12] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [13–16]. Cheng and Mordeson [17] introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of the Kramosil and Michálek type [14]. In 2003, Bag and Samanta [18] modified the definition of Cheng and Mordeson by removing a regular condition. Also, they investigated a decomposition theorem of a fuzzy norm into a family to crisp norms and gave some properties of fuzzy norm. The fuzzy stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning these problems [17, 19–23].

In the sequel, we use the definitions and some basic facts concerning fuzzy Banach spaces given in Bag and Samanta [18].

Definition 1.1 Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if, for all $x, y \in X$ and $s, t \in \mathbb{R}$, N satisfies the following conditions:

(N1) $N(x,t) = 0$ for all $t \leq 0$;

(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;

(N3) $N(cx,t) = N(x, t/|c|)$ for all $c \in \mathbb{R}$ with $c \neq 0$;

(N4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;

(N5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;

(N6) for all $x \in X$ with $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed linear space*.

Example 1.2 Let $(X, \|\cdot\|)$ be a normed linear space and let $\alpha, \beta > 0$. Then

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta \|x\|}, & t > 0, x \in X; \\ 0, & t \leq 0, x \in X, \end{cases}$$

is a fuzzy norm on X.

Definition 1.3 Let (X, N) be a fuzzy normed linear space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for any $\epsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that $N(x_n - x, t) > 1 - \epsilon$ for all $n \geq n_0$. In this case, x is called the *limit* of the sequence $\{x_n\}$, which is denoted by $x = \lim_{n\to\infty} x_n$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for any $\epsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $n \geq n_0$ and $p \in \mathbb{Z}^+$.

(3) If every Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed linear space is called a *fuzzy Banach space*.

A mapping $f : X \to Y$ between fuzzy normed linear spaces X and Y is said to be *continuous* at a point $x_0 \in X$ if, for any sequence $\{x_n\}$ in X converging to a point $x_0 \in X$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at every point $x \in X$, then f is said to be *continuous* on X.

Example 1.4 Let $N : \mathbb{R} \times \mathbb{R} \to [0,1]$ be a fuzzy norm on $\mathbb{R}$ defined by

$$N(x,t) = \begin{cases} \frac{t}{t + \|x\|}, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

Then $(\mathbb{R}, N)$ is a fuzzy Banach space.

In this paper, we establish the generalized Hyers-Ulam stability problem of a radical quadratic functional equation $f(\sqrt{x^2+y^2}) = f(x) + f(y)$ in fuzzy Banach spaces via the direct and fixed point methods.

2 Fuzzy stability of the radical quadratic functional equations

In this section, we study a fuzzy version of the generalized Hyers-Ulam stability of functional equation which approximate uniformly a radical quadratic mapping in fuzzy Banach spaces.

2.1 The direct method

Theorem 2.1 *Let $\ell \in \{-1,1\}$ be fixed, $(\mathcal{Y}, N)$ be a fuzzy Banach space and $\phi : \mathbb{R}^2 \to [0,\infty)$ be a mapping such that*

$$\Phi(x,y) := \sum_{n=\frac{1-\ell}{2}}^{\infty} \frac{1}{2^{\ell n}} \phi\bigl(2^{\frac{\ell n}{2}}x, 2^{\frac{\ell n}{2}}y\bigr) + \phi\bigl(2^{\frac{\ell n+1}{2}}x, 0\bigr) < \infty \tag{2.1}$$

for all $x, y \in \mathbb{R}$. Suppose that $f : \mathbb{R} \to \mathcal{Y}$ is a mapping with $f(0) = 0$ such that, for all $t > 0$,

$$\lim_{t\to\infty} N\bigl(f\bigl(\sqrt{x^2+y^2}\bigr) - f(x) - f(y), t\phi(x,y)\bigr) = 1 \tag{2.2}$$

uniformly on $\mathbb{R}^2$. Then there exists a unique quadratic mapping $Q : \mathbb{R} \to \mathcal{Y}$ such that, if there exist $\delta > 0$ and $\alpha > 0$ such that

$$N\bigl(f\bigl(\sqrt{x^2+y^2}\bigr) - f(x) - f(y), \delta\phi(x,y)\bigr) \geq \alpha \tag{2.3}$$

for all $x, y \in \mathbb{R}$, then

$$N\left(f(x) - Q(x), \frac{\delta}{2}\Phi(x,x)\right) \geq \alpha \tag{2.4}$$

for all $x \in \mathbb{R}$. Furthermore, the quadratic mapping $Q : \mathbb{R} \to \mathcal{Y}$ is a unique mapping such that, for all $t > 0$,

$$N\bigl(f(x) - Q(x), t\Phi(x,x)\bigr) = 1 \tag{2.5}$$

uniformly on $\mathbb{R}$.

Proof Assume that $\ell = 1$. For any $\epsilon > 0$, by (2.2), we can find some $t_0 > 0$ such that

$$N\bigl(f\bigl(\sqrt{x^2+y^2}\bigr) - f(x) - f(y), t\phi(x,y)\bigr) \geq 1 - \epsilon \tag{2.6}$$

for all $x, y \in \mathbb{R}$ and $t \geq t_0$. Replacing x and y by $\frac{x+y}{\sqrt{2}}$ and $\frac{x-y}{\sqrt{2}}$ in (2.6), respectively, we have

$$N\left(f\bigl(\sqrt{x^2+y^2}\bigr) - f\left(\frac{x+y}{\sqrt{2}}\right) - f\left(\frac{x-y}{\sqrt{2}}\right), t\phi\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right)\right) \geq 1 - \epsilon \tag{2.7}$$

for all $x, y \in \mathbb{R}$ and $t \geq t_0$. It follows from (2.6), (2.7), and (N4) that

$$N\left(f(x)+f(y)-f\left(\frac{x+y}{\sqrt{2}}\right)-f\left(\frac{x-y}{\sqrt{2}}\right), t\left(\phi(x,y)+\phi\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right)\right)\right) \geq 1-\epsilon \tag{2.8}$$

for all $x, y \in \mathbb{R}$ and $t \geq t_0$. Letting $y = x$ in (2.8), we have

$$N\left(2f(x)-f\left(2^{\frac{1}{2}}x\right), t\hat{\phi}(x,x)\right) \geq 1-\epsilon \tag{2.9}$$

for all $x \in \mathbb{R}$ and $t \geq t_0$, where $\hat{\phi}(x,x) = \phi(x,x) + \phi(2^{\frac{1}{2}}x, 0)$. By induction on n, we have

$$N\left(2^n f(x)-f\left(2^{\frac{n}{2}}x\right), t\sum_{k=0}^{n-1} 2^{n-k-1}\hat{\phi}\left(2^{\frac{k}{2}}x, 2^{\frac{k}{2}}x\right)\right) \geq 1-\epsilon \tag{2.10}$$

for all $x \in \mathbb{R}$, $t \geq t_0$ and $n \in \mathbb{Z}^+$. Let $t = t_0$. Replacing n and x by p and $2^{\frac{n}{2}}x$ in (2.10), respectively, we have

$$N\left(\frac{f(2^{\frac{n}{2}}x)}{2^n}-\frac{f(2^{\frac{n+p}{2}}x)}{2^{n+p}}, \frac{t_0}{2^{n+p}}\sum_{k=0}^{p-1} 2^{p-k-1}\hat{\phi}\left(2^{\frac{n+k}{2}}x, 2^{\frac{n+k}{2}}x\right)\right) \geq 1-\epsilon \tag{2.11}$$

for all $n \geq 0$ and $p > 0$. It follows from (2.1) and the equality

$$\sum_{k=0}^{p-1} \frac{1}{2^{n+k+1}}\hat{\phi}\left(2^{\frac{n+k}{2}}x, 2^{\frac{n+k}{2}}x\right) = \frac{1}{2}\sum_{k=n}^{n+p-1}\frac{1}{2^k}\hat{\phi}\left(2^{\frac{k}{2}}x, 2^{\frac{k}{2}}x\right)$$

that, for any $\delta > 0$, there exists some $n_0 \in \mathbb{Z}^+$ such that

$$\frac{t_0}{2}\sum_{k=n}^{n+p-1}\frac{1}{2^k}\hat{\phi}\left(2^{\frac{k}{2}}x, 2^{\frac{k}{2}}x\right) < \delta$$

for all $n \geq n_0$ and $p > 0$. Now, it follows from (2.11) that

$$\begin{aligned} &N\left(\frac{f(2^{\frac{n}{2}}x)}{2^n}-\frac{f(2^{\frac{n+p}{2}}x)}{2^{n+p}}, \delta\right) \\ &\quad \geq N\left(\frac{f(2^{\frac{n}{2}}x)}{2^n}-\frac{f(2^{\frac{n+p}{2}}x)}{2^{n+p}}, \frac{t_0}{2^{n+p}}\sum_{k=0}^{p-1} 2^{p-k-1}\hat{\phi}\left(2^{\frac{n+k}{2}}x, 2^{\frac{n+k}{2}}x\right)\right) \geq 1-\epsilon \end{aligned} \tag{2.12}$$

for all $n \geq n_0$ and $p > 0$. Thus the sequence $\{\frac{f(2^{\frac{n}{2}}x)}{2^n}\}$ is a Cauchy sequence in a fuzzy Banach space $\mathcal{Y}$ and so it converges to some $Q(x) \in \mathcal{Y}$. We can define a mapping $Q : \mathbb{R} \to \mathcal{Y}$ by

$$Q(x) = \lim_{n\to\infty}\frac{f(2^{\frac{n}{2}}x)}{2^n},$$

that is, $\lim_{n\to\infty} N(\frac{f(2^{\frac{n}{2}}x)}{2^n} - Q(x), t) = 1$ for all $x \in \mathbb{R}$ and $t > 0$. Let $x, y \in \mathbb{R}$, $t > 0$ and $0 < \epsilon < 1$. Since $\lim_{n\to\infty} \frac{1}{2^n}\hat{\phi}(2^{\frac{n}{2}}x, 2^{\frac{n}{2}}y) = 0$, there exists $n_1 \in \mathbb{Z}^+$ with $n_1 > n_0$ such that

$$t_0\hat{\phi}\left(2^{\frac{n}{2}}x, 2^{\frac{n}{2}}y\right) < \frac{2^n t}{4}$$

for all $n \geq n_1$. Then, by (N4), we have

$$\begin{aligned} &N\bigl(Q\bigl(\sqrt{x^2+y^2}\bigr)-Q(x)-Q(y),t\bigr)\\ &\quad\geq \min\biggl\{N\biggl(Q(x+y)-\frac{1}{2^n}f\bigl(\sqrt{2^nx^2+2^ny^2}\bigr),\frac{t}{4}\biggr),N\biggl(Q(x)-\frac{1}{2^n}f\bigl(2^{\frac{n}{2}}x\bigr),\frac{t}{4}\biggr),\\ &\qquad N\biggl(Q(y)-\frac{1}{2^n}f\bigl(2^{\frac{n}{2}}y\bigr),\frac{t}{4}\biggr),N\biggl(f\bigl(\sqrt{2^nx^2+2^ny^2}\bigr)-f\bigl(2^{\frac{n}{2}}x\bigr)-f\bigl(2^{\frac{n}{2}}y\bigr),\frac{2^nt}{4}\biggr)\biggr\} \end{aligned} \tag{2.13}$$

for all $n \geq n_1$. Since the first three terms on the right-hand side of the above inequality tend to 1 as $n \to \infty$ and

$$N\bigl(f\bigl(\sqrt{2^nx^2+2^ny^2}\bigr)-f\bigl(2^{\frac{n}{2}}x\bigr)-f\bigl(2^{\frac{n}{2}}y\bigr),t_0\hat{\phi}\bigl(2^{\frac{n}{2}},2^{\frac{n}{2}}y\bigr)\bigr)\geq 1-\epsilon,$$

we have

$$N\bigl(Q\bigl(\sqrt{x^2+y^2}\bigr)-Q(x)-Q(y),t\bigr)\geq 1-\epsilon$$

for all $x, y \in \mathbb{R}$, $t > 0$ and $0 < \epsilon < 1$. It follows from (N2) that $Q(\sqrt{x^2+y^2}) = Q(x) + Q(y)$ for all $x, y \in \mathbb{R}$. This means that Q is a quadratic mapping [10].

Now, suppose that (2.3) holds for some $\delta > 0$ and $\alpha > 0$. Then assume that

$$\psi_n(x,y)=\sum_{k=0}^{n-1}\frac{1}{2^{k+1}}\hat{\phi}\bigl(2^{\frac{k}{2}}x,2^{\frac{k}{2}}y\bigr)$$

for all $x, y \in \mathbb{R}$. For all $x \in \mathbb{R}$, by a similar method to the beginning of the proof, we have

$$N\Biggl(2^nf(x)-f\bigl(2^{\frac{n}{2}}x\bigr),\delta\sum_{k=0}^{n-1}2^{n-k-1}\hat{\phi}\bigl(2^{\frac{k}{2}}x,2^{\frac{k}{2}}x\bigr)\Biggr)\geq\alpha \tag{2.14}$$

for all $n \in \mathbb{Z}^+$. Let $t > 0$. Then we have

$$\begin{aligned} &N\bigl(f(x)-Q(x),\delta\psi_n(x,x)+t\bigr)\\ &\quad\geq\min\biggl\{N\biggl(f(x)-\frac{f(2^{\frac{n}{2}}x)}{2^n},\delta\psi_n(x,x)\biggr),N\biggl(\frac{f(2^{\frac{n}{2}}x)}{2^n}-Q(x),t\biggr)\biggr\}. \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15) and using the fact $\lim_{n\to\infty}N(\frac{f(2^{\frac{n}{2}}x)}{2^n}-Q(x),t)=1$, we obtain

$$N\bigl(f(x)-Q(x),\delta\psi_n(x,x)+t\bigr)\geq\alpha \tag{2.16}$$

for large enough $n \in \mathbb{Z}^+$. It follows from the continuity of the function $N(f(x) - Q(x), \cdot)$ that

$$N\biggl(f(x)-Q(x),\frac{\delta}{2}\Phi(x,x)+t\biggr)\geq\alpha.$$

Letting $t \to 0$, we conclude (2.5).

Next, assume that there exists another quadratic mapping T which satisfies (2.5). For any $\epsilon > 0$, by applying (2.5) for the mappings Q and T, we can find some $t_0 > 0$ such that

$$N\left(f(x) - Q(x), \frac{t}{2}\Phi(x,x)\right) \geq 1 - \epsilon, \qquad N\left(f(x) - T(x), \frac{t}{2}\Phi(x,x)\right) \geq 1 - \epsilon$$

for all $x \in \mathbb{R}$ and $t \geq t_0$. Fix $x \in \mathbb{R}$ and $c > 0$. Then we find some $n_0 \in \mathbb{Z}^+$ such that

$$t_0 \sum_{k=n}^{\infty} \frac{1}{2^k} \hat{\phi}\big(2^{\frac{k}{2}}x, 2^{\frac{k}{2}}y\big) < \frac{c}{2}$$

for all $x, y \in \mathbb{R}$ and $n \geq n_0$. It follows from

$$\begin{aligned}
\sum_{k=n}^{\infty} \frac{1}{2^k} \hat{\phi}\big(2^{\frac{k}{2}}x, 2^{\frac{k}{2}}y\big) &= \frac{1}{2^n} \sum_{k=n}^{\infty} \frac{1}{2^{k-n}} \hat{\phi}\big(2^{k-n}\big(2^{\frac{n}{2}}x\big), 2^{k-n}\big(2^{\frac{n}{2}}y\big)\big) \\
&= \frac{1}{2^n} \sum_{m=0}^{\infty} \frac{1}{2^m} \hat{\phi}\big(2^m\big(2^{\frac{n}{2}}x\big), 2^m\big(2^{\frac{n}{2}}y\big)\big) \\
&= \frac{1}{2^n} \Phi\big(2^{\frac{n}{2}}x, 2^{\frac{n}{2}}y\big)
\end{aligned}$$

that

$$\begin{aligned}
&N\big(Q(x) - T(x), c\big) \\
&\quad \geq \min\left\{N\left(\frac{f(2^{\frac{n}{2}}x)}{2^n} - Q(x), \frac{c}{2}\right), N\left(T(x) - \frac{f(2^{\frac{n}{2}}x)}{2^n}, \frac{c}{2}\right)\right\} \\
&\quad = \min\big\{N\big(f\big(2^{\frac{n}{2}}x\big) - Q\big(2^{\frac{n}{2}}x\big), 2^{n-1}c\big), N\big(T\big(2^{\frac{n}{2}}x\big) - f\big(2^{\frac{n}{2}}x\big), 2^{n-1}c\big)\big\} \\
&\quad \geq \min\left\{N\left(f\big(2^{\frac{n}{2}}x\big) - Q\big(2^{\frac{n}{2}}x\big), 2^n t_0 \sum_{k=n}^{\infty} \frac{1}{2^k} \hat{\phi}\big(2^{\frac{k}{2}}x, 2^{\frac{k}{2}}x\big)\right),\right. \\
&\qquad \left. N\left(T\big(2^{\frac{n}{2}}x\big) - f\big(2^{\frac{n}{2}}x\big), 2^n t_0 \sum_{k=n}^{\infty} \frac{1}{2^k} \hat{\phi}\big(2^{\frac{k}{2}}x, 2^{\frac{k}{2}}x\big)\right)\right\} \\
&\quad \geq \min\big\{N\big(f\big(2^{\frac{n}{2}}x\big) - Q\big(2^{\frac{n}{2}}x\big), t_0 \Phi\big(2^{\frac{n}{2}}x, 2^{\frac{n}{2}}x\big)\big), \\
&\qquad N\big(T\big(2^{\frac{n}{2}}x\big) - f\big(2^{\frac{n}{2}}x\big), t_0 \Phi\big(2^{\frac{n}{2}}x, 2^{\frac{n}{2}}x\big)\big)\big\} \\
&\quad \geq 1 - \epsilon
\end{aligned}$$

for all $x, y \in \mathbb{R}$ and $c > 0$. Thus we have $N(Q(x) - T(x), c) = 1$ for all $c > 0$ and so $Q(x) = T(x)$ for all $x \in \mathbb{R}$.

For the case $\ell = -1$, we can state the proof in the same method as in the first case. In the case, the mapping Q is defined by $Q(x) = \lim_{n\to\infty} 2^n f(2^{-\frac{n}{2}}x)$. This completes the proof. □

Corollary 2.2 *Let $(\mathcal{Y}, N)$ be a fuzzy Banach space, θ and $p \in \mathbb{R}$ with $p < 2$ be positive real numbers. Suppose that $f : \mathbb{R} \to \mathcal{Y}$ is a mapping with $f(0) = 0$ such that, for all $t > 0$,*

$$\lim_{t\to\infty} N\big(f\big(\sqrt{x^2 + y^2}\big) - f(x) - f(y), t\theta\big(|x|^p + |y|^p\big)\big) = 1 \tag{2.17}$$

uniformly on $\mathbb{R}$. *Then the limit* $Q(x) = \lim_{n\to\infty} \frac{f(2^{\frac{n}{2}}x)}{2^n}$ *exists for all* $x \in X$ *and there exists a unique quadratic mapping* $Q : \mathbb{R} \to \mathcal{Y}$ *such that*

$$\lim_{t\to\infty} N\left(f(x) - Q(x), \frac{2(2+2^{\frac{p}{2}})}{2-2^{\frac{p}{2}}}\theta|x|^p t\right) = 1 \tag{2.18}$$

uniformly on $\mathbb{R}$.

Proof The proof follows from Theorem 2.1 by taking $\phi(x,y) = \theta(|x|^p + |y|^p)$ for all $x, y \in \mathbb{R}$. □

Corollary 2.3 *Let* $(\mathcal{Y}, N)$ *be a fuzzy Banach space and* $\psi : [0,\infty) \to [0,\infty)$ *be a mapping such that, for all* $s, t > 0$,

(a) $\psi(ts) = \psi(t)\psi(s)$;
(b) $\psi(\sqrt{2}) < 2$.

Suppose that $f : \mathbb{R} \to \mathcal{Y}$ *is a mapping with* $f(0) = 0$ *such that, for all* $t > 0$,

$$\lim_{t\to\infty} N\left(f\left(\sqrt{x^2+y^2}\right) - f(x) - f(y), t\theta\left(\psi\left(|x|\right) + \psi\left(|y|\right)\right)\right) = 1 \tag{2.19}$$

uniformly on $\mathbb{R}^2$, *where* $\theta > 0$ *is fixed. Then the limit* $Q(x) = \lim_{n\to\infty} \frac{f(2^{\frac{n}{2}}x)}{2^n}$ *exists for all* $x \in \mathbb{R}$ *and defines a quadratic mapping* $Q : \mathbb{R} \to \mathcal{Y}$ *such that, for all* $t > 0$,

$$\lim_{t\to\infty} N\left(f(x) - Q(x), \frac{2(2+\psi(\sqrt{2}))}{2-\psi(\sqrt{2})}\theta\psi\left(|x|\right)t\right) = 1 \tag{2.20}$$

uniformly on $\mathbb{R}$.

Proof The proof follows from Theorem 2.1 by taking $\phi(x,y) = \theta(\psi(|x|) + \psi(|y|))$ for all $x, y \in \mathbb{R}$. □

2.2 The fixed point method

Recall that a mapping $d : X^2 \to [0, +\infty]$ is called a *generalized metric* on a nonempty set X if

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

A set X with the generalized metric d is called a *generalized metric space*.

In [24], Diaz and Margolis proved the following fixed point theorem, which plays an important role for the main results in this section.

Theorem 2.4 [24] *Suppose that* (Ω, d) *is a complete generalized metric space and* $T : \Omega \to \Omega$ *is a strictly contractive mapping with Lipshitz constant* L. *Then, for any* $x \in \Omega$, *either* $d(T^n x, T^{n+1}x) = \infty$ *for all* $n \geq 0$ *or there exists a positive integer* n_0 *such that*

(1) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(2) *the sequence* $\{T^n x\}$ *is convergent to a fixed point* y^* *of* T;
(3) y^* *is the unique fixed point of* T *in the set* $\Lambda = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Lambda$.

Theorem 2.5 *Let $(\mathcal{Y}, N)$ be a fuzzy Banach space and $\phi : \mathbb{R}^2 \to [0,\infty)$ be a mapping such that there exists $L < 1$ with*

$$\phi\big(2^{\frac{1}{2}}x, 2^{\frac{1}{2}}y\big) \le 2L\phi(x,y) \tag{2.21}$$

for all $x, y \in \mathbb{R}$. If $f : \mathbb{R} \to \mathcal{Y}$ is a mapping with $f(0) = 0$ and

$$N\big(f\big(\sqrt{x^2+y^2}\big) - f(x) - f(y), t\big) \ge \frac{t}{t + \phi(x,y)} \tag{2.22}$$

for all $x, y \in \mathbb{R}$ and $t > 0$, then the limit $Q(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^{\frac{n}{2}}x)$ exists for all $x \in \mathbb{R}$ and a unique quadratic mapping $Q : \mathbb{R} \to \mathcal{Y}$ satisfies the inequality

$$N\big(f(x) - Q(x), t\big) \ge \frac{(1-L)t}{(1-L)t + \hat{\phi}(x,x)} \tag{2.23}$$

for all $x \in \mathbb{R}$, where $\hat{\phi}(x,x) = \phi(x,x) + \phi(2^{\frac{1}{2}}x, 0)$.

Proof Letting x and y by $\frac{x+y}{\sqrt{2}}$ and $\frac{x-y}{\sqrt{2}}$ in (2.22), respectively, we have

$$N\left(f\big(\sqrt{x^2+y^2}\big) - f\left(\frac{x+y}{\sqrt{2}}\right) - f\left(\frac{x-y}{\sqrt{2}}\right), t\right) \ge \frac{t}{t + \phi(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}})} \tag{2.24}$$

for all $x, y \in \mathbb{R}$ and $t \ge t_0$. It follows from (2.22), (2.24), and (N4) that

$$\begin{aligned}
&N\left(f(x) + f(y) - f\left(\frac{x+y}{\sqrt{2}}\right) - f\left(\frac{x-y}{\sqrt{2}}\right), 2t\right) \\
&\quad\ge \min\left\{N\big(f(x) + f(y) - f\big(\sqrt{x^2+y^2}\big), t\big),\right. \\
&\qquad \left. N\left(f\big(\sqrt{x^2+y^2}\big) - f\left(\frac{x+y}{\sqrt{2}}\right) - f\left(\frac{x-y}{\sqrt{2}}\right), t\right)\right\} \\
&\quad\ge \min\left\{\frac{t}{t + \phi(x,y)}, \frac{t}{t + \phi(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}})}\right\} \\
&\quad\ge \frac{t}{t + \phi(x,y) + \phi(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}})}
\end{aligned} \tag{2.25}$$

for all $x, y \in \mathbb{R}$ and $t \ge t_0$. Letting $y = x$ in (2.25), we have

$$N\left(f(x) - \frac{1}{2}f\big(2^{\frac{1}{2}}x\big), t\right) \ge \frac{t}{t + \hat{\phi}(x,x)} \tag{2.26}$$

for all $x \in \mathbb{R}$ and $t \ge t_0$, where $\hat{\phi}(x,y) = \phi(x,y) + \phi(2^{\frac{1}{2}}x, 0)$.

Let Ω be a set of all mapping from $\mathbb{R}$ to $\mathcal{Y}$ and introduce a generalized metric on Ω as follows:

$$d(g,h) = \inf\left\{\mu \in [0,\infty) : N\big(g(x) - h(x), \mu t\big) \ge \frac{t}{t + \hat{\phi}(x,x)}, \forall x \in \mathbb{R}, t > 0\right\}.$$

It is easy to show that (Ω, d) is a generalized complete metric space [25]. We consider the mapping $T : \Omega \to \Omega$ defined by

$$Tg(x) = \frac{1}{2} g\big(2^{\frac{1}{2}} x\big)$$

for all $g \in \Omega$ and $x \in \mathbb{R}$. Let $g, h \in \Omega$ such that $d(g,h) \le \mu$. Then we have

$$N\big(Tg(x) - Th(x), t\mu L\big) = N\big(g\big(2^{\frac{1}{2}} x\big) - h\big(2^{\frac{1}{2}} x\big), 2t\mu L\big) \ge \frac{t}{t + \hat{\phi}(x,x)}$$

for all $x \in \mathbb{R}$, and so

$$d(Tg, Th) \le Ld(g,h)$$

for all $g, h \in \Omega$. This means that T is a strictly contractive self-mapping of Ω with the Lipschitz constant L.

It follows from (2.26) that $d(f, Tf) \le 1 < \infty$. Now, it follows from Theorem 2.4 that the sequence $\{T^n f\}$ converges to a unique fixed point Q of T. So there exists a fixed point Q of T in Ω such that

$$Q(x) = \lim_{n\to\infty} \frac{1}{2^n} f\big(2^{\frac{n}{2}} x\big) \tag{2.27}$$

for all $x \in \mathbb{R}$ since $\lim_{n\to\infty} d(T^n, Q) = 0$. Again, using the fixed point method, since Q is the unique fixed point of T in $\Omega^* = \{g \in \Omega : d(f,g) < \infty\}$, we have

$$d(f,Q) \le \frac{1}{1-L} d(f, Tf) \le \frac{1}{1-L},$$

which gives

$$N\big(f(x) - Q(x), t\big) \ge \frac{(1-L)t}{(1-L)t + \hat{\phi}(x,x)}$$

for all $x \in \mathbb{R}$ and $t > 0$. Further, we have

$$\begin{aligned} &N\big(Q\big(\sqrt{x^2+y^2}\big) - Q(x) - Q(y), t\big) \\ &\quad \ge \lim_{n\to\infty} N\big(f\big(\sqrt{2^n x^2 + 2^n y^2}\big) - f\big(2^{\frac{n}{2}} x\big) - f\big(2^{\frac{n}{2}} y\big), 2^n t\big) \\ &\quad \ge \lim_{n\to\infty} \frac{t}{t + L^n \hat{\phi}(x,y)} = 1 \end{aligned} \tag{2.28}$$

for all $x, y \in \mathbb{R}$ and $t > 0$. It follows from (N2) and $N(Q(\sqrt{x^2+y^2}) - Q(x) - Q(y), t) \ge 1$ that $Q(\sqrt{x^2+y^2}) = Q(x) + Q(y)$ for all $x, y \in \mathbb{R}$. This means that Q is a quadratic mapping on $\mathbb{R}$. This completes the proof. □

Theorem 2.6 *Let $(\mathcal{Y}, N)$ be a fuzzy Banach space and $\phi : \mathbb{R}^2 \to [0,\infty)$ be a mapping such that there exists $L < 1$ with*

$$\phi\bigg(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\bigg) \le \frac{L}{2}\phi(x,y) \tag{2.29}$$

for all $x, y \in \mathbb{R}$. *If* $f : \mathbb{R} \to \mathcal{Y}$ *is a mapping with* $f(0) = 0$ *and* (2.22), *then the limit* $Q(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^{\frac{n}{2}}})$ *exists for all* $x \in \mathbb{R}$ *and there exists a unique quadratic mapping* $Q : \mathbb{R} \to \mathcal{Y}$ *satisfying the inequality*

$$N\big(f(x) - Q(x), t\big) \geq \frac{(1-L)t}{(1-L)t + L\hat{\phi}(x,x)} \tag{2.30}$$

for all $x \in \mathbb{R}$ *and* $t > 0$, *where* $\hat{\phi}(x,x) = \phi(x,x) + \phi(2^{\frac{1}{2}}x, 0)$.

Proof It follows from (2.26) that

$$N\left(f(x) - 2f\left(\frac{x}{\sqrt{2}}\right), Lt\right) \geq \frac{t}{t + \hat{\phi}(x,x)} \tag{2.31}$$

for all $x \in \mathbb{R}$ and $t \geq t_0$, where $\hat{\phi}(x,y) = \phi(x,y) + \phi(2^{\frac{1}{2}}x, 0)$. Let Ω and d be as in the proof of Theorem 2.5. Then (Ω, d) becomes a generalized complete metric space and we consider the mapping $T : \Omega \to \Omega$ defined by

$$(Tg)(x) = 2g\left(\frac{x}{\sqrt{2}}\right),$$

$x \in \mathbb{R}$. So, we have $d(Tg, Th) \leq Ld(g,h)$ for all $g, h \in \Omega$. It follows from Theorem 2.4 that there exists a unique mapping $Q : \mathbb{R} \to \mathcal{Y}$ in the set $\{g \in \Omega : d(f,g) < \infty\}$ which is a unique fixed point of T such that

$$Q(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^{\frac{n}{2}}}\right)$$

for all $x \in \mathbb{R}$. Also, from (2.31) we have $d(f, Tf) \leq L$. So, we can conclude that

$$d(f, Q) \leq \frac{1}{1-L} d(f, Tf) \leq \frac{L}{1-L},$$

which implies the inequality (2.30). The remaining assertion goes through in a similar way to the corresponding part of Theorem 2.4. This completes the proof. □

Corollary 2.7 *Let* $(\mathcal{Y}, N)$ *be a fuzzy Banach space and* $\theta, p \neq 2$ *be positive real numbers. Suppose that* $f : \mathbb{R} \to \mathcal{Y}$ *is a mapping with* $f(0) = 0$ *such that, for all* $t > 0$,

$$N\big(f\big(\sqrt{x^2 + y^2}\big) - f(x) - f(y), t\big) \geq \frac{t}{t + \theta(|x|^p + |y|^p)} \tag{2.32}$$

uniformly on $\mathbb{R}$. *Then there exists a unique quadratic mapping* $Q : \mathbb{R} \to \mathcal{Y}$ *such that*

$$N\big(f(x) - Q(x), t\big) \geq \begin{cases} \frac{(2-2^{\frac{p}{2}})t}{(2-2^{\frac{p}{2}})t + 2(2+2^{\frac{p}{2}})\theta|x|^p}, & p < 2, \\ \frac{(2-2^{\frac{p}{2}})t}{(2-2^{\frac{p}{2}})t + 2^{\frac{p}{2}}(2+2^{\frac{p}{2}})\theta|x|^p}, & p > 2, \end{cases} \tag{2.33}$$

uniformly on $\mathbb{R}$.

Proof Taking $\phi(x,y) = \theta(|x|^p + |y|^p)$ for all $x, y \in \mathbb{R}$ and choosing $L = 2^{\frac{p}{2}}$, we have the desired result. □

Remark 2.8 The radical quadratic functional equation $f(\sqrt{x^2+y^2}) = f(x) + f(y)$ is not stable for $p = 2$ [11].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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