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Ulam type stability problems for alternative homomorphisms

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Abstract
We introduce an alternative homomorphism with respect to binary operations and investigate the Ulam type stability problem for such a mapping. The obtained results apply to Ulam type stability problems for several important functional equations.
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1 Introduction

In 1940, SM Ulam proposed the following stability problem: Given an approximately additive mapping, can one find the strictly additive mapping near it? A year later, DH Hyers gave an affirmative answer to this problem for additive mappings between Banach spaces. Subsequently many mathematicians came to deal with this problem (*cf.* [1–5]).

We introduce an alternative homomorphism from a set X with two binary operations $\circ$ and $*$ to another set E with two binary operations $\diamond$ and $\star$ defined by

$$f(x \circ y) \star f(x * y) = f(x) \diamond f(y) \quad (\forall x, y \in X),$$

and we investigate the Ulam type stability problem for such a mapping when E is a complete metric space. In particular, if $s \star t = s$ for all $s, t \in E$, then our results imply the stability results obtained in [6]. Also the method used in the paper have already applied for some other equations (*cf.* [7–15]).

One consequence of Banach's fixed point theorem

A fixed point theorem has played an important role in the stability problem (*cf.* [16]). The authors used an easy consequence of Banach's fixed point theorem in [6]. It will serve again in this paper. Here we review it.

Let X be a set and (E, d) a complete metric space. Fix two mappings $f : X \to E$ and $\varphi : X \to \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of all nonnegative real numbers. Denote by $\Delta_{f,\varphi}$ the set of all mappings $u : X \to E$ such that there exists a finite constant K_u satisfying

$$d\big(u(x), f(x)\big) \le K_u \varphi(x) \quad (\forall x \in X).$$

For any $u, v \in \Delta_{f,\varphi}$, we define

$$\rho_{f,\varphi}(u,v) = \inf\bigl\{K \geq 0 : d\bigl(u(x), v(x)\bigr) \leq K\varphi(x)\ (\forall x \in X)\bigr\}.$$

Then $(\Delta_{f,\varphi}, \rho_{f,\varphi})$ is a complete metric space which contains f.

Now, fix three mappings $\sigma : X \to X$, $\tau : E \to E$ and $\varepsilon : X \times X \to \mathbb{R}^+$. For any mapping $u : X \to E$, we define the mapping $T_{\sigma,\tau}u : X \to E$ by

$$(T_{\sigma,\tau}u)(x) = \tau\bigl(u(\sigma x)\bigr) \quad (x \in X).$$

Also, we consider three quantities:

$$\alpha_{\sigma,\varepsilon} = \inf\bigl\{K \geq 0 : \varepsilon(\sigma x, \sigma y) \leq K\varepsilon(x,y)\ (x,y \in X)\bigr\},$$
$$\beta_{\sigma,\varphi} = \inf\bigl\{K \geq 0 : \varphi(\sigma x) \leq K\varphi(x)\ (x \in X)\bigr\},$$
$$\gamma_\tau = \inf\bigl\{K \geq 0 : d(\tau s, \tau t) \leq Kd(s,t)\ (s,t \in E)\bigr\}.$$

If $\alpha_{\sigma,\varepsilon} < \infty$, $\beta_{\sigma,\varphi} < \infty$ and $\gamma_\tau < \infty$, then we have

$$\varepsilon(\sigma x, \sigma y) \leq \alpha_{\sigma,\varepsilon}\varepsilon(x,y) \quad (\forall x,y \in X),$$
$$\varphi(\sigma x) \leq \beta_{\sigma,\varphi}\varphi(x) \quad (\forall x \in X),$$
$$d(\tau s, \tau t) \leq \gamma_\tau d(s,t) \quad (\forall s,t \in E),$$

respectively. We will use these inequalities throughout this paper.

We now state our fixed point theorem.

Lemma A ([6, Proposition 2.1]) *Let X be a set and (E,d) a complete metric space. Suppose that four mappings $f : X \to E$, $\varphi : X \to \mathbb{R}^+$, $\sigma : X \to X$ and $\tau : E \to E$ satisfy*

$$T_{\sigma,\tau}f \in \Delta_{f,\varphi}, \qquad \beta_{\sigma,\varphi} < \infty, \qquad \gamma_\tau < \infty \quad \textit{and} \quad \beta_{\sigma,\varphi}\gamma_\tau < 1.$$

Then $T_{\sigma,\tau}(\Delta_{f,\varphi}) \subseteq \Delta_{f,\varphi}$ and $T_{\sigma,\tau}$ has a unique fixed point f_∞ in $\Delta_{f,\varphi}$. Moreover,

$$\lim_{n\to\infty} d\bigl(\bigl(T^n_{\sigma,\tau}f\bigr)(x), f_\infty(x)\bigr) = 0 \quad \textit{and} \quad d\bigl(f(x), f_\infty(x)\bigr) \leq \frac{\rho_{f,\varphi}(T_{\sigma,\tau}f, f)}{1 - \beta_{\sigma,\varphi}\gamma_\tau}\varphi(x)$$

for all $x \in X$.

2 A stability of alternative homomorphisms

Let $(X, \circ, *)$ be a set X with two binary operations $\circ$ and $*$. Let $(E, d, \diamond, \star)$ be a complete metric space (E,d) with two binary operations $\diamond$ and $\star$. Given $f : X \to E$, we consider the following commutative diagram:

$$\begin{array}{ccc} X \times X & \xrightarrow{(f\circ)\times(f*)} & E \times E \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle \star} \\ E \times E & \xrightarrow[\diamond]{} & E. \end{array} \tag{1}$$

This means that

$$f(x \circ y) \star f(x * y) = f(x) \diamond f(y) \quad (\forall x, y \in X). \tag{2}$$

In particular, if $s \star t = s$ for all $s, t \in E$, then (1) and (2) become

$$\begin{array}{ccc} X \times X & \xrightarrow{\ \circ\ } & X \\ {\scriptstyle f \times f}\downarrow & & \downarrow{\scriptstyle f} \\ E \times E & \xrightarrow[\ \diamond\ ]{} & E \end{array}$$

and

$$f(x \circ y) = f(x) \diamond f(y) \quad (\forall x, y \in X).$$

In other words, f is a homomorphism from X to E. Thus, if a mapping $f : X \to E$ satisfies (2), then we say that f is an *alternative homomorphism*.

In this section, we establish two general settings, on which we can give an affirmative answer to the Ulam type stability problem for the commutative diagram (1). These settings have a property such as duality, that is, each of them works as a complement of the other.

Let us describe the first setting. For $\varepsilon : X \times X \to \mathbb{R}^+$ and $\delta : X \to \mathbb{R}^+$, we consider the following three conditions:

(i) The square operator $x \mapsto x \circ x$ is an automorphism of X with respect to $\circ$ and $*$. We denote by σ the inverse mapping of this automorphism.

(ii) The binary operations $\diamond$ and $\star$ on E are continuous. The square operator $\tau : s \mapsto s \diamond s$ is an endomorphism of E with respect to $\diamond$ and $\star$.

(iii) $\alpha \equiv \alpha_{\sigma,\varepsilon} < \infty$, $\beta \equiv \beta_{\sigma,\delta} < \infty$, $\gamma \equiv \gamma_\tau < \infty$ and $\gamma \max\{\alpha, \beta\} < 1$.

Under the above conditions, we show the Ulam type stability for the commutative diagram (1), as follows.

Theorem 1 *Let* $(X, \circ, *)$ *and* $(E, d, \diamond, \star)$ *be as above. Suppose that four mappings* $\sigma : X \to X$, $\tau : E \to E$, $\varepsilon : X \times X \to \mathbb{R}^+$ *and* $\delta : X \to \mathbb{R}^+$ *satisfy* (i), (ii), *and* (iii). *If a mapping* $f : X \to E$ *satisfies*

$$d\bigl(f(x \circ y) \star f(x * y), f(x) \diamond f(y)\bigr) \le \varepsilon(x, y) \quad (\forall x, y \in X), \tag{3}$$

$$d\bigl(f(x) \star f(\sigma x * \sigma x), f(x)\bigr) \le \delta(x) \quad (\forall x \in X), \tag{4}$$

then there exists a mapping $f_\infty : X \to E$ *such that*

$$f_\infty(x \circ y) \star f_\infty(x * y) = f_\infty(x) \diamond f_\infty(y) \quad (\forall x, y \in X), \tag{5}$$

$$f_\infty(x) \star f_\infty(\sigma x * \sigma x) = f_\infty(x) \quad (\forall x \in X), \tag{6}$$

$$d\bigl(f(x), f_\infty(x)\bigr) \le \frac{\alpha \varepsilon(x, x) + \delta(x)}{1 - \gamma \max\{\alpha, \beta\}} \quad (\forall x \in X). \tag{7}$$

Moreover, if a mapping $g : X \to E$ *satisfies* (5), (6), *and*

$$\exists K_g \ge 0 : d\bigl(f(x), g(x)\bigr) \le K_g \bigl\{\alpha \varepsilon(x, x) + \delta(x)\bigr\} \quad (\forall x \in X), \tag{8}$$

then $g = f_\infty$.

Proof For simplicity, we write $T = T_{\sigma,\tau}$. We note that α, β, and γ are finite by (iii). Suppose that $f : X \to E$ satisfies (3) and (4). Put $\varphi(x) = \alpha\varepsilon(x,x) + \delta(x)$ for all $x \in X$. To apply Lemma A to f and φ, we first observe that $Tf \in \Delta_{f,\varphi}$. Fix $x \in X$. Replacing x and y in (3) by σx, we get

$$d\bigl(f(\sigma x \circ \sigma x) \star f(\sigma x * \sigma x), f(\sigma x) \diamond f(\sigma x)\bigr) \le \varepsilon(\sigma x, \sigma x).$$

Since

$$\begin{aligned} &\sigma x \circ \sigma x = \sigma^{-1}(\sigma x) = x, \\ &f(\sigma x) \diamond f(\sigma x) = \tau\bigl(f(\sigma x)\bigr) = (Tf)(x), \end{aligned}$$

and

$$\varepsilon(\sigma x, \sigma x) \le \alpha\varepsilon(x,x),$$

it follows that

$$d\bigl(f(x) \star f(\sigma x * \sigma x), (Tf)(x)\bigr) \le \alpha\varepsilon(x,x).$$

Using this and (4), we have

$$\begin{aligned} d\bigl((Tf)(x), f(x)\bigr) &\le d\bigl((Tf)(x), f(x) \star f(\sigma x * \sigma x)\bigr) + d\bigl(f(x) \star f(\sigma x * \sigma x), f(x)\bigr) \\ &\le \alpha\varepsilon(x,x) + \delta(x) \\ &= \varphi(x). \end{aligned}$$

Hence $Tf \in \Delta_{f,\varphi}$ and $\rho_{f,\varphi}(Tf,f) \le 1$.

We next estimate the quantity $\beta_{\sigma,\varphi}$. For $x \in X$, we have

$$\begin{aligned} \varphi(\sigma x) &= \alpha\varepsilon(\sigma x, \sigma x) + \delta(\sigma x) \\ &\le \alpha^2\varepsilon(x,x) + \beta\delta(x) \\ &\le \max\{\alpha,\beta\}\bigl(\alpha\varepsilon(x,x) + \delta(x)\bigr) \\ &= \max\{\alpha,\beta\}\varphi(x). \end{aligned}$$

Hence $\beta_{\sigma,\varphi} \le \max\{\alpha,\beta\}$ and $\beta_{\sigma,\varphi}\gamma_\tau \le \gamma \max\{\alpha,\beta\} < 1$ by (iii).

Thus we can apply Lemma A. As a consequence, T has a unique fixed point $f_\infty \in \Delta_{f,\varphi}$. Moreover,

$$\lim_{n\to\infty} d\bigl((T^n f)(x), f_\infty(x)\bigr) = 0 \tag{9}$$

and

$$d\bigl(f(x), f_\infty(x)\bigr) \le \frac{\rho_{f,\varphi}(Tf,f)}{1 - \beta_{\sigma,\varphi}\gamma_\tau}\varphi(x) \tag{10}$$

for all $x \in X$. Since $\rho_{f,\varphi}(Tf,f) \le 1$ and $\beta_{\sigma,\varphi}\gamma_\tau \le \gamma \max\{\alpha,\beta\} < 1$, (10) implies (7).

Here we show (5). If $x, y \in X$ and $n \in \mathbb{N}$, then we have

$$\begin{aligned}
&d\big(f_\infty(x \circ y) \star f_\infty(x * y), f_\infty(x) \diamond f_\infty(y)\big) \\
&\quad\leq d\big(f_\infty(x \circ y) \star f_\infty(x * y), \big(T^n f\big)(x \circ y) \star \big(T^n f\big)(x * y)\big) \\
&\qquad + d\big(\big(T^n f\big)(x \circ y) \star \big(T^n f\big)(x * y), \big(T^n f\big)(x) \diamond \big(T^n f\big)(y)\big) \\
&\qquad + d\big(\big(T^n f\big)(x) \diamond \big(T^n f\big)(y), f_\infty(x) \diamond f_\infty(y)\big).
\end{aligned} \tag{11}$$

We will see that the right hand side of (11) tends to 0 as $n \to \infty$. The first and third terms on the right hand side tend to 0 as $n \to \infty$, because of (9) and the continuity of $\star$ and $\diamond$ in (ii). Moreover, the second term, say $A_n(x, y)$, is estimated as follows: By (i), (ii), and (3), we have

$$\begin{aligned}
A_n(x, y) &= d\big(\tau^n\big(f\big(\sigma^n(x \circ y)\big)\big) \star \tau^n\big(f\big(\sigma^n(x * y)\big)\big), \tau^n\big(f\big(\sigma^n x\big)\big) \diamond \tau^n\big(f\big(\sigma^n y\big)\big)\big) \\
&= d\big(\tau^n\big(f\big(\sigma^n x \circ \sigma^n y\big)\big) \star \tau^n\big(f\big(\sigma^n x * \sigma^n y\big)\big), \tau^n\big(f\big(\sigma^n x\big) \diamond f\big(\sigma^n y\big)\big)\big) \\
&= d\big(\tau^n\big(f\big(\sigma^n x \circ \sigma^n y\big) \star f\big(\sigma^n x * \sigma^n y\big)\big), \tau^n\big(f\big(\sigma^n x\big) \diamond f\big(\sigma^n y\big)\big)\big) \\
&\leq \gamma^n d\big(f\big(\sigma^n x \circ \sigma^n y\big) \star f\big(\sigma^n x * \sigma^n y\big), f\big(\sigma^n x\big) \diamond f\big(\sigma^n y\big)\big) \\
&\leq \gamma^n \varepsilon\big(\sigma^n x, \sigma^n y\big) \\
&\leq \gamma^n \alpha^n \varepsilon(x, y),
\end{aligned}$$

where τ^n and σ^n denote the n-fold compositions of endomorphisms τ and σ, respectively. Since $\gamma\alpha < 1$ by (iii), it follows that $A_n(x, y) \to 0$ as $n \to \infty$. Thus the right hand side of (11) tends to 0, and we obtain (5).

Next, we show (6). For $x \in X$, we replace x and y in (5) by σx to get

$$f_\infty(\sigma x \circ \sigma x) \star f_\infty(\sigma x * \sigma x) = f_\infty(\sigma x) \diamond f_\infty(\sigma x).$$

Since $\sigma x \circ \sigma x = x$ and

$$f_\infty(\sigma x) \diamond f_\infty(\sigma x) = \tau\big(f_\infty(\sigma x)\big) = (T f_\infty)(x) = f_\infty(x),$$

we obtain (6).

Finally, we show the last statement. Since g satisfies (5) and (6), we have

$$\begin{aligned}
(Tg)(x) &= \tau\big(g(\sigma x)\big) = g(\sigma x) \diamond g(\sigma x) \\
&= g(\sigma x \circ \sigma x) \star g(\sigma x * \sigma x) \\
&= g(x) \star g(\sigma x * \sigma x) \\
&= g(x)
\end{aligned}$$

for all $x \in X$. This says that g is a fixed point of T. Also, by (8), we have $g \in \Delta_{f,\varphi}$. Thus the uniqueness of a fixed point of T in $\Delta_{f,\varphi}$ implies that $g = f_\infty$. □

The next corollary is obtained in [6].

Corollary 1 ([6, Corollary 3.2]) *Let X be a set with a binary operation $\circ$ such that the square operation $x \mapsto x \circ x$ is an automorphism of X with respect to $\circ$ and E a complete metric space with a continuous binary operation $\diamond$ such that the square operation $\tau : s \mapsto s \diamond s$ is an endomorphism of E with respect to $\diamond$. Let $\varepsilon : X \times X \to \mathbb{R}^+$ and suppose that $\alpha \equiv \alpha_{\sigma,\varepsilon} < \infty$, $\gamma \equiv \gamma_\tau < \infty$ and $\gamma\alpha < 1$, where σ denotes the inverse mapping of the square operation $x \mapsto x \circ x$. If a mapping $f : X \to E$ satisfies*

$$d\big(f(x \circ y), f(x) \diamond f(y)\big) \le \varepsilon(x,y) \quad (\forall x, y \in X),$$

then there exists a unique mapping $f_\infty : X \to E$ such that

$$f_\infty(x \circ y) = f_\infty(x) \diamond f_\infty(y) \quad \text{and} \quad d\big(f(x), f_\infty(x)\big) \le \frac{\alpha}{1 - \alpha\gamma}\varepsilon(x,x)$$

for all $x, y \in X$.

Proof Consider the case that $* = \circ$ and $s \star t = s$ for $s, t \in E$, in Theorem 1. In this case, τ is clearly an endomorphism of E with respect to $\star$. Therefore the corollary follows immediately from Theorem 1 with $\delta = 0$. □

Now we turn to another setting. Let $(X, \circ, *)$ and $(E, d, \diamond, \star)$ be as in the first part of this section. For $\varepsilon : X \times X \to \mathbb{R}^+$ and $\delta : X \to \mathbb{R}^+$, we consider the following three conditions:

- (iv) The square operator $\tilde{\sigma} : x \mapsto x \circ x$ is an endomorphism of X with respect to $\circ$ and $*$.
- (v) The binary operations $\diamond$ and $\star$ on E are continuous. The square operator $s \mapsto s \diamond s$ is an automorphism of E with respect to $\diamond$ and $\star$. We denote by $\tilde{\tau}$ the inverse mapping of this automorphism.
- (vi) $\tilde{\alpha} \equiv \alpha_{\tilde{\sigma},\varepsilon} < \infty$, $\tilde{\beta} \equiv \beta_{\tilde{\sigma},\delta} < \infty$, $\tilde{\gamma} \equiv \gamma_{\tilde{\tau}} < \infty$, and $\tilde{\gamma}\max\{\tilde{\alpha}, \tilde{\beta}\} < 1$.

Under the above conditions, we show the Ulam type stability for the commutative diagram (1), as follows.

Theorem 2 *Let $(X, \circ, *)$ and $(E, d, \diamond, \star)$ be as above. Suppose that four mappings $\tilde{\sigma} : X \to X$, $\tilde{\tau} : E \to E$, $\varepsilon : X \times X \to \mathbb{R}^+$ and $\delta : X \to \mathbb{R}^+$ satisfy* (iv), (v), *and* (vi). *If a mapping $f : X \to E$ satisfies* (3) *and*

$$d\big(f(x \circ x) \star f(x * x), f(x \circ x)\big) \le \delta(x) \quad (\forall x \in X), \tag{12}$$

then there exists a mapping $f_\infty : X \to E$ satisfying (5)

$$f_\infty(x \circ x) \star f_\infty(x * x) = f_\infty(x \circ x) \quad (\forall x \in X), \tag{13}$$

$$d\big(f(x), f_\infty(x)\big) \le \frac{\tilde{\gamma}\{\varepsilon(x,x) + \delta(x)\}}{1 - \tilde{\gamma}\max\{\tilde{\alpha}, \tilde{\beta}\}} \quad (\forall x \in X). \tag{14}$$

Moreover, if a mapping $g : X \to E$ satisfies (13), (14), *and*

$$\exists K_g \ge 0 : d\big(f(x), g(x)\big) \le K_g \tilde{\gamma}\big\{\varepsilon(x,x) + \delta(x)\big\} \quad (\forall x \in X), \tag{15}$$

then $g = f_\infty$.

Proof For simplicity, we write $\tilde{T} = T_{\tilde{\sigma},\tilde{\tau}}$, that is, $(\tilde{T}f)(x) = \tilde{\tau}(f(\tilde{\sigma}x))$ for $x \in X$. We note that $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are finite by (vi). Suppose that $f : X \to E$ satisfies (3) and (12). Put $\tilde{\varphi}(x) = \tilde{\gamma}\{\varepsilon(x,x) + \delta(x)\}$ for all $x \in X$. To apply Lemma A to f and $\tilde{\varphi}$, we first observe that $\tilde{T}f \in \Delta_{f,\tilde{\varphi}}$. Fix $x \in X$. Since $\tilde{\tau}(f(x) \diamond f(x)) = f(x)$, it follows from (3) and (12) that

$$\begin{aligned}
&d\big((\tilde{T}f)(x),f(x)\big) \\
&\quad = d\big(\tilde{\tau}\big(f(\tilde{\sigma}x)\big),f(x)\big) \\
&\quad = d\big(\tilde{\tau}\big(f(x\circ x)\big),\tilde{\tau}\big(f(x)\diamond f(x)\big)\big) \\
&\quad \le \tilde{\gamma}\, d\big(f(x\circ x),f(x)\diamond f(x)\big) \\
&\quad \le \tilde{\gamma}\big\{d\big(f(x\circ x),f(x\circ x)\star f(x*x)\big) + d\big(f(x\circ x)\star f(x*x),f(x)\diamond f(x)\big)\big\} \\
&\quad \le \tilde{\gamma}\big\{\delta(x) + \varepsilon(x,x)\big\} \\
&\quad = \tilde{\varphi}(x).
\end{aligned}$$

Hence $\tilde{T}f \in \Delta_{f,\tilde{\varphi}}$ and $\rho_{f,\tilde{\varphi}}(\tilde{T}f,f) \le 1$.

We next estimate the quantity $\beta_{\tilde{\sigma},\tilde{\varphi}}$. For $x \in X$, we have

$$\begin{aligned}
\tilde{\varphi}(\tilde{\sigma}x) &= \tilde{\gamma}\big\{\varepsilon(\tilde{\sigma}x,\tilde{\sigma}x) + \delta(\tilde{\sigma}x)\big\} \\
&\le \tilde{\gamma}\big\{\tilde{\alpha}\varepsilon(x,x) + \tilde{\beta}\delta(x)\big\} \\
&\le \tilde{\gamma}\max\{\tilde{\alpha},\tilde{\beta}\}\big\{\varepsilon(x,x) + \delta(x)\big\} \\
&= \max\{\tilde{\alpha},\tilde{\beta}\}\tilde{\varphi}(x).
\end{aligned}$$

Hence $\beta_{\tilde{\sigma},\tilde{\varphi}} \le \max\{\tilde{\alpha},\tilde{\beta}\}$ and $\beta_{\tilde{\sigma},\tilde{\varphi}}\gamma_{\tilde{\tau}} \le \tilde{\gamma}\max\{\tilde{\alpha},\tilde{\beta}\} < 1$ by (vi).

Thus we can apply Lemma A. As a consequence, $\tilde{T}$ has a unique fixed point $f_\infty \in \Delta_{f,\tilde{\varphi}}$. Moreover,

$$\lim_{n\to\infty} d\big(\big(\tilde{T}^n f\big)(x),f_\infty(x)\big) = 0 \tag{16}$$

and

$$d\big(f(x),f_\infty(x)\big) \le \frac{\rho_{f,\tilde{\varphi}}(\tilde{T}f,f)}{1-\beta_{\tilde{\sigma},\tilde{\varphi}}\gamma_{\tilde{\tau}}}\tilde{\varphi}(x) \tag{17}$$

for all $x \in X$. Since $\rho_{f,\tilde{\varphi}}(\tilde{T}f,f) \le 1$ and $\beta_{\tilde{\sigma},\tilde{\varphi}}\gamma_{\tilde{\tau}} \le \tilde{\gamma}\max\{\tilde{\alpha},\tilde{\beta}\} < 1$, (17) implies (14).

Here we show (5). If $x,y \in X$ and $n \in \mathbb{N}$, then we have

$$\begin{aligned}
&d\big(f_\infty(x\circ y)\star f_\infty(x*y),f_\infty(x)\diamond f_\infty(y)\big) \\
&\quad \le d\big(f_\infty(x\circ y)\star f_\infty(x*y),\big(\tilde{T}^n f\big)(x\circ y)\star\big(\tilde{T}^n f\big)(x*y)\big) \\
&\qquad + d\big(\big(\tilde{T}^n f\big)(x\circ y)\star\big(\tilde{T}^n f\big)(x*y),\big(\tilde{T}^n f\big)(x)\diamond\big(\tilde{T}^n f\big)(y)\big) \\
&\qquad + d\big(\big(\tilde{T}^n f\big)(x)\diamond\big(\tilde{T}^n f\big)(y),f_\infty(x)\diamond f_\infty(y)\big).
\end{aligned}$$

Letting $n \to \infty$, the first and third terms on the right hand side tend to 0, because of (16) and the continuity of $\star$ and $\diamond$ in (v). Moreover, the second term, say $\tilde{A}_n(x,y)$, is estimated as follows: By (iv), (v), and (3),

$$\begin{aligned}\tilde{A}_n(x,y) &= d\big(\tilde{\tau}^n\big(f\big(\tilde{\sigma}^n(x\circ y)\big)\big)\star\tilde{\tau}^n\big(f\big(\tilde{\sigma}^n(x*y)\big)\big),\tilde{\tau}^n\big(f\big(\tilde{\sigma}^n x\big)\big)\diamond\tilde{\tau}^n\big(f\big(\tilde{\sigma}^n y\big)\big)\big)\\ &= d\big(\tilde{\tau}^n\big(f\big(\tilde{\sigma}^n x\circ\tilde{\sigma}^n y\big)\big)\star\tilde{\tau}^n\big(f\big(\tilde{\sigma}^n x*\tilde{\sigma}^n y\big)\big),\tilde{\tau}^n\big(f\big(\tilde{\sigma}^n x\big)\diamond f\big(\tilde{\sigma}^n y\big)\big)\big)\\ &= d\big(\tilde{\tau}^n\big(f\big(\tilde{\sigma}^n x\circ\tilde{\sigma}^n y\big)\star f\big(\tilde{\sigma}^n x*\tilde{\sigma}^n y\big)\big),\tilde{\tau}^n\big(f\big(\tilde{\sigma}^n x\big)\diamond f\big(\tilde{\sigma}^n y\big)\big)\big)\\ &\le \tilde{\gamma}^n d\big(f\big(\tilde{\sigma}^n x\circ\tilde{\sigma}^n y\big)\star f\big(\tilde{\sigma}^n x*\tilde{\sigma}^n y\big),f\big(\tilde{\sigma}^n x\big)\diamond f\big(\tilde{\sigma}^n y\big)\big)\\ &\le \tilde{\gamma}^n\varepsilon\big(\tilde{\sigma}^n x,\tilde{\sigma}^n y\big)\\ &\le \tilde{\gamma}^n\tilde{\alpha}^n\varepsilon(x,y),\end{aligned}$$

where $\tilde{\tau}^n$ and $\tilde{\sigma}^n$ denote the n-fold compositions of endomorphisms $\tilde{\tau}$ and $\tilde{\sigma}$, respectively. Since $\tilde{\gamma}\tilde{\alpha}<1$ by (vi), it follows that $\tilde{A}_n(x,y)\to 0$ as $n\to\infty$. Thus we obtain (5).

Next, we show (13). Replacing y in (5) by x, we have

$$f_\infty(x\circ x)\star f_\infty(x*x)=f_\infty(x)\diamond f_\infty(x). \tag{18}$$

Also since

$$\tilde{\tau}\big(f_\infty(x\circ x)\big)=\tilde{\tau}\big(f_\infty(\tilde{\sigma}x)\big)=(\tilde{T}f_\infty)(x)=f_\infty(x)=\tilde{\tau}\big(f_\infty(x)\diamond f_\infty(x)\big),$$

it follows that

$$f_\infty(x\circ x)=f_\infty(x)\diamond f_\infty(x).$$

Combining with (18), we obtain (13).

Finally, we show the last statement. Since g satisfies (14) and (13), we have

$$g(\tilde{\sigma}x)=g(x\circ x)=g(x\circ x)\star g(x*x)=g(x)\diamond g(x)=\tilde{\tau}^{-1}\big(g(x)\big),$$

that is, $(\tilde{T}g)(x)=g(x)$ for all $x\in X$. This says that g is a fixed point of $\tilde{T}$. Also, by (15), we have $g\in\Delta_{f,\tilde{\varphi}}$. Hence the uniqueness of a fixed point of $\tilde{T}$ in $\Delta_{f,\tilde{\varphi}}$ implies that $g=f_\infty$. □

The next corollary is obtained in [6].

Corollary 2 ([6, Corollary 3.5]) *Let X be a set with a binary operation $\circ$ such that the square operation $\tilde{\sigma}:x\mapsto x\circ x$ is an endomorphism of X with respect to $\circ$ and E a complete metric space with a continuous binary operation $\diamond$ such that the square operation $s\mapsto s\diamond s$ is an automorphism of E with respect to $\diamond$. Let $\varepsilon:X\times X\to\mathbb{R}^+$ and suppose that $\tilde{\alpha}\equiv\alpha_{\tilde{\sigma},\varepsilon}<\infty$, $\tilde{\gamma}\equiv\gamma_{\tilde{\tau}}<\infty$ and $\tilde{\gamma}\tilde{\alpha}<1$, where $\tilde{\tau}$ denotes the inverse mapping of the square operation $s\mapsto s\diamond s$. If a mapping $f:X\to E$ satisfies*

$$d\big(f(x\circ y),f(x)\diamond f(y)\big)\le\varepsilon(x,y)\quad(\forall x,y\in X),$$

then there exists a unique mapping $f_\infty:X\to E$ such that

$$f_\infty(x\circ y)=f_\infty(x)\diamond f_\infty(y)\quad\textit{and}\quad d\big(f(x),f_\infty(x)\big)\le\frac{\tilde{\gamma}}{1-\tilde{\alpha}\tilde{\gamma}}\varepsilon(x,x)$$

for all $x,y\in X$.

Proof Consider the case that $* = \circ$ and $s \star t = s$ for $s, t \in E$, in Theorem 2. Then $\tilde{\tau}$ is clearly an endomorphism of E with respect to $\star$. Therefore the corollary follows immediately from Theorem 2 with $\delta = 0$. □

3 Application I

The Ulam type stability problem for Euler-Lagrange type additive mappings has been investigated in [17]. Here we take up the following Euler-Lagrange type mapping $f : X \to E$ satisfying

$$f(ax + by) + f(bx + ay) + (a + b)\big(f(-x) + f(-y)\big) = 0 \quad (\forall x, y \in X), \tag{19}$$

where X is a complex normed space, E a complex Banach space and $a, b \in \mathbb{C}$ with $a + b \neq 0$. The following is an Ulam type stability result for this mapping.

Corollary 3 (*cf.* [17, Theorem 2.1]) *Let* $\varepsilon : X \times X \to \mathbb{R}^+$ *and suppose that*

(vii) $\exists K \geq 0 : |a + b|K < 1$ *and* $\varepsilon(x, y) \leq K\varepsilon(-(a + b)x, -(a + b)y)$ $(\forall x, y \in X)$.

If a mapping $f : X \to E$ *satisfies*

$$\big\| f(ax + by) + f(bx + ay) + (a + b)\big(f(-x) + f(-y)\big)\big\| \leq \varepsilon(x, y) \quad (\forall x, y \in X), \tag{20}$$

then there exists a unique mapping $f_\infty : X \to E$ *satisfying* (19) *and*

$$\big\| f(x) - f_\infty(x)\big\| \leq \frac{K}{2(1 - |a + b|K)}\varepsilon(-x, -x) \quad (\forall x \in X). \tag{21}$$

Proof Put $u = -x$, $v = -y$ for each $x, y \in X$. Under these transformations, (20) changes into the following estimate:

$$\left\| \frac{1}{2}\big\{f(-au - bv) + f(-bu - av)\big\} + \frac{a + b}{2}\big\{f(u) + f(v)\big\}\right\| \leq \varepsilon_1(u, v) \quad (\forall u, v \in X), \tag{22}$$

where $\varepsilon_1(u, v) = \frac{1}{2}\varepsilon(-u, -v)$ $(\forall u, v \in X)$.

Now we define $u \circ v = -au - bv$, $u * v = -bu - av$ for each $u, v \in X$. In this case, we can easily see that the square operator $u \mapsto u \circ u$ is an endomorphism of X with respect to $\circ$ and $*$. Also since $a + b \neq 0$, this endomorphism is bijective and so automorphic. We denote by σ the inverse mapping of this automorphism. Moreover, we define $s \diamond t = -\frac{1}{2}(a + b)(s + t)$, $s \star t = \frac{1}{2}(s + t)$ for each $s, t \in E$. Then we can also see that the binary operations $\diamond$ and $\star$ on E are continuous and the square operator $\tau : s \mapsto s \diamond s$ is an automorphism of E with respect to $\diamond$ and $\star$. Note that (22) changes into the following:

$$\big\| f(u \circ v) \star f(u * v) - f(u) \diamond f(v)\big\| \leq \varepsilon_1(u, v) \quad (\forall u, v \in X). \tag{23}$$

Since $x \circ x = x * x$ for all $x \in X$, it follows that $\sigma x * \sigma x = \sigma x \circ \sigma x = \sigma^{-1}\sigma x = x$ for all $x \in X$. Also, since $s \star s = s$ for all $s \in E$, it follows that $f(x) \star f(\sigma x * \sigma x) = f(x) \star f(x) = f(x)$ for all $x \in X$ and then (4) holds with $\delta = 0$. Moreover, $\beta_{\sigma,\delta} = 0$ must hold with $\delta = 0$. It is also obvious that $\gamma_\tau = |a + b|$ from the definition of τ. We also note that $\alpha_{\sigma,\varepsilon_1} \leq K$ from the second condition of (vii) and hence $\gamma_\tau \alpha_{\sigma,\varepsilon_1} \le |a+b|K < 1$ from the first condition of (vii). Therefore, by Theorem 1, there exists a unique mapping $f_\infty : X \to E$ such that

$$f_\infty(u \circ v) \star f_\infty(u * v) = f_\infty(u) \diamond f_\infty(v) \quad (\forall u, v \in X),$$

namely, (19) holds and

$$\|f(u) - f_\infty(u)\| \le \frac{\alpha_{\sigma,\varepsilon_1}\varepsilon_1(u,u)}{1 - \gamma_\tau \max\{\alpha_{\sigma,\varepsilon_1}, \beta_{\sigma,\delta}\}} \le \frac{K}{2(1 - |a+b|K)}\varepsilon(-u,-u) \quad (\forall u \in X),$$

and so (21) holds. □

The following is also an Ulam type stability result for the mapping satisfying (19).

Corollary 4 (*cf.* [17, Theorem 2.2]) *Let* $\varepsilon : X \times X \to \mathbb{R}^+$ *and suppose that*

(viii) $\exists K \ge 0 : K < |a+b|$ *and* $\varepsilon(-(a+b)x, -(a+b)y) \le K\varepsilon(x,y)$ $(\forall x, y \in X)$.

If a mapping $f : X \to E$ *satisfies* (20), *then there exists a unique mapping* $f_\infty : X \to E$ *satisfying* (19) *and*

$$\|f(x) - f_\infty(x)\| \le \frac{1}{2(|a+b| - K)}\varepsilon(-x,-x) \quad (\forall x \in X). \tag{24}$$

Proof As observed in the proof of Corollary 3, (20) changes into (22). Now we define $u \circ v = -au - bv$, $u * v = -bu - av$ for each $u, v \in X$. In this case, we can easily see that the square operator $\tilde{\sigma} : u \mapsto u \circ u$ is an endomorphism of X with respect to $\circ$ and $*$. Moreover, we define $s \diamond t = -\frac{1}{2}(a+b)(s+t)$, $s \star t = \frac{1}{2}(s+t)$ for each $s, t \in E$. Then we can also see that the binary operations $\diamond$ and $\star$ on E are continuous and the square operator $s \mapsto s \diamond s$ is an endomorphism of E with respect to $\diamond$ and $\star$. Also since $a + b \ne 0$, this endomorphism is bijective and so automorphic. We denote by $\tilde{\tau}$ the inverse mapping of this automorphism. Note that (22) changes into (23). Since $x \circ x = x * x$ $(\forall x \in X)$ and $s \star s = s$ $(\forall s \in E)$, it follows that $f(x \circ x) \star f(x * x) = f(x \circ x)$ for all $x \in X$ and then (12) holds with $\delta = 0$.

Moreover, $\beta_{\tilde{\sigma},\delta} = 0$ must hold with $\delta = 0$. It is also obvious that $\gamma_{\tilde{\tau}} = |a+b|^{-1}$ from the definition of $\tilde{\tau}$. We also note that $\alpha_{\tilde{\sigma},\varepsilon_1} \le K$ from the second condition of (viii) and hence $\gamma_{\tilde{\tau}}\alpha_{\tilde{\sigma},\varepsilon_1} \le |a+b|^{-1}K < 1$ from the first condition of (viii).

Therefore, by Theorem 2, there exists a unique mapping $f_\infty : X \to E$ such that

$$f_\infty(u \circ v) \star f_\infty(u * v) = f_\infty(u) \diamond f_\infty(v) \quad (\forall u, v \in X),$$

namely, (19) holds and

$$\begin{aligned}\|f(u) - f_\infty(u)\| &\le \frac{\gamma_{\tilde{\tau}}\varepsilon_1(u,u)}{1 - \gamma_{\tilde{\tau}} \max\{\alpha_{\tilde{\sigma},\varepsilon_1}, \beta_{\tilde{\sigma},\delta}\}} \\ &\le \frac{|a+b|^{-1}}{2(1 - |a+b|^{-1}K)}\varepsilon(-u,-u) \\ &= \frac{1}{2(|a+b| - K)}\varepsilon(-u,-u) \quad (\forall u \in X),\end{aligned}$$

and so (24) holds. □

Corollary 5 (*cf.* [17, Corollary 2.3]) *Suppose that* $|a+b| \neq 1$, $\delta, p, q \geq 0$ *and* $p+q \neq 1$. *If a mapping* $f : X \to E$ *satisfies*

$$\left\| f(ax+by) + f(bx+ay) + (a+b)\{f(-x)+f(-y)\} \right\| \leq \delta \|x\|^p \|y\|^q$$

for all $x, y \in X$, *then there exists a unique mapping* $f_\infty : X \to E$ *satisfying* (19) *and*

$$\left\| f(x) - f_\infty(x) \right\| \leq \frac{\delta}{2(||a+b|^{p+q} - |a+b||} \|x\|^{p+q} \quad (\forall x \in X).$$

Proof Put $\varepsilon(x,y) = \delta \|x\|^p \|y\|^q$ for each $x, y \in X$.

(a) The case where either

$$\begin{cases} |a+b| > 1, \\ p+q > 1, \end{cases}$$

or

$$\begin{cases} |a+b| < 1, \\ p+q < 1. \end{cases}$$

Put $K = |a+b|^{-(p+q)}$. Then K satisfies (vii). Note also that

$$\frac{K}{2(1-|a+b|K)} \varepsilon(-x,-x) = \frac{\delta}{2(|a+b|^{p+q} - |a+b|)} \|x\|^{p+q}$$

for all $x \in X$. Then the desired result follows from Corollary 3.

(b) The case where either

$$\begin{cases} |a+b| > 1, \\ p+q < 1, \end{cases}$$

or

$$\begin{cases} |a+b| < 1, \\ p+q > 1. \end{cases}$$

Put $K = |a+b|^{p+q}$. Then K satisfies (viii). Note also that

$$\frac{1}{2(|a+b| - K)} \varepsilon(-x,-x) = \frac{\delta}{2(|a+b| - |a+b|^{p+q})} \|x\|^{p+q}$$

for all $x \in X$. Then the desired result follows from Corollary 4. □

4 Application II

Let $(X, +)$ be an Abelian group. In [18], the following result has been shown by A. Simon and P. Volkmann.

Lemma B ([18, Théorème 1)] *A mapping $f : X \to \mathbb{R}$ satisfies*

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y) \quad (\forall x, y \in X), \tag{25}$$

if and only if $f(x) = |\pi(x)|$ $(\forall x \in X)$ for some additive function $\pi : X \to \mathbb{R}$.

In this section, we deal with the Ulam type stability problem for Equation (25). Put $x \circ y = x + y$ and $x * y = x - y$ for each $x, y \in X$. Moreover, put $s \diamond t = s + t$ and $s \star t = \max\{s, t\}$ for each $s, t \in \mathbb{R}$. Then (25) changes into (2). Also we can easily see that the square operation $\tilde{\sigma} : x \mapsto x \circ x$ is endomorphic with respect to $\circ$ and $*$ and that the square operator $s \mapsto s \diamond s$ is automorphic with respect to $\diamond$ and $\star$. Denote by $\tilde{\tau}$ the inverse mapping of this automorphism. In this case, it is obvious that $\tilde{\tau}(s) = \frac{1}{2}s$ for each $s \in \mathbb{R}$ and hence $\gamma_{\tilde{\tau}} = 1/2$.

Now let ε be a nonnegative constant and suppose that $f : X \to \mathbb{R}$ satisfies

$$\left|\max\{f(x+y), f(x-y)\} - \{f(x) + f(y)\}\right| \le \varepsilon \quad (\forall x, y \in X). \tag{26}$$

Putting $x = y = 0$ in (26), we obtain

$$\left|f(0)\right| \le \varepsilon. \tag{27}$$

Also, putting $x = y$ in (26), we obtain

$$-\varepsilon + f(0) \le -\varepsilon + \max\{f(x+x), f(0)\} \le 2f(x) \quad (\forall x \in X). \tag{28}$$

Combining (27) and (28), we obtain

$$-\varepsilon \le f(x) \quad (\forall x \in X). \tag{29}$$

Put $\delta = 2\varepsilon$. By (27) and (28), we obtain

$$0 \le \max\{f(x+x), f(0)\} - f(x+x) \le \varepsilon + \varepsilon = \delta \quad (\forall x \in X),$$

and hence (12) holds. Moreover, note that $\alpha_{\tilde{\sigma},\varepsilon} = \beta_{\tilde{\sigma},\delta} = 1$ since ε and δ are constant. Then Lemma B and Theorem 2 easily imply the following.

Corollary 6 *Let X be an Abelian group and ε a nonnegative constant. If $f : X \to \mathbb{R}$ satisfies* (26), *then there exists an additive mapping $\pi : X \to \mathbb{R}$ such that*

$$\left|f(x) - \left|\pi(x)\right|\right| \le 3\varepsilon \quad (\forall x \in X).$$

For the related results, see [19, 20].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this paper. They read and approved the final manuscript.

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