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Generalized (ξ, α)-expansive mappings and related fixed-point theorems

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Abstract
In this paper, we introduce a new class of expansive mappings called generalized (ξ, α)-expansive mappings and investigate the existence of a fixed point for the mappings in this class. We conclude that several fixed-point theorems can be considered as a consequence of main results. Moreover, some examples are given to illustrate the usability of the obtained results.
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1 Introduction

Fixed-point theory has attracted many mathematicians since it provides a simple proof for the existence and uniqueness of the solutions to various mathematical models (integral and partial differential equations, variational inequalities etc.). After the celebrated results of Banach [1], fixed-point theory became one of the most interesting topics in nonlinear analysis. Consequently, a number of the papers have appeared since then; see *e.g.* [2–10] and references therein. Among them, we mention the α-ψ-contractive mapping, which was introduced by Samet *et al.* [9] via α-admissible mappings. In this paper, the authors established various fixed-point theorems for such mappings in complete metric spaces. Furthermore, Samet *et al.* [9] stated that several existing results can be concluded from their main results. For the sake of completeness, we recall some basic definitions and fundamental results.

Definition 1.1 [9] Let φ denote the family of all functions $\psi : [0, +\infty) \to [0, +\infty)$ which satisfy:

(i) $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.
(ii) ψ is non-decreasing.

Definition 1.2 [9] Let (X, d) be a metric space and $T : X \to X$ be a given self mapping. T is said to be an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \varphi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi\big(d(x, y)\big)$$

for all $x, y \in X$.

Definition 1.3 [9] Let $T: X \to X$ and $\alpha: X \times X \to [0, +\infty)$. T is said to be α-admissible if

$$x, y \in X, \quad \alpha(x,y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Now, we give some examples of α-admissible mappings.

Example 1.4 Let X be the set of all non-negative real numbers. Let us define the mapping $\alpha: X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \geq y, \\ 0 & \text{if } x < y \end{cases}$$

and define the mapping $T: X \to X$ by $Tx = x^2$ for all $x \in X$. Then T is α-admissible.

In what follows, we present the main results of Samet *et al.* [9].

Theorem 1.1 [9] *Let (X,d) be a complete metric space and $T: X \to X$ be an α-ψ-contractive mapping satisfying the following conditions*:

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *T is continuous.*

Then T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 1.2 [9] *Let (X,d) be a complete metric space and $T: X \to X$ be an α-ψ-contractive mapping satisfying the following conditions*:

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then T has a fixed point.

Samet *et al.* [9] added the following condition (H) to the hypotheses of Theorem 1.1 and Theorem 1.2 to assure the uniqueness of the fixed point:

(H) For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.

Afterwards, Karapınar and Samet [10] generalized these notions to obtain further fixed-point results in the setting of complete metric space.

Definition 1.5 [10] Let (X,d) be a metric space and $T: X \to X$ be a given mapping. We say that T is a generalized α-ψ-contractive mapping if there exist two functions $\alpha: X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$, we have

$$\alpha(x,y) d(Tx, Ty) \leq \psi\big(M(x,y)\big), \tag{1}$$

where $M(x,y) = \max\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

Theorem 1.3 [10] *Let (X,d) be a complete metric space. Suppose that $T: X \to X$ is a generalized α-ψ-contractive mapping and satisfies the following conditions*:

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *T is continuous*.

Then there exists $u \in X$ such that $Tu = u$.

Theorem 1.4 [10] *Let (X,d) be a complete metric space. Suppose that $T : X \to X$ is a generalized α-ψ-contractive mapping and the following conditions hold*:
(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

Then there exists $u \in X$ such that $Tu = u$.

Theorem 1.5 [10] *Adding condition* (H) *to the hypotheses of Theorem* 1.3 (*resp. Theorem* 1.4), *we find that u is the unique fixed point of T.*

On the other hand, Shahi *et al.* [11] introduced a new category of expansive mappings called (ξ,α)-expansive mappings as a complement of the concept of α-ψ-contractive type mappings. The authors in [11] also studied several fixed-point theorems for these mappings in the context of complete metric spaces.

We recollect the notion of (ξ,α)-expansive mappings as follows. Let χ denote all functions $\xi : [0,+\infty) \to [0,+\infty)$ which satisfy the following properties:

(ξ_{i}) ξ is non-decreasing,
(ξ_{ii}) $\sum_{n=1}^{+\infty} \xi^n(t) < +\infty$ for each $t > 0$, where ξ^n is the nth iterate of ξ,
(ξ_{iii}) $\xi(s+t) = \xi(s) + \xi(t)$, $\forall s,t \in [0,+\infty)$.

Lemma 1.6 [9] *If $\xi : [0,+\infty) \to [0,+\infty)$ is a non-decreasing function, then for each $t > 0$, $\lim_{n\to+\infty} \xi^n(t) = 0$ implies $\xi(t) < t$.*

Definition 1.6 [11] Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an (ξ,α)-expansive mapping if there exist two functions $\xi \in \chi$ and $\alpha : X \times X \to [0,+\infty)$ such that

$$\xi\bigl(d(Tx,Ty)\bigr) \geq \alpha(x,y)d(x,y) \tag{2}$$

for all $x,y \in X$.

Remark 1.1 If $T : X \to X$ is an expansion mapping, then T is an (ξ,α)-expansive mapping, where $\alpha(x,y) = 1$ for all $x,y \in X$ and $\xi(a) = ka$ for all $a \geq 0$ and some $k \in [0,1)$.

The main result of Shahi *et al.* [11] is the following.

Theorem 1.7 [11] *Let (X,d) be a complete metric space and $T : X \to X$ be a bijective, (ξ,α)-expansive mapping satisfying the following conditions*:
(i) *T^{-1} is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$*;

(iii) *T is continuous.*

Then T has a fixed point, that is, there exists $u \in X$ such that $Tu = u$.

Motivated by the above ideas, we aim to give a new concept of generalized (ξ,α)-expansive mappings. The results proved in this paper extend and generalize many existing results in the literature. We also illustrate some examples to support our statements.

2 Main results

We begin this section with the following definition.

Definition 2.1 Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a generalized (ξ,α)-expansive mapping if there exists two functions $\xi \in \chi$ and $\alpha : X \times X \to [0,+\infty)$ such that for all $x, y \in X$, we have

$$\xi\bigl(d(Tx, Ty)\bigr) \geq \alpha(x,y)m(x,y), \tag{3}$$

where $m(x,y) = \min\{d(x,y), d(x,Tx), d(y,Ty)\}$.

Theorem 2.1 *Let (X,d) be a complete metric space and $T : X \to X$ be a bijective, generalized (ξ,α)-expansive mapping satisfying the following conditions*:

(i) *T^{-1} is α-admissible*;

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$*;

(iii) *T is continuous.*

Then T has a fixed point, that is, there exists $u \in X$ such that $Tu = u$.

Proof Let $x_0 \in X$ be such that $\alpha(x_0, T^{-1}x_0) \geq 1$. Let us define the sequence $\{x_n\}$ in X by

$$x_n = Tx_{n+1}, \quad \text{for all } n \in \mathbb{N}.$$

Now, if $x_n = x_{n+1}$ for any $n \in \mathbb{N}$, one sees that x_n is a fixed point of T from the definition. Without loss of generality, we can suppose $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. Since T^{-1} is an α-admissible mapping and $\alpha(x_0, T^{-1}x_0) \geq 1$, we deduce that $\alpha(T^{-1}x_0, T^{-1}x_1) = \alpha(x_1, x_2) \geq 1$. Continuing this process, we get

$$\alpha(x_n, x_{n+1}) \geq 1, \tag{4}$$

for all $n \in \mathbb{N} \cup \{0\}$. Applying inequality (3) with $x = x_n$, $y = x_{n+1}$, we obtain

$$d(x_{n-1}, x_n) > \xi\bigl(d(Tx_n, Tx_{n+1})\bigr) \geq \alpha(x_n, x_{n+1})m(x_n, x_{n+1}).$$

Owing to the fact that $\alpha(x_n, x_{n+1}) \geq 1$ for all n, we have

$$d(x_{n-1}, x_n) > \xi\bigl(d(Tx_n, Tx_{n+1})\bigr) \geq \min\bigl\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\bigr\}.$$

Now, if $\min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} = d(x_{n-1}, x_n)$ for some $n \in \mathbb{N}$, then

$$d(x_{n-1}, x_n) > \xi\bigl(d(Tx_n, Tx_{n+1})\bigr) \geq d(x_{n-1}, x_n),$$

which is a contradiction. Hence, for all $n \in \mathbb{N}$, we obtain

$$\xi\bigl(d(x_{n-1},x_n)\bigr) \geq d(x_n,x_{n+1}).$$

By induction, we have

$$\xi^n\bigl(d(x_0,x_1)\bigr) \geq d(x_n,x_{n+1}).$$

For any $n > m \geq 0$, we infer that

$$\begin{aligned} d(x_m,x_n) &\leq d(x_m,x_{m+1}) + d(x_{m+1},x_{m+2}) + d(x_{m+2},x_{m+3}) + \cdots + d(x_{n-1},x_n) \\ &\leq \xi^m\bigl(d(x_0,x_1)\bigr) + \cdots + \xi^{n-1}\bigl(d(x_0,x_1)\bigr). \end{aligned}$$

From (ξ_{ii}), it follows that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X,d). So, there exists $u \in X$ such that $x_n \to u$ as $n \to +\infty$. From the continuity of T, it follows that $x_n = Tx_{n+1} \to Tu$ as $n \to +\infty$. Owing to the uniqueness of the limit, we get $u = Tu$, that is, u is a fixed point of T. This completes the proof. □

In the sequel, we prove that Theorem 2.1 still holds for T not necessarily continuous, assuming the following condition:

(P) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $\{x_n\} \to x \in X$ as $n \to +\infty$, then

$$\alpha\bigl(T^{-1}x_n, T^{-1}x\bigr) \geq 1, \tag{5}$$

for all n.

Theorem 2.2 *If in Theorem* 2.1 *we replace the continuity of T by the condition* (P), *then the result holds true.*

Proof Following the proof of Theorem 2.1, we see that $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $\{x_n\} \to u \in X$ as $n \to +\infty$. Now, in view of condition (P), we infer that

$$\alpha\bigl(T^{-1}x_n, T^{-1}u\bigr) \geq 1 \tag{6}$$

for all $n \in \mathbb{N}$. Owing to inequalities (3) and (4), we get

$$\begin{aligned} m\bigl(T^{-1}x_n, T^{-1}u\bigr) &\leq \alpha\bigl(T^{-1}x_n, T^{-1}u\bigr)m\bigl(T^{-1}x_n, T^{-1}u\bigr) \\ &\leq \xi\bigl(d(x_n,u)\bigr). \end{aligned}$$

Letting $n \to +\infty$ in the above inequality and due to the continuity of ξ at $t = 0$, we obtain

$$m\bigl(u, T^{-1}u\bigr) = 0.$$

That is, either $d(u, T^{-1}u) = 0$ or $d(u, Tu) = 0$. This implies that u is a fixed point of T. □

We shall present some examples to illustrate the validity of our results.

Example 2.2 Let $X = [0, +\infty)$ endowed with the metric

$$d(x,y) = \begin{cases} \max\{x,y\} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Define the mappings $T : X \to X$ and $\xi \in \chi$ by $T(x) = x^2$ for all $x \in X$. Consider the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Clearly, T is continuous and generalized (ξ, α)-expansive mapping with $\xi(a) = \frac{a}{2}$ for all $a \geq 0$. In fact, for all $x, y \in X$, we have

$$\frac{1}{2} d(Tx, Ty) \geq \alpha(x,y) m(x,y).$$

Moreover, there exists $x_0 = 0 \in X$ such that $\alpha(x_0, T^{-1}x_0) = \alpha(0,0) \geq 1$. Now, we proceed to show that T^{-1} is α-admissible. Let $x, y \in X$ such that $\alpha(x,y) \geq 1$ implying that $x = y$. Now, by the definition of T^{-1} and α, we obtain $T^{-1}x = T^{-1}y$. Thus, $\alpha(T^{-1}x, T^{-1}y) \geq 1$, that is, T^{-1} is α-admissible. Now, all the conditions of Theorem 2.1 are satisfied. Consequently, T has a fixed point. In this example, 0 and 1 are two fixed points of T.

Now, we give an example involving a function T that is not continuous.

Example 2.3 Let $X = [0, +\infty)$ endowed with the standard metric $d(x,y) = |x - y|$ for all $x, y \in X$. Define the mappings $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x < 1, \\ x & \text{if } x \geq 1 \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \geq 1, \\ 0 & \text{if } x, y \in [0,1). \end{cases}$$

Owing to the discontinuity of T at 1, Theorem 2.1 is not applicable in this case. Clearly, T is a generalized (ξ, α)-expansive mapping with $\xi(a) = \frac{a}{2}$ for all $a \geq 0$. In fact, for all $x, y \in X$, we infer that

$$\frac{1}{2} d(Tx, Ty) \geq \alpha(x,y) m(x,y).$$

Moreover, there exists $x_0 = 1 \in X$ such that $\alpha(x_0, T^{-1}x_0) = \alpha(1,1) \geq 1$. Now, we need to show that T^{-1} is α-admissible. Let $x, y \in X$ such that $\alpha(x,y) \geq 1$ implying that $x \geq 1$ and $y \geq 1$. By the definition of T^{-1} and α, we get $\alpha(T^{-1}x, T^{-1}y) \geq 1$. Thus, T^{-1} is α-admissible.

Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to x \in X$ as $n \to +\infty$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all n, in view of definition of α, we get $x_n \geq 1$ for all n and $x \geq 1$. Thus, $\alpha(T^{-1}x_n, T^{-1}x) = 1$.

Therefore, all the conditions of Theorem 2.2 are satisfied, and so T has a fixed point. Here, 0 and 1 are two fixed points of T.

If we take $\alpha(x,y) = 1$ in Theorem 2.1, we get the following result.

Corollary 2.3 *Let (X,d) be a complete metric space and $T : X \to X$ be a bijective map. Suppose that T satisfies the following condition*:

$$\xi\bigl(d(Tx,Ty)\bigr) \geq m(x,y), \tag{7}$$

where $m(x,y) = \min\{d(x,y), d(x,Tx), d(y,Ty)\}$ and $\xi \in \chi$. Suppose also that

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$;*
(ii) *T is continuous.*

Then T has a fixed point, that is, there exists $u \in X$ such that $Tu = u$.

Remark 2.1 Let (X,d) be a metric space and $T : X \to X$ be a map. Then the following inequality is evidently satisfied:

$$\begin{aligned}\xi\bigl(d(Tx,Ty)\bigr) &\geq \alpha(x,y)\bigl[ad(x,y) + bd(x,Tx) + bd(y,Ty)\bigr]\\ &\geq \alpha(x,y)\bigl[d(x,y) + d(x,Tx) + d(y,Ty)\bigr]\\ &\geq \alpha(x,y)d(x,y)\\ &\geq \alpha(x,y)\min\bigl\{d(x,y), d(x,Tx), d(y,Ty)\bigr\},\end{aligned}$$

where $a,b,c > 1$.

As a consequence of the observation above, one can deduce the following results from Theorem 2.1.

Corollary 2.4 *Let (X,d) be a complete metric space and $T : X \to X$ be a bijective. Suppose that T satisfies the following condition*:

$$\xi\bigl(d(Tx,Ty)\bigr) \geq \alpha(x,y)\bigl[ad(x,y) + bd(x,Tx) + cd(y,Ty)\bigr], \tag{8}$$

where $a + b + c > 1$ and $\xi \in \chi$. Suppose also that

(i) *T^{-1} is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$;*
(iii) *T is continuous.*

Then T has a fixed point, that is, there exists $u \in X$ such that $Tu = u$.

Corollary 2.5 *Let (X,d) be a complete metric space and $T : X \to X$ be a bijective map. Suppose that T satisfies for $\xi \in \chi$*

$$\xi\bigl(d(Tx,Ty)\bigr) \geq \alpha(x,y)\bigl[d(x,y) + d(x,Tx) + d(y,Ty)\bigr]. \tag{9}$$

We suppose also that

(i) *T^{-1} is α-admissible;*

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, T^{-1}x_0) \geq 1$;
(iii) T *is continuous.*
Then T *has a fixed point, that is, there exists* $u \in X$ *such that* $Tu = u$.

Corollary 2.6 *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a bijective map. Suppose that* T *satisfies for* $\xi \in \chi$

$$\xi\big(d(Tx, Ty)\big) \geq \alpha(x,y)d(x,y). \tag{10}$$

Suppose also that
(i) T^{-1} *is* α*-admissible*;
(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, T^{-1}x_0) \geq 1$;
(iii) T *is continuous.*
Then T *has a fixed point, that is, there exists* $u \in X$ *such that* $Tu = u$.

Corollary 2.7 *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a bijective map. Suppose that* T *is continuous, satisfying the following condition*:

$$\xi\big(d(Tx, Ty)\big) \geq d(x,y), \tag{11}$$

where $\xi \in \chi$. *Then* T *has a fixed point, that is, there exists* $u \in X$ *such that* $Tu = u$.

Proof By taking $\alpha(x,y) = 1$ for all $x, y \in X$ in Corollary 2.6, we get the proof of this corollary. □

Corollary 2.8 *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a bijective map. Suppose that* T *is continuous, satisfying the following condition*:

$$d(Tx, Ty) \geq ad(x,y), \tag{12}$$

where $a > 1$. *Then* T *has a fixed point, that is, there exists* $u \in X$ *such that* $Tu = u$.

Proof By taking $\xi(t) = kt$, where $k < 1$ and $a = \frac{1}{k}$ in Corollary 2.7, we get the proof of this corollary. □

Remark 2.2 If we replace the continuity assumption of T by the condition (P) in Corollary 2.5, Corollary 2.4, Corollary 2.6, Corollary 2.7,Corollary 2.8, then the result holds true.

3 Consequences

3.1 Fixed-point theorems on metric spaces endowed with a partial order

Recently, there have been tremendous growth in the study of fixed-point problems of contractive mappings in metric spaces endowed with a partial order. The first result in this direction was given by Turinici [12], where he generalized the Banach contraction principle in partially ordered sets. Some applications of Turinici's theorem to matrix equations were demonstrated by Ran and Reurings [13]. Later, numerous important results had been obtained concerning the existence of a fixed point for contraction type mappings in partially ordered metric spaces by Bhaskar and Lakshmikantham [4], Nieto and Lopez [7, 14], Agarwal *et al.* [15], Lakshmikantham and Ćirić [6] and Samet [16] etc. In this section, we will deduce some fixed-point results on a metric space endowed with a partial order. For this, we require the following concepts.

Definition 3.1 Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a given mapping. We say that T is non-decreasing with respect to $\preceq$ if

$$x, y \in X, \quad x \preceq y \quad \Rightarrow \quad Tx \preceq Ty. \tag{13}$$

Definition 3.2 Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be non-decreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all n.

Definition 3.3 Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that $(X, \preceq, d)$ is regular if for every non-decreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\{x_{n(k)}\} \preceq x$ for all k.

Now, we have the following result.

Corollary 3.1 *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a bijective mapping such that T^{-1} is a non-decreasing mapping with respect to $\preceq$ satisfying the following condition for all $x, y \in X$ with $x \succeq y$:*

$$\xi\big(d(Tx, Ty)\big) \geq m(x, y), \tag{14}$$

where $\xi \in \chi$ and

$$m(x, y) = \min\big\{d(x, y), d(x, Tx), d(y, Ty)\big\}.$$

Suppose also that

(i) *there exists $x_0 \in X$ such that $x_0 \preceq T^{-1}x_0$;*
(ii) *T is continuous or $(X, \preceq, d)$ is regular.*

Then T has a fixed point, that is, there exists $u \in X$ such that $Tu = u$.

Proof Let us define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y \text{ or } x \succeq y, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, T is a generalized (ξ, α)-expansive mapping, that is,

$$\xi\big(d(Tx, Ty)\big) \geq \alpha(x, y)m(x, y), \tag{15}$$

for all $x, y \in X$. In view of condition (i), we have $\alpha(x_0, T^{-1}x_0) \geq 1$. Owing to the monotonicity of T^{-1}, we get

$$\alpha(x, y) \geq 1 \quad \Rightarrow \quad x \succeq y \quad \text{or} \quad x \preceq y \quad \Rightarrow \quad T^{-1}x \succeq T^{-1}y \quad \text{or}$$
$$T^{-1}x \preceq T^{-1}y \quad \Rightarrow \quad \alpha\big(T^{-1}x, T^{-1}y\big) \geq 1.$$

This shows that T^{-1} is α-admissible. Now, if T is continuous, the existence of a fixed point follows from Theorem 2.1. Suppose now that $(X,\preceq,d)$ is regular. Assume that $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in X$ as $n\to\infty$. Due to the regularity hypotheses, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)}\preceq x$ for all k. Now, in view of the definition of α, we obtain $\alpha(x_{n(k)},x)\geq 1$ for all k. Thus, we get the existence of a fixed point in this case from Theorem 2.2. □

Regarding Lemma 2.1, the following is a natural consequence of the above corollary.

Corollary 3.2 *Let $(X,\preceq)$ be a partially ordered set and d be a metric on X such that (X,d) is complete. Let $T:X\to X$ be a bijective mapping such that T^{-1} is a non-decreasing mapping with respect to $\preceq$ satisfying the following condition for all $x,y\in X$ with $x\succeq y$:*

$$\xi\bigl(d(Tx,Ty)\bigr)\geq ad(x,y)+bd(x,Tx)+cd(y,Ty), \tag{16}$$

where $a+b+c>1$. Suppose also that
 (i) *there exists $x_0\in X$ such that $x_0\preceq T^{-1}x_0$;*
 (ii) *T is continuous or $(X,\preceq,d)$ is regular.*
Then T has a fixed point, that is, there exists $u\in X$ such that $Tu=u$.

Corollary 3.3 *Let $(X,\preceq)$ be a partially ordered set and d be a metric on X such that (X,d) is complete. Let $T:X\to X$ be a bijective mapping such that T^{-1} be a non-decreasing mapping with respect to $\preceq$ satisfying the following condition for all $x,y\in X$ with $x\succeq y$:*

$$\xi\bigl(d(Tx,Ty)\bigr)\geq d(x,y), \tag{17}$$

where $\xi\in\chi$. Suppose also that
 (i) *there exists $x_0\in X$ such that $x_0\preceq T^{-1}x_0$;*
 (ii) *T is continuous or $(X,\preceq,d)$ is regular.*
Then T has a fixed point, that is, there exists $u\in X$ such that $Tu=u$.

Proof If we take $m(x,y)=d(x,y)$ in Corollary 3.1, then we get the proof of this corollary. □

Corollary 3.4 *Let $(X,\preceq)$ be a partially ordered set and d be a metric on X such that (X,d) is complete. Let $T:X\to X$ be a bijective mapping such that T^{-1} is a non-decreasing mapping with respect to $\preceq$ satisfying the following condition for all $x,y\in X$ with $x\succeq y$:*

$$d(Tx,Ty)\geq kd(x,y), \tag{18}$$

where $k>1$. Suppose also that
 (i) *there exists $x_0\in X$ such that $x_0\preceq T^{-1}x_0$;*
 (ii) *T is continuous or $(X,\preceq,d)$ is regular.*
Then T has a fixed point, that is, there exists $u\in X$ such that $Tu=u$.

Proof If we take $\xi(t)=at$, where $a<1$ and $k=\frac{1}{a}$ in Corollary 3.3, then we get the proof of this corollary. □

3.2 Fixed-point theorems for cyclic contractive mappings

Kirk *et al.* [17] in 2003 generalized the Banach contraction mapping principle by introducing cyclic representations and cyclic contractions. A mapping $T : A \cup B \to A \cup B$ is called cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$, where A, B are nonempty subsets of a metric space (X,d). Moreover, T is called a cyclic contraction if there exists $k \in (0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x \in A$ and $y \in B$. It is to be noted that although a contraction is continuous, cyclic contractions need not be. This is one of the important benefits of this theorem. In the last decade, various authors have used the cyclic representations and cyclic contractions to derive various fixed-point results. See for example [18–23].

Corollary 3.5 *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X,d) and $T : Y \to Y$ be a given bijective mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold*:

(I) $T^{-1}(A_1) \subseteq A_2$ *and* $T^{-1}(A_2) \subseteq A_1$;

(II) *there exists a function* $\xi \in \chi$ *such that*

$$\xi\big(d(Tx,Ty)\big) \geq m(x,y), \quad \forall (x,y) \in A_1 \times A_2. \tag{19}$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Proof As A_1 and A_2 are closed subsets of the complete metric space (X,d), then (Y,d) is complete. Let us define the mapping

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (A_1 \times A_2), \\ 0 & \text{otherwise.} \end{cases}$$

In view of (II) and the definition of α, we infer that

$$\xi\big(d(Tx,Ty)\big) \geq m(x,y), \quad \forall (x,y) \in A_1 \times A_2 \tag{20}$$

for all $x, y \in Y$. Thus, T is a generalized (ξ,α)-expansive mapping.

Let $(x,y) \in Y \times Y$ such that $\alpha(x,y) \geq 1$. If $(x,y) \in A_1 \times A_2$, from (I), $(T^{-1}x, T^{-1}y) \in A_2 \times A_1$, which implies that $\alpha(T^{-1}x, T^{-1}y) \geq 1$. Therefore, in all cases, we have $\alpha(T^{-1}x, T^{-1}y) \geq 1$. This implies that T^{-1} is α-admissible. Also, in view of (I), for any $u \in A_1$, we get $(u, T^{-1}u) \in A_1 \times A_2$, thereby implying that $\alpha(u, T^{-1}u) \geq 1$.

Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. From the definition of α, we have

$$(x_n, x_{n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1), \quad \forall n. \tag{21}$$

As $(A_1 \times A_2) \cup (A_2 \times A_1)$ is a closed set with respect to the Euclidean metric, we infer that

$$(x,x) \in (A_1 \times A_2) \cup (A_2 \times A_1), \tag{22}$$

which implies that $x \in A_1 \cap A_2$. So, we obtain from the definition of α the result that $\alpha(x_n, x) \geq 1$ for all n. □

From Lemma 2.1, one get deduce the following result.

Corollary 3.6 *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X,d) and $T: Y \to Y$ be a given bijective mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold*:

(I) $T^{-1}(A_1) \subseteq A_2$ *and* $T^{-1}(A_2) \subseteq A_1$;

(II) *there exist constants a, b, c such that*

$$\xi\big(d(Tx,Ty)\big) \geq ad(x,y) + bd(x,Tx) + cd(y,Ty), \quad \text{where } a+b+c>1, \tag{23}$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Corollary 3.7 *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X,d) and $T: Y \to Y$ be a given bijective mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold*:

(I) $T^{-1}(A_1) \subseteq A_2$ *and* $T^{-1}(A_2) \subseteq A_1$;

(II) *there exists a function $\xi \in \chi$ such that*

$$\xi\big(d(Tx,Ty)\big) \geq d(x,y), \quad \forall (x,y) \in A_1 \times A_2. \tag{24}$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Proof If we take $m(x,y) = d(x,y)$ in Corollary 3.5, then we get the proof of this corollary. □

Corollary 3.8 *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X,d) and $T: Y \to Y$ be a given bijective mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold*:

(I) $T^{-1}(A_1) \subseteq A_2$ *and* $T^{-1}(A_2) \subseteq A_1$;

(II) *there exists a constant $k > 1$ such that*

$$d(Tx,Ty) \geq kd(x,y), \quad \forall (x,y) \in A_1 \times A_2. \tag{25}$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Proof By taking $\xi(t) = at$, where $a < 1$ and $k = \frac{1}{a}$ in Corollary 3.7, we get the proof of this corollary. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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References

1. Banach, S: Sur les operations dans les ensembles abstraits et leur application aux equations itegrales. Fundam. Math. **3**, 133-181 (1922)
2. Caccioppoli, R: Un teorema generale sull-esistenza di elementi uniti in una trasformazione funzionale. Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. **11**, 794-799 (1930) (in Italian)
3. Kannan, R: Some results on fixed points. Bull. Calcutta Math. Soc. **10**, 71-76 (1968)
4. Bhaskar, TG, Lakshmikantham, V: Fixed point theory in partially ordered metric spaces and applications. Nonlinear Anal. **65**, 1379-1393 (2006)
5. Branciari, A: A fixed point theorem for mappings satisfying a general contractive condition of integral type. Int. J. Math. Math. Sci. **29**, 531-536 (2002)
6. Lakshmikantham, V, Ćirić, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**, 4341-4349 (2009)
7. Nieto, JJ, Lopez, RR: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-239 (2005)
8. Saadati, R, Vaezpour, SM: Monotone generalized weak contractions in partially ordered metric spaces. Fixed Point Theory **11**, 375-382 (2010)
9. Samet, B, Vetro, C, Vetro, P: Fixed point theorem for α-ψ contractive type mappings. Nonlinear Anal. **75**, 2154-2165 (2012)
10. Karapınar, E, Samet, B: Generalized α-ψ-contractive type mappings and related fixed point theorems with applications. Abstr. Appl. Anal. **2012**, Article ID 793486 (2012). doi:10.1155/2012/793486
11. Shahi, P, Kaur, J, Bhatia, SS: Fixed point theorems for (ξ,α)-expansive mappings in complete metric spaces. Fixed Point Theory Appl. **2012**, Article ID 157 (2012)
12. Turinici, M: Abstract comparison principles and multivariable Gronwall-Bellman inequalities. J. Math. Anal. Appl. **117**, 100-127 (1986)
13. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**, 1435-1443 (2004)
14. Nieto, JJ, Lopez, RR: Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. Acta Math. Sin. Engl. Ser. **23**, 2205-2212 (2007)
15. Agarwal, RP, El-Gebeily, MA, O'Regan, D: Generalized contractions in partially ordered metric spaces. Appl. Anal. **87**, 1-8 (2008)
16. Samet, B: Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces. Nonlinear Anal. TMA (2010). doi:10.1016/j.na.2010.02.026
17. Kirk, WA, Srinivasan, PS, Veeramani, P: Fixed points for mappings satisfying cyclical contractive conditions. Fixed Point Theory **4**, 79-89 (2003)
18. Agarwal, RP, Alghamdi, MA, Shahzad, N: Fixed point theory for cyclic generalized contractions in partial metric spaces. Fixed Point Theory Appl. **2012**, Article ID 40 (2012)
19. Karapınar, E: Fixed point theory for cyclic weak ϕ-contraction. Appl. Math. Lett. **24**, 822-825 (2011)
20. Karapınar, E, Sadaranagni, K: Fixed point theory for cyclic $(\phi - \psi)$-contractions. Fixed Point Theory Appl. **2011**, Article ID 69 (2011)
21. Pacurar, M, Rus, IA: Fixed point theory for cyclic φ-contractions. Nonlinear Anal. **72**, 1181-1187 (2010)
22. Petric, MA: Some results concerning cyclic contractive mappings. Gen. Math. **18**, 213-226 (2010)
23. Rus, IA: Cyclic representations and fixed points. Ann. "Tiberiu Popoviciu" Sem. Funct. Equ. Approx. Convexity **3**, 171-178 (2005)

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