Journal of Inequalities and Applications
a SpringerOpen Journal

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A discussion on generalized almost contractions via rational expressions in partially ordered metric spaces

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Abstract
The main purpose of this paper is to give some fixed point results for mappings involving generalized (ϕ, ψ)-contractions in partially ordered metric spaces. Our results generalize, extend, and unify several well-known comparable results in the literature (Jaggi in Indian J. Pure Appl. Math. 8(2):223-230, 1977, Harjani *et al.* in Nonlinear Anal. 71:3403-3410, 2009, Luong and Thuan in Fixed Point Theory Appl. 2011:46, 2011). The presented results are supported by three illustrative examples.

MSC: 46N40; 47H10; 54H25; 46T99

Keywords: ordered set; metric space; fixed point

1 Introduction and preliminaries

The Banach contraction mapping principle [1] is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its application in a vast number of branches of mathematics. Generalizations of this principle have been investigated heavily (see Jaggi [2], Harjani *et al.* [3], Luong and Thuan [4]). In particular, in 1977, Jaggi [2] proved the following theorem satisfying a contractive condition of a rational type.

Theorem 1 *Let (X,d) be a complete metric space. Let $T : X \to X$ be a continuous mapping such that*

$$d(Tx, Ty) \le \alpha \frac{d(x, Tx)d(y, Ty)}{d(x,y)} + \beta d(x,y) \tag{1.1}$$

for all distinct points $x, y \in X$ where $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$. Then T has a unique fixed point.

Existence of fixed point in partially ordered sets has been recently studied in [3–53].

Recently, Harjani *et al.* [3] proved the ordered version of Theorem 1. Very recently, Luong and Thuan [4] generalized the results of [3] and proved the following.

Theorem 2 *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a metric d such that (X, d) is a metric space. Let $T : X \to X$ be a non-decreasing mapping such that*

$$d(Tx, Ty) \leq M(x, y) - \psi\bigl(M(x, y)\bigr) \tag{1.2}$$

for all distinct points $x, y \in X$ with $y \leq x$ where $\psi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with the property that $\psi(t) = 0$ if and only if $t = 0$, and

$$M(x, y) = \max\left\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\right\}. \tag{1.3}$$

Also, assume either

(i) *T is continuous or*

(ii) *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Set $\Phi = \{\phi \mid \phi : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing with $\phi(t) = 0$ if and only if $t = 0\}$ and $\Psi = \{\psi \mid \psi : [0, \infty) \to [0, \infty)$ is lower semi continuous, $\psi(t) > 0$ for all $t > 0$, and $\psi(0) = 0\}$. For some work on the class of Φ or the class of Ψ, we refer the reader to [21, 51, 54].

In 2004, Berinde [55] introduced an almost contraction, a new class of contractive type mappings which exhibits totally different features more than the one of the particular results incorporated [1, 16, 39, 50], *i.e.*, an almost contraction generally does not have a unique fixed point; see Example 1 in [55]. Thereafter, many authors presented several interesting and useful facts about almost contractions; see [42, 56–59].

The purpose of this article is to generalize the above results for a mapping $T : X \to X$ involving a generalized (ϕ, ψ)-almost contraction. Some examples are also presented to show that our results are effective.

2 Main result

Our essential result is given as follows.

Theorem 3 *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a metric d such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping which satisfies the inequality*

$$\begin{aligned}\phi\bigl(d(Tx, Ty)\bigr) &\leq \phi\bigl(M(x, y)\bigr) - \psi\bigl(M(x, y)\bigr) \\ &\quad + L\min\bigl\{d(x, Ty), d(y, Tx), d(x, Tx), d(y, Ty)\bigr\}\end{aligned} \tag{2.1}$$

for all distinct points $x, y \in X$ with $y \leq x$ where $\phi \in \Phi$, $\psi \in \Psi$, $L \geq 0$ and

$$M(x, y) = \max\left\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\right\}.$$

Also, assume either

(i) *T is continuous or*

(ii) *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Proof Let $x_0 \in X$ such that $x_0 \preceq Tx_0$. We define a sequence $\{x_n\}$ in X as follows:

$$x_n = Tx_{n-1} \quad \text{for } n \geq 1. \tag{2.2}$$

Since T is a non-decreasing mapping together with (2.2), we have $x_2 = Tx_1$. Inductively, we obtain

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_{n-1} \preceq x_n \preceq x_{n+1} \preceq \cdots. \tag{2.3}$$

Assume that there exists n_0 such that $x_{n_0} = x_{n_0+1}$. Since $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then T has a fixed point. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Thus, by (2.3) we have

$$x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n \prec x_{n+1} \prec \cdots. \tag{2.4}$$

Regarding (2.4), the condition (2.1) implies that

$$\begin{aligned}
\phi\bigl(d(x_n,x_{n+1})\bigr) &= \phi\bigl(d(Tx_{n-1},Tx_n)\bigr) \\
&\leq \phi\bigl(M(x_{n-1},x_n)\bigr) - \psi\bigl(M(x_{n-1},x_n)\bigr) \\
&\quad + L\min\bigl\{d(x_{n-1},Tx_n), d(Tx_{n-1},x_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n)\bigr\} \\
&\leq \phi\bigl(M(x_{n-1},x_n)\bigr) - \psi\bigl(M(x_{n-1},x_n)\bigr) \\
&\quad + L\min\bigl\{d(x_{n-1},x_{n+1}), d(x_n,x_n), d(x_{n-1},x_n), d(x_n,x_{n+1})\bigr\} \\
&= \phi\bigl(M(x_{n-1},x_n)\bigr) - \psi\bigl(M(x_{n-1},x_n)\bigr),
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
M(x_{n-1},x_n) &= \max\left\{\frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{d(x_{n-1},x_n)}, d(x_{n-1},x_n)\right\} \\
&= \max\bigl\{d(x_n,x_{n+1}), d(x_{n-1},x_n)\bigr\}.
\end{aligned}$$

Suppose that $M(x_{n-1},x_n) = d(x_n,x_{n+1})$ for some $n \geq 1$. Then the inequality (2.5) turns into

$$\phi\bigl(d(x_n,x_{n+1})\bigr) \leq \phi\bigl(d(x_n,x_{n+1})\bigr) - \psi\bigl(d(x_n,x_{n+1})\bigr).$$

Regarding (2.4) and the property of ψ, this is a contradiction. Thus, $M(x_{n-1},x_n) = d(x_{n-1},x_n)$ for all $n \geq 1$. Therefore, the inequality (2.5) yields

$$\phi\bigl(d(x_n,x_{n+1})\bigr) \leq \phi\bigl(d(x_{n-1},x_n)\bigr) - \psi\bigl(d(x_{n-1},x_n)\bigr) < \phi\bigl(d(x_{n-1},x_n)\bigr). \tag{2.6}$$

Since ϕ is non-decreasing, we have $d(x_n,x_{n+1}) \leq d(x_{n-1},x_n)$. Consequently, $\{d(x_{n-1},x_n)\}$ is a decreasing sequence of positive real numbers which is bounded below. So, there exists $\alpha \geq 0$ such that $\lim_{n\to\infty} d(x_{n-1},x_n) = \alpha$. We claim that $\alpha = 0$. Suppose, to the contrary, that $\alpha > 0$. By taking the limit of the supremum in the relation $\phi(d(x_n,x_{n+1})) \leq \phi(d(x_{n-1},x_n)) - \psi(d(x_{n-1},x_n))$, as $n \to \infty$, we get

$$\phi(\alpha) \leq \phi(\alpha) - \psi(\alpha) < \phi(\alpha),$$

which is a contradiction. Hence, we conclude that $\alpha = 0$, that is,

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{2.7}$$

We prove that the sequence $\{x_n\}$ is Cauchy in X. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. So, there exists $\varepsilon > 0$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \tag{2.8}$$

where $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ are subsequences of $\{x_n\}$ with

$$n(k) > m(k) \geq k. \tag{2.9}$$

Moreover, $n(k)$ is chosen to be the smallest integer satisfying (2.8). Thus, we have

$$d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.10}$$

By the triangle inequality, we get

$$\begin{aligned} \varepsilon \leq d(x_{m(k)}, x_{n(k)}) &\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ &< \varepsilon + d(x_{n(k)-1}, x_{n(k)}). \end{aligned}$$

Keeping (2.7) in mind and letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{2.11}$$

Due to the triangle inequality, we have

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \tag{2.12}$$

and

$$d(x_{m(k)-1}, x_{n(k)-1}) \leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1}). \tag{2.13}$$

By using (2.7), (2.11), and letting $n \to \infty$ in (2.12) and (2.13), we get

$$\lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{2.14}$$

Analogously, we derive

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon \quad \text{and} \quad \lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon. \tag{2.15}$$

Since $m(k) < n(k)$ we have $x_{m(k)-1} < x_{n(k)-1}$. By (2.1) we have

$$\begin{aligned} &\phi\bigl(d(x_{m(k)}, x_{n(k)})\bigr) \\ &\quad = \phi\bigl(d(Tx_{m(k)-1}, Tx_{n(k)-1})\bigr) \end{aligned}$$

$$\begin{aligned}
&\le \phi\bigl(M(x_{m(k)-1}, x_{n(k)-1})\bigr) - \psi\bigl(M(x_{m(k)-1}, x_{n(k)-1})\bigr) \\
&\quad + L\min\bigl\{d(x_{n(k)-1}, Tx_{m(k)-1}), d(x_{m(k)-1}, Tx_{n(k)-1}), \\
&\quad d(x_{m(k)-1}, Tx_{m(k)-1}), d(x_{n(k)-1}, Tx_{n(k)-1})\bigr\} \\
&\le \phi\bigl(M(x_{m(k)-1}, x_{n(k)-1})\bigr) - \psi\bigl(M(x_{m(k)-1}, x_{n(k)-1})\bigr) \\
&\quad + L\min\bigl\{d(x_{n(k)-1}, x_{m(k)}), d(x_{m(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)})\bigr\},
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
&M(x_{m(k)-1}, x_{n(k)-1}) \\
&\quad = \max\left\{\frac{d(x_{m(k)-1}, Tx_{m(k)-1})d(x_{n(k)-1}, Tx_{n(k)-1})}{d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\} \\
&\quad = \max\left\{\frac{d(x_{m(k)-1}, x_{m(k)})d(x_{n(k)-1}, x_{n(k)})}{d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\}.
\end{aligned} \tag{2.17}$$

Letting $n \to \infty$ in (2.16) (and hence in (2.17)), and taking (2.7), (2.11), (2.14), and (2.15) into account, we obtain

$$\phi(\varepsilon) \le \phi\bigl(\max\{0, \varepsilon\}\bigr) - \psi\bigl(\max\{0, \varepsilon\}\bigr) + L\min\{\varepsilon, \varepsilon, 0, 0\} < \phi(\varepsilon), \tag{2.18}$$

which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete metric space, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

We will show that z is a fixed point of T. Assume that (i) holds. Then by the continuity of T, we have

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T\Bigl(\lim_{n\to\infty} x_{n-1}\Bigr) = Tz.$$

Suppose that (ii) holds. Since $\{x_n\}$ is a non-decreasing sequence and $\lim_{n\to\infty} x_n = z$ then $z = \sup\{x_n\}$. Hence, $x_n \preceq z$ for all $n \in \mathbb{N}$. Since T is a non-decreasing mapping, we conclude that $Tx_n \preceq Tz$, or equivalently,

$$x_n \preceq x_{n+1} \preceq Tz \quad \text{for all } n \in \mathbb{N}. \tag{2.19}$$

Then $z = \sup\{x_n\}$, and we get $z \preceq Tz$.

To this end, we construct a new sequence $\{y_n\}$ as follows:

$$y_0 = z \quad \text{and} \quad y_n = Ty_{n-1} \quad \text{for all } n \ge 1.$$

Since $z \preceq Tz$, we have $y_0 \preceq Ty_0 = y_1$. Hence we find that $\{y_n\}$ is a non-decreasing sequence. By repeating the discussion above, one can conclude that $\{y_n\}$ is Cauchy. Thus there exists $y \in X$ such that $\lim_{n\to\infty} y_n = y$. By (ii), we have $y = \sup\{y_n\}$ and so we have $y_n \preceq y$. From (2.19), we get

$$x_n < z = y_0 \preceq Tz = Ty_0 \preceq y_n \preceq y \quad \text{for all } n \in \mathbb{N}. \tag{2.20}$$

If $z = y$ then the proof is finished. Suppose that $z \neq y$. On account of (2.20), the expression (2.1) implies that

$$\begin{aligned}
\phi\bigl(d(x_{n+1}, y_{n+1})\bigr) = {} & \phi\bigl(d(Tx_n, Ty_n)\bigr) \\
& \leq \phi\bigl(M(x_n, y_n)\bigr) - \psi\bigl(M(x_n, y_n)\bigr) \\
& \quad + L \min\bigl\{d(x_n, Ty_n), d(y_n, Tx_n), d(x_n, Tx_n), d(y_n, Ty_n)\bigr\} \\
& \leq \phi\bigl(M(x_n, y_n)\bigr) - \psi\bigl(M(x_n, y_n)\bigr) \\
& \quad + L \min\bigl\{d(x_n, y_{n+1}), d(y_n, x_{n+1}), d(x_n, x_{n+1}), d(y_n, y_{n+1})\bigr\},
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
M(x_n, y_n) &= \max\left\{\frac{d(x_n, Tx_n)d(y_n, Ty_n)}{d(x_n, y_n)}, d(x_n, y_n)\right\} \\
&= \max\left\{\frac{d(x_n, x_{n+1})d(y_n, y_{n+1})}{d(x_n, y_n)}, d(x_n, y_n)\right\}.
\end{aligned} \tag{2.22}$$

Letting $n \to \infty$ in (2.21) (and hence (2.22)), we obtain

$$\phi\bigl(d(y, z)\bigr) \leq \phi\bigl(d(y, z)\bigr) - \psi\bigl(d(y, z)\bigr) < \phi\bigl(d(y, z)\bigr)$$

which is a contradiction. So $y = z$ and we have $z \leq Tz \leq z$, then $Tz = z$. □

If we take $L = 0$ in Theorem 3 we get the following result.

Theorem 4 *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a metric d such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping which satisfies the inequality*

$$\phi\bigl(d(Tx, Ty)\bigr) \leq \phi\bigl(M(x, y)\bigr) - \psi\bigl(M(x, y)\bigr) \tag{2.23}$$

for all distinct $x, y \in X$ with $y \leq x$ where $\phi \in \Phi$, $\psi \in \Psi$ and

$$M(x, y) = \max\left\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\right\}.$$

Also, assume either
(i) *T is continuous or*
(ii) *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$.*
If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Other corollaries could be derived.

Corollary 5 *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a metric d such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that*

$$d(Tx, Ty) \leq M(x, y) - \psi\bigl(M(x, y)\bigr) + L \min\bigl\{d(x, Ty), d(y, Tx), d(x, Tx), d(y, Ty)\bigr\} \tag{2.24}$$

for all distinct $x, y \in X$ with $y \leq x$ where $\psi \in \Psi$, $L \geq 0$ and

$$M(x,y) = \max\left\{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}, d(x,y)\right\}.$$

Also, assume either

(i) *T is continuous or*

(ii) *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Proof Take $\phi(t) = t$ in Theorem 3. □

Corollary 6 *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a metric d X such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that*

$$d(Tx, Ty) \leq kM(x,y) + L\min\{d(x,Ty), d(y,Tx), d(x,Tx), d(y,Ty)\}, \tag{2.25}$$

for all distinct $x, y \in X$ with $y \leq x$ where $L \geq 0$ and

$$M(x,y) = \max\left\{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}, d(x,y)\right\}.$$

Also, assume either

(i) *T is continuous or*

(ii) *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Proof Take $\psi(t) = (1-k)\psi(t)$ for all $t \in [0,\infty)$ in Corollary 5. □

Corollary 7 *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a metric d such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that*

$$d(Tx, Ty) \leq \alpha\frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y) \tag{2.26}$$

for all distinct $x, y \in X$ with $y \leq x$ where $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$. Also, assume either

(i) *T is continuous or*

(ii) *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Proof Take $L = 0$ and $k = \alpha + \beta$ for all $t \in [0,\infty)$ in Corollary 6. Indeed,

$$\begin{aligned} d(Tx, Ty) &\leq \alpha\frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y) \\ &\leq (\alpha+\beta)\max\left\{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}, d(x,y)\right\}. \end{aligned} \tag{2.27}$$

□

Theorem 8 *In addition to the hypotheses of Theorem* 3, *assume that*

$$\text{for every } x, y \in X \text{ there exists } z \in X \text{ that is comparable to } x \text{ and } y, \tag{2.28}$$

then T has a unique fixed point.

Proof Suppose, to the contrary, that x and y are fixed points of T where $x \neq y$. By (2.28), there exists a point $z \in X$ which is comparable with x and y. Without loss of generality, we choose $z \preceq x$. We construct a sequence $\{z_n\}$ as follows:

$$z_0 = z \quad \text{and} \quad z_n = Tz_{n-1} \quad \text{for all } n \geq 1. \tag{2.29}$$

Since T is non-decreasing, $z \preceq x$ implies $Tz \preceq Tx = x$. By induction, we get $z_n \preceq x$.

If $x = z_{N_0}$ for some $N_0 \geq 1$ then $z_n = Tz_{n-1} = Tx = x$ for all $n \geq N_0 - 1$. So $\lim_{n\to\infty} z_n = x$. Analogously, we get $\lim_{n\to\infty} z_n = y$, which completes the proof.

Consider the other case, that is, $x \neq z_n$ for all $n = 0, 1, 2, \ldots$. Then, by (2.1), we observe that

$$\begin{aligned} \phi\bigl(d(x, z_n)\bigr) &= \phi\bigl(d(Tx, Tz_{n-1})\bigr) \\ &\leq \phi\bigl(M(x, z_{n-1})\bigr) - \psi\bigl(M(x, z_{n-1})\bigr) \\ &\quad + L \min\bigl\{d(x, Tx), d(z_{n-1}, Tz_{n-1}), d(x, Tz_{n-1}), d(z_{n-1}, Tz_{n-1})\bigr\} \\ &= \phi\bigl(M(x, z_{n-1})\bigr) - \psi\bigl(M(x, z_{n-1})\bigr) \end{aligned} \tag{2.30}$$

for all distinct $x, y \in X$ with $y \preceq x$ where $\phi \in \Phi$, $\psi \in \Psi$ and

$$\begin{aligned} M(x, z_{n-1}) &= \max\left\{\frac{d(x, Tx)d(z_{n-1}, Tz_{n-1})}{d(x, z_{n-1})}, d(x, z_{n-1})\right\} \\ &= \max\left\{\frac{d(x, x)d(z_{n-1}, z_n)}{d(x, z_{n-1})}, d(x, z_{n-1})\right\} \\ &= d(x, z_{n-1}). \end{aligned} \tag{2.31}$$

Thus,

$$\phi\bigl(d(x, z_n)\bigr) \leq \phi\bigl(d(x, z_{n-1})\bigr) - \psi\bigl(d(x, z_{n-1})\bigr) < \phi\bigl(d(x, z_n)\bigr),$$

which is a contradiction. This ends the proof. □

Remark

- Corollary 5 is a generalization of Theorem 2.1 of Luong and Thuan [4].
- Corollary 7 (with $L = 0$) corresponds to Theorem 2.2 and Theorem 2.3 of Harjani, López and Sadarangani [3].
- Theorem 2.28 generalizes Theorem 2.4 of Luong and Thuan [4].

Now, we give some examples illustrating our results.

Example 9 Let $X = \{4,5,6\}$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$, and $\preceq := \{(4,4),(5,5),(6,6),(6,4)\}$. Consider the mapping

$$T = \begin{pmatrix} 4 & 5 & 6 \\ 4 & 6 & 4 \end{pmatrix}.$$

We define the functions $\phi, \psi : [0,+\infty) \to [0,+\infty)$ by $\phi(t) = 2t$ and $\psi(t) = \frac{3}{2}t$. Now, we will check that all the hypotheses required by Theorem 4 (Theorem 3 with $L = 0$) are satisfied.

First, X has the property: if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$. Indeed, let $\{z_n\}$ be a non-decreasing sequence in X with respect to $\preceq$ such that $z_n \to z \in X$ as $n \to +\infty$. We have $z_n \preceq z_{n+1}$ for all $n \in \mathbb{N}$.

- If $z_0 = 4$, then $z_0 = 4 \preceq z_1$. From the definition of $\preceq$, we have $z_1 = 4$. By induction, we get $z_n = 4$ for all $n \in \mathbb{N}$ and $z = 4$. Then $z_n \preceq z$ for all $n \in \mathbb{N}$ and $z = \sup\{z_n\}$.
- If $z_0 = 5$, then $z_0 = 5 \preceq z_1$. From the definition of $\preceq$, we have $z_1 = 5$. By induction, we get $z_n = 5$ for all $n \in \mathbb{N}$ and $z = 5$. Then $z_n \preceq z$ for all $n \in \mathbb{N}$ and $z = \sup\{z_n\}$.
- If $z_0 = 6$, then $z_0 = 6 \preceq z_1$. From the definition of $\preceq$, we have $z_1 \in \{6,4\}$. By induction, we get $z_n \in \{6,4\}$ for all $n \in \mathbb{N}$. Suppose that there exists $p \geq 1$ such that $z_p = 4$. From the definition of $\preceq$, we get $z_n = z_p = 4$ for all $n \geq p$. Thus, we have $z = 4$ and $z_n \preceq z$ for all $n \in \mathbb{N}$. Now, suppose that $z_n = 6$ for all $n \in \mathbb{N}$. In this case, we get $z = 6$ and $z_n \preceq z$ for all $n \in \mathbb{N}$ and $z = \sup\{z_n\}$.

Thus, we proved that in all cases, we have $z = \sup\{z_n\}$.

Let $x,y \in X$ such that $x \preceq y$ and $x \neq y$, so we have only $x = 6$ and $y = 4$. In particular

$$d(T6,T4) = 0 \quad \text{and} \quad M(6,4) = 2,$$

so (2.23) holds easily. On the other hand, it is obvious that T is a non-decreasing mapping with respect to $\preceq$ and there exists $x_0 = 6$ such that $x_0 \preceq Tx_0$. All the hypotheses of Theorem 4 are verified and $u = 4$ is a fixed point of T.

Note that Theorem 1 is not applicable. Indeed, taking $x = 4$ and $y = 5$

$$d(T4,T5) = 2 > \beta = \alpha \frac{d(4,T4)d(5,T5)}{d(4,5)} + \beta d(4,5),$$

for any $\alpha, \beta \geq 0$ such that $\alpha + \beta < 1$. Also, we could not apply Theorem 2 in this example. Indeed, for $x = 6$ and $y = 4$ (that is, $x \neq y$ and $x \preceq y$), we have

$$0 = d(T6,T4) > M(T6,T4) - \psi\big(M(T6,T4)\big) = -1.$$

Example 10 Let $X = [0,\infty)$ be endowed with the Euclidean metric and the order $\preceq$ given as follows:

$$x \preceq y \quad \Longleftrightarrow \quad (x = y) \text{ or } (x,y \geq 1, x \leq y).$$

Define $T : X \to X$ by $Tx = x$ if $0 \leq x < 1$ and $Tx = 0$ if $x \geq 1$. Define the functions $\phi, \psi : [0,+\infty) \to [0,+\infty)$ by $\phi(t) = 4t$ and $\psi(t) = 3t$.

Take $x \preceq y$ and $x \neq y$. It means that $1 \leq x < y$. In particular, $d(Tx,Ty) = 0$ and $M(x,y) = y - x$. This implies that (2.23) holds. It is easy that X satisfies the property: if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$. Also, the other conditions of Theorem 4 are satisfied and $u = 0$ is a fixed point of T.

Notice that we cannot apply Theorem 1 (since T is not continuous) nor Theorem 2 to this example. Indeed, letting $x \preceq y$ and $x \neq y$ (that is, $1 \le x < y$), we have

$$d(Tx, Ty) = 0 > M(x,y) - \psi\big(M(x,y)\big) = -2(y - x).$$

Example 11 Let $X = \{(0,1),(1,0),(1,1)\} \subset \mathbb{R}^2$ with the Euclidean distance d_2. (X, d_2) is, obviously, a complete metric space. Moreover, we consider the order $\le$ in X given by $R = \{(x,x), x \in X\} \cup \{((0,1),(1,1))\}$. We also consider $T : X \to X$ given by $T((0,1)) = (0,1)$, $T((1,0)) = (1,0)$ and $T((1,1)) = (0,1)$. Take $\phi(t) = 3t$ and $\psi(t) = 2t$. Obviously, T is a continuous and non-decreasing mapping since $(0,1) \le (1,1)$ and $T(0,1) = (0,1) \le T(1,1) = (0,1)$. Let $x \le y$ and $x \neq y$, then necessarily $x = (0,1)$ and $y = (1,1)$. Then

$$d_2(Tx, Ty) = d_2\big((0,1),(0,1)\big) = 0 \quad \text{and} \quad M(x,y) = \sqrt{2},$$

so (2.23) holds. Also, $(0,1) \le T((0,1))$, therefore all conditions in Theorem 4 hold and there are two fixed points which are $(0,1)$ and $(1,0)$. The non-uniqueness follows from the fact that the partial order $\le$ is not total.

Note that Theorem 1 is not applicable. Indeed, taking $x = (0,1)$ and $y = (1,0)$

$$d_2(Tx, Ty) = \sqrt{2} > (\alpha + \beta)\sqrt{2} = \alpha \frac{d_2(x, Tx) d_2(y, Ty)}{d_2(x,y)} + \beta d_2(x,y),$$

for any $\alpha, \beta \ge 0$ such that $\alpha + \beta < 1$. Also, we could not apply Theorem 2 in this example. Indeed, for $x = (0,1)$ and $y = (1,1)$ we have

$$0 = d_2(Tx, Ty) > \sqrt{2} - 2\sqrt{2} = M(x,y) - \psi\big(M(x,y)\big).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Acknowledgements
The authors express their gratitude to the referees for constructive and useful remarks and suggestions.

Received: 9 October 2013 Accepted: 15 May 2014 Published: 02 Jun 2014

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10.1186/1029-242X-2014-219
Cite this article as: Mustafa et al.: A discussion on generalized almost contractions via rational expressions in partially ordered metric spaces. *Journal of Inequalities and Applications* 2014, 2014:219