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Hybrid method for a class of accretive variational inequalities involving nonexpansive mappings

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Abstract

In this paper we use contractions to regularize a class of accretive variational inequalities and prove the strongly convergence in Banach spaces. We extend the result of Lu *et al.* (Nonlinear Anal. 71:1032-1041, 2009) to the framework of Banach spaces.

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1 Introduction

Let H be a Hilbert space, C be a nonempty closed convex subset of H, and $F : C \to H$ a nonlinear mapping. The set of fixed points of F is denoted by $\mathrm{Fix}(F)$, *i.e.*, $\mathrm{Fix}(F) = \{x \in C : Fx = x\}$. A monotone variational inequality problem is to find a point x^* with the property

$$x^* \in C, \text{such that } \langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1}$$

where F is a monotone operator.

Recently, Lu *et al.* [1] were concerned with a special class of variational inequalities in which the mapping F is the complement of a nonexpansive mapping and the constraint set is the set of fixed points of another nonexpansive mapping. Namely, they considered the following type of monotone variational inequality (VI) problem:

$$\text{Find } x^* \in \mathrm{Fix}(T), \text{such that } \langle (I - V)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T), \tag{2}$$

where $T, V : C \to C$ are nonexpansive mappings and $\mathrm{Fix}(T) \neq \emptyset$.

Hybrid methods for solving VI (1) were studied by Yamada [2], where F is Lipschitzian and strongly monotone. However, his methods do not apply to the variational inequality (2) since the mapping $I - V$ fails, in general, to be strongly monotone, though it is Lipschitzian. Therefore, other hybrid methods have to be sought. Recently, Moudafi and Mainge [3] studied VI (2) by regularizing the mapping $tS + (1 - t)T$ and defined $\{x_{s,t}\}$ as the unique fixed point of the equation

$$x_{s,t} = sf(x_{s,t}) + (1 - s)\big[tS(x_{s,t}) + (1 - t)T(x_{s,t})\big], \quad s, t \in (0, 1). \tag{3}$$

Since Moudafi and Mainge's regularization depends on t, the convergence of the scheme (3) is more complicated. Very recently, Lu *et al.* [1] studied VI (2) by regularizing the mapping S and defined $\{x_{s,t}\}$ as the unique fixed point of the equation

$$x_{s,t} = s\big[tf(x_{s,t}) + (1-t)S(x_{s,t})\big] + (1-s)T(x_{s,t}), \quad s,t \in (0,1). \tag{4}$$

Note that Lu *et al.*'s regularization (4) no longer depends on t.

Motivated and inspired by the result of Lu *et al.* [1], we put forward a question: Can this implicit hybrid method [1] in Hilbert spaces be extended to the framework of Banach spaces? In this paper, we give a positive answer.

Throughout this paper, we always assume that E is a real Banach space. Let C be a nonempty closed convex subset of E. Let $F: C \to E$ be a nonlinear mapping.

In this paper, we consider the following type of accretive variational inequality problem:

$$x^* \in \mathrm{Fix}(T), \text{such that } \big\langle (I-S)x^*, j\big(x-x^*\big)\big\rangle \ge 0, \quad \forall x \in \mathrm{Fix}(T), \tag{5}$$

where $S, T: C \to C$ are two nonexpansive mappings with the set of fixed point $\mathrm{Fix}(T) \neq \emptyset$. Let Ω denote the set of solutions of VI (5) and assume that Ω is nonempty.

2 Preliminaries

Let E be a real Banach space and J be the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \big\{x^* \in E^* : \langle x, x^*\rangle = \|x\| \big\|x^*\big\|, \|x\| = \big\|x^*\big\|\big\}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*.

Let C be a nonempty closed convex subset of a real Banach space E. Recall the following concepts of mappings.

(i) A mapping $f: C \to C$ is a ρ-contraction if $\rho \in [0,1)$ and the following property is satisfied:

$$\big\|f(x) - f(y)\big\| \le \rho\|x-y\|, \quad \forall x,y \in C.$$

(ii) A mapping $T: C \to C$ is nonexpansive provided

$$\|Tx - Ty\| \le \|x-y\|, \quad \forall x,y \in C.$$

(iii) A mapping $F: C \to E$ is

(a) accretive if for any $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that

$$\big\langle Fx - Fy, j(x-y)\big\rangle \ge 0;$$

(b) strictly accretive if F is accretive and the equality in (a) holds if and only if $x = y$;

(c) β-strongly accretive if for any $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that

$$\big\langle Fx - Fy, j(x-y)\big\rangle \ge \beta\|x-y\|^2$$

for some real constant $\beta > 0$.

Let $\varphi : [0,\infty) := R^+ \to R^+$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. This function φ is called a gauge function. The duality mapping $J_\varphi : E \to E^*$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|), \|x^*\| = \varphi(\|x\|)\}, \quad \forall x \in E.$$

In the case that $\varphi(t) = t$, $J_\varphi = J$, where J is the normalized duality mapping. Clearly, the relation $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|}J(x)$, $\forall x \neq 0$ holds (see [4]).

Following Browder [4], we say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single valued and weak-to-weak* sequentially continuous (*i.e.*, if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weakly* to $J_\varphi(x)$). It is well known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)\,d\tau, \quad t > 0,$$

then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad \forall x \in E,$$

where ∂ denotes the sub-differential in the sense of convex analysis.

Remark 2.1 If J_φ is weak-to-weak* sequentially continuous, then J is strong-to-weak* sequentially continuous.

Indeed, if $x_n \to x$ strongly, then $x_n \to x$ weakly, $J_\varphi(x_n)$ converges weakly* to $J_\varphi(x)$ and $\varphi(\|x_n\|) \to \varphi(\|x\|)$ strongly. Since $J_\varphi(x)\frac{\|x\|}{\varphi(\|x\|)} = J(x)$, $\forall x \neq 0$, for any $y \in E$, we have

$$\begin{aligned}
&\left|\langle y, J(x_n)\rangle - \langle y, J(x)\rangle\right| \\
&\quad = \left|\left\langle y, J_\varphi(x_n)\frac{\|x_n\|}{\varphi(\|x_n\|)}\right\rangle - \left\langle y, J_\varphi(x)\frac{\|x\|}{\varphi(\|x\|)}\right\rangle\right| \\
&\quad \le \left|\left\langle y, J_\varphi(x_n)\frac{\|x_n\|}{\varphi(\|x_n\|)}\right\rangle - \left\langle y, J_\varphi(x_n)\frac{\|x\|}{\varphi(\|x\|)}\right\rangle\right| + \left|\left\langle y, J_\varphi(x_n)\frac{\|x\|}{\varphi(\|x\|)}\right\rangle - \left\langle y, J_\varphi(x)\frac{\|x\|}{\varphi(\|x\|)}\right\rangle\right| \\
&\quad \le \|y\|\varphi(\|x_n\|)\frac{|\|x_n\|\varphi(\|x\|) - \|x\|\varphi(\|x_n\|)|}{\varphi(\|x_n\|)\varphi(\|x\|)} + \frac{\|x\|}{\varphi(\|x\|)}\left|\langle y, J_\varphi(x_n)\rangle - \langle y, J_\varphi(x)\rangle\right| \\
&\quad \le \|y\|\frac{\|x_n\||\varphi(\|x\|) - \varphi(\|x_n\|)| + |\|x_n\| - \|x\||\varphi(\|x_n\|)}{\varphi(\|x\|)} \\
&\qquad + \frac{\|x\|}{\varphi(\|x\|)}\left|\langle y, J_\varphi(x_n)\rangle - \langle y, J_\varphi(x)\rangle\right|.
\end{aligned}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty}\langle y, J(x_n)\rangle = \langle y, J(x)\rangle,$$

i.e., J is strong-to-weak* sequentially continuous.

Lemma 2.1 ([5, Lemma 2.1]) *Assume that a Banach space E has a weakly continuous duality mapping J_φ with a gauge φ. For all $x, y \in E$, the following inequality holds*:

$$\Phi(\|x+y\|) \le \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

In particular, for all $x, y \in E$,

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

Lemma 2.2 (see [6]) *Let C be a nonempty closed convex subset of a real Banach space E. Assume that $F : C \to E$ is accretive and weakly continuous along segments*; *that is $F(x + ty) \rightharpoonup F(x)$ as $t \to 0$. Then the variational inequality*

$$x^* \in C, \quad \langle Fx^*, j(x - x^*)\rangle \ge 0, \quad \forall x \in C$$

is equivalent to the dual variational inequality

$$x^* \in C, \quad \langle Fx, j(x - x^*)\rangle \ge 0, \quad \forall x \in C.$$

3 Main results

In this section, we introduce an implicit algorithm and prove this algorithm converges strongly to x^* which solves VI (5). Let C be a nonempty closed convex subset of a real Banach space E. Let $f : C \to C$ be a contraction and $S, T : C \to C$ be two nonexpansive mappings. For $s, t \in (0,1)$, we define the following mapping:

$$x \mapsto W_{s,t}x := s[tf(x) + (1-t)Sx] + (1-s)Tx.$$

It is obvious that $W_{s,t} : C \to C$ is a contraction. So the contraction $W_{s,t}$ has a unique fixed point which is denoted $x_{s,t}$. Namely,

$$x_{s,t} = s[tf(x_{s,t}) + (1-t)S(x_{s,t})] + (1-s)T(x_{s,t}), \quad s, t \in (0,1). \tag{6}$$

Theorem 3.1 *Let C be a nonempty closed convex subset of a reflexive Banach space E which has a weakly continuous duality map $J_\varphi(x)$ with the gauge φ. Let $f : C \to C$ be a contraction with constant $\rho > 0$ and $S, T : C \to C$ be two nonexpansive mappings with $\mathrm{Fix}(T) \neq \emptyset$. Suppose that the solution set Ω of VI (5) is nonempty. Let, for each $(s,t) \in (0,1)^2$, $\{x_{s,t}\}$ be defined implicitly by (6). Then, for each fixed $t \in (0,1)$, the net $\{x_{s,t}\}$ converges in norm, as $s \to 0$, to a point $x_t \in \mathrm{Fix}(T)$. Moreover, as $t \to 0$, the net $\{x_t\}$ converges in norm to the unique solution x^* of the following variational inequality*:

$$x^* \in \Omega, \quad \langle (I-f)x^*, J(x - x^*)\rangle \ge 0, \quad \forall x \in \Omega. \tag{7}$$

Hence, for each null sequence $\{t_n\}$ in $(0,1)$, there exists another null sequence $\{s_n\}$ in $(0,1)$, such that the sequence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$.

Proof Step 1. For each fixed $t \in (0,1)$, the net $\{x_{s,t}\}$ is bounded.

For any $z \in \operatorname{Fix}(T)$, we have

$$\begin{aligned}
&\bigl\|s\bigl[t\bigl(f(x_{s,t})-f(z)\bigr)+(1-t)\bigl(S(x_{s,t})-S(z)\bigr)\bigr]+(1-s)\bigl(T(x_{s,t})-z\bigr)\bigr\| \\
&\quad\le s\bigl\|t\bigl(f(x_{s,t})-f(z)\bigr)+(1-t)\bigl(S(x_{s,t})-S(z)\bigr)\bigr\|+(1-s)\bigl\|T(x_{s,t})-Tz\bigr\| \\
&\quad\le st\bigl\|f(x_{s,t})-f(z)\bigr\|+(1-t)s\bigl\|S(x_{s,t})-S(z)\bigr\|+(1-s)\|x_{s,t}-z\| \\
&\quad\le st\rho\|x_{s,t}-z\|+(1-t)s\|x_{s,t}-z\|+(1-s)\|x_{s,t}-z\| \\
&\quad=\bigl(1-st(1-\rho)\bigr)\|x_{s,t}-z\|.
\end{aligned}$$

Combining the above inequality and Lemma 2.1, we obtain

$$\begin{aligned}
\Phi\bigl(\|x_{s,t}-z\|\bigr) &= \Phi\bigl(\bigl\|s\bigl[t\bigl(f(x_{s,t})-f(z)\bigr)+(1-t)\bigl(S(x_{s,t})-S(z)\bigr)\bigr] \\
&\quad+(1-s)\bigl(T(x_{s,t})-z\bigr)+st\bigl(f(z)-z\bigr)+s(1-t)\bigl(S(z)-z\bigr)\bigr\|\bigr) \\
&\le \Phi\bigl(\bigl\|s\bigl[t\bigl(f(x_{s,t})-f(z)\bigr)+(1-t)\bigl(S(x_{s,t})-S(z)\bigr)\bigr]+(1-s)\bigl(T(x_{s,t})-z\bigr)\bigr\|\bigr) \\
&\quad+st\bigl\langle f(z)-z,J_\varphi(x_{s,t}-z)\bigr\rangle+(1-t)s\bigl\langle S(z)-z,J_\varphi(x_{s,t}-z)\bigr\rangle \\
&\le \Phi\bigl(\bigl(1-st(1-\rho)\bigr)\|x_{s,t}-z\|\bigr)+st\bigl\langle f(z)-z,J_\varphi(x_{s,t}-z)\bigr\rangle \\
&\quad+(1-t)s\bigl\langle S(z)-z,J_\varphi(x_{s,t}-z)\bigr\rangle \\
&\le \bigl(1-st(1-\rho)\bigr)\Phi\bigl(\|x_{s,t}-z\|\bigr)+st\bigl\langle f(z)-z,J_\varphi(x_{s,t}-z)\bigr\rangle \\
&\quad+(1-t)s\bigl\langle S(z)-z,J_\varphi(x_{s,t}-z)\bigr\rangle,
\end{aligned}$$

which implies that

$$\Phi\bigl(\|x_{s,t}-z\|\bigr)\le\frac{t}{t(1-\rho)}\bigl\langle f(z)-z,J_\varphi(x_{s,t}-z)\bigr\rangle+\frac{1-t}{t(1-\rho)}\bigl\langle S(z)-z,J_\varphi(x_{s,t}-z)\bigr\rangle. \tag{8}$$

Taking $\varphi(t)=t$, then $J_\varphi=J$ and $\Phi(t)=\frac{t^2}{2}$, from (8) we have

$$\begin{aligned}
\|x_{s,t}-z\|^2 &\le \frac{2t}{t(1-\rho)}\bigl\langle f(z)-z,J(x_{s,t}-z)\bigr\rangle+\frac{2(1-t)}{t(1-\rho)}\bigl\langle S(z)-z,J(x_{s,t}-z)\bigr\rangle \\
&\le \frac{2t}{t(1-\rho)}\bigl\|f(z)-z\bigr\|\|x_{s,t}-z\|+\frac{2(1-t)}{t(1-\rho)}\bigl\|S(z)-z\bigr\|\|x_{s,t}-z\|,
\end{aligned} \tag{9}$$

which implies that

$$\|x_{s,t}-z\|\le\frac{2}{t(1-\rho)}\max\bigl\{\bigl\|f(z)-z\bigr\|,\bigl\|S(z)-z\bigr\|\bigr\}.$$

So for each fixed $t\in(0,1)$, $\{x_{s,t}\}$ is bounded, furthermore $\{f(x_{s,t})\}$, $\{S(x_{s,t})\}$ and $\{T(x_{s,t})\}$ are all bounded.

Step 2. $x_{s,t}\to x_t\in\operatorname{Fix}(T)$ as $s\to 0$.

From (6) and the boundedness of the sequences $\{f(x_{s,t})\}$, $\{S(x_{s,t})\}$ and $\{T(x_{s,t})\}$, for each fixed $t\in(0,1)$ we have

$$\|x_{s,t}-Tx_{s,t}\|=s\bigl\|tf(x_{s,t})+(1-t)S(x_{s,t})-Tx_{s,t}\bigr\|\to 0\quad(s\to 0). \tag{10}$$

Assume that $\{s_n\} \subset (0,1)$ is such that $s_n \to 0$ $(n \to \infty)$. From (8), for any $z \in \mathrm{Fix}(T)$, we have

$$\Phi(\|x_{s_n,t} - z\|) \leq \frac{t}{t(1-\rho)}\langle f(z) - z, J_\varphi(x_{s_n,t} - z)\rangle + \frac{1-t}{t(1-\rho)}\langle S(z) - z, J_\varphi(x_{s_n,t} - z)\rangle. \tag{11}$$

Since $\{x_{s_n,t}\}$ is bounded, without loss of generality, we may assume that $\{x_{s_n,t}\}$ converges weakly to a point x_t as $n \to \infty$. This together with (10) implies that $x_t \in \mathrm{Fix}(T)$. Taking $z = x_t$ in (11), we have

$$\Phi(\|x_{s_n,t} - x_t\|) \leq \frac{1}{t(1-\rho)}\langle tf(x_t) + (1-t)S(x_t) - x_t, J_\varphi(x_{s_n,t} - x_t)\rangle. \tag{12}$$

Since J_φ is weakly continuous, it follows from (12) that $\Phi(\|x_{s_n,t} - x_t\|) \to 0$ as $n \to \infty$, which implies that $x_{s_n,t} \to x_t$ strongly. This has proved the relative norm compactness of the net $\{x_{s,t}\}$ as $s \to 0$.

Taking $s = s_n$ in (9), we have

$$\|x_{s_n,t} - z\|^2 \leq \frac{2t}{t(1-\rho)}\langle f(z) - z, J(x_{s_n,t} - z)\rangle + \frac{2(1-t)}{t(1-\rho)}\langle S(z) - z, J(x_{s_n,t} - z)\rangle.$$

Since J_φ is weakly continuous, then by Remark 2.1, J is strong-to-weak* sequentially continuous. Let $s_n \to 0$ in the above inequality, we have

$$\|x_t - z\|^2 \leq \frac{2t}{t(1-\rho)}\langle f(z) - z, J(x_t - z)\rangle + \frac{2(1-t)}{t(1-\rho)}\langle S(z) - z, J(x_t - z)\rangle.$$

Hence we obtain

$$x_t \in \mathrm{Fix}(T), \quad \langle tf(z) + (1-t)S(z) - z, J(x_t - z)\rangle \geq 0, \quad \forall z \in \mathrm{Fix}(T).$$

This together with Lemma 2.2, we have

$$x_t \in \mathrm{Fix}(T), \quad \langle tf(x_t) + (1-t)S(x_t) - x_t, J(x_t - z)\rangle \geq 0, \quad \forall z \in \mathrm{Fix}(T). \tag{13}$$

Next, we prove that the entire net $\{x_{s,t}\}$ converges strongly to x_t as $s \to 0$. We assume that $x_{s'_n,t} \to x'_t$ where $s'_n \to 0$. Similar to the above proof, we have $x'_t \in \mathrm{Fix}(T)$ and

$$x'_t \in \mathrm{Fix}(T), \quad \langle tf(x'_t) + (1-t)S(x'_t) - x'_t, J(x'_t - z)\rangle \geq 0, \quad \forall z \in \mathrm{Fix}(T). \tag{14}$$

Taking $z = x'(t)$ and $z = x_t$ in (13) and (14), respectively, we have

$$t\langle f(x_t) - x_t, J(x_t - x'_t)\rangle + (1-t)\langle S(x_t) - x_t, J(x_t - x'_t)\rangle \geq 0,$$
$$t\langle f(x'_t) - x'_t, J(x'_t - x_t)\rangle + (1-t)\langle S(x'_t) - x'_t, J(x'_t - x_t)\rangle \geq 0.$$

Adding up the above two inequalities yields

$$t\langle (I-f)x_t - (I-f)x'_t, J(x_t - x'_t)\rangle + (1-t)\langle (I-S)x_t - (I-S)x'_t, J(x_t - x'_t)\rangle \leq 0.$$

Since

$$\big\langle (I-f)x_t-(I-f)x_t',J\big(x_t-x_t'\big)\big\rangle \geq (1-\rho)\big\|x_t-x_t'\big\|^2,$$
$$\big\langle (I-S)x_t-(I-S)x_t',J\big(x_t-x_t'\big)\big\rangle \geq 0,$$

we obtain

$$(1-\rho)\big\|x_t-x_t'\big\|^2 \leq 0,$$

i.e., $x_t = x_t'$. So the entire net $\{x_{s,t}\}$ converges in norm to $x_t \in \operatorname{Fix}(T)$ as $s \to 0$.

Step 3. The net $\{x_t\}$ is bounded.

For any $y \in \Omega$, taking $z = y$ in (13), we have

$$\big\langle tf(x_t) + (1-t)S(x_t) - x_t, J(x_t-y)\big\rangle \geq 0,$$

which together with the fact of $y \in \Omega$ implies that

$$\begin{aligned} &t\big\langle (I-f)x_t-(I-f)y,J(x_t-y)\big\rangle + (1-t)\big\langle (I-S)x_t-(I-S)y,J(x_t-y)\big\rangle \\ &\quad\leq t\big\langle f(y)-y,J(x_t-y)\big\rangle + (1-t)\big\langle Sy-y,J(x_t-y)\big\rangle \\ &\quad\leq t\big\langle f(y)-y,J(x_t-y)\big\rangle. \end{aligned} \tag{15}$$

Since $I-f$ is strongly accretive and $I-S$ is accretive, we obtain

$$\big\langle (I-f)x_t-(I-f)y,J(x_t-y)\big\rangle \geq (1-\rho)\|x_t-y\|^2, \tag{16}$$
$$\big\langle (I-S)x_t-(I-S)y,J(x_t-y)\big\rangle \geq 0. \tag{17}$$

It follows from (15)-(17) that

$$\begin{aligned} \|x_t-y\|^2 &\leq \frac{1}{1-\rho}\big\langle f(y)-y,J(x_t-y)\big\rangle \\ &\leq \frac{1}{1-\rho}\big\|f(y)-y\big\|\,\|x_t-y\|. \end{aligned} \tag{18}$$

Hence we have

$$\|x_t-y\| \leq \frac{1}{1-\rho}\big\|f(y)-y\big\|, \quad \forall t \in (0,1).$$

Step 4. The net $x_t \to x^* \in \Omega$ which solves VI (7).

First, the uniqueness of the solution of VI (7) is obvious. We denote the unique solution by x^*.

Next we prove that $\omega_w(x_t) \subset \Omega$, *i.e.*, if $\{t_n\}$ is a null sequence in $(0,1)$ such that $x_{t_n} \to x'$ weakly as $n \to \infty$, then $x' \in \Omega$. Indeed, since $\{x_t\} \subset \operatorname{Fix}(T)$, then $x' \in \operatorname{Fix}(T)$. Since $I-S$ is accretive, for any $z \in \operatorname{Fix}(T)$ we have

$$\big\langle (I-S)z,J(z-x_t)\big\rangle \geq \big\langle (I-S)x_t,J(z-x_t)\big\rangle. \tag{19}$$

It follows from (13) that

$$\langle (I-S)x_t, J(z-x_t)\rangle \geq \frac{t}{1-t}\langle (I-f)x_t, J(x_t-z)\rangle. \tag{20}$$

By virtue of (19) and (20), we have

$$\langle (I-S)z, J(z-x_t)\rangle \geq \frac{t}{1-t}\langle (I-f)x_t, J(x_t-z)\rangle,$$

furthermore, we get

$$\begin{aligned}\langle (I-S)z, J_\varphi(z-x_t)\rangle &\geq \frac{t}{(1-t)}\langle (I-f)x_t, J_\varphi(x_t-z)\rangle,\\ &\geq -\frac{t}{(1-t)}\big(\|x_t\| + \|f(x_t)\|\big)\varphi\big(\|x_t-z\|\big).\end{aligned}$$

Letting $t = t_n \to 0$ $(n \to \infty)$ in the above inequality, since $\{x_t\}$ is bounded and φ is a continuous strictly increasing function, we have

$$\langle (I-S)z, J_\varphi(z-x')\rangle \geq 0, \quad \forall z \in \operatorname{Fix}(T).$$

This implies that

$$\langle (I-S)z, J(z-x')\rangle \geq 0, \quad \forall z \in \operatorname{Fix}(T),$$

hence from the above inequality and Lemma 2.2, we have

$$\langle (I-S)x', J(z-x')\rangle \geq 0, \quad \forall z \in \operatorname{Fix}(T),$$

i.e., $x' \in \Omega$.

Next we show that x' is the solution of VI (7). Taking $y = x'$ and $t = t_n$ in (18), we obtain

$$\begin{aligned}\|x_{t_n}-x'\|^2 &\leq \frac{1}{1-\rho}\langle f(x')-x', J(x_{t_n}-x')\rangle\\ &= \frac{1}{1-\rho}\langle f(x')-x', J_\varphi(x_{t_n}-x')\rangle\frac{\|x_{t_n}-x'\|}{\varphi(\|x_{t_n}-x'\|)},\end{aligned}$$

which implies that

$$\|x_{t_n}-x'\|\varphi\big(\|x_{t_n}-x'\|\big) \leq \frac{1}{1-\rho}\langle f(x')-x', J_\varphi(x_{t_n}-x')\rangle. \tag{21}$$

Since $x_{t_n} \to x'$ weakly and J_φ is weakly continuous, let $t_n \to 0$ in (21), we get

$$\|x_{t_n}-x'\|\varphi\big(\|x_{t_n}-x'\|\big) \to 0 \quad (n \to \infty),$$

which together with the property of φ implies that $x_{t_n} \to x'$ in norm. It follows from (15) and (17) that

$$\langle (I-f)x_t, J(x_t-y)\rangle \leq 0. \tag{22}$$

Since J is strong-to-weak* sequentially continuous and f is a contraction, we have

$$\begin{aligned}
&\bigl|\langle (I-f)x_{t_n}, J(x_{t_n}-y)\rangle - \langle (I-f)x', J(x'-y)\rangle\bigr| \\
&\quad \le \bigl|\langle (I-f)x_{t_n}, J(x_{t_n}-y)\rangle - \langle (I-f)x', J(x_{t_n}-y)\rangle\bigr| \\
&\qquad + \bigl|\langle (I-f)x', J(x_{t_n}-y)\rangle - \langle (I-f)x', J(x'-y)\rangle\bigr| \\
&\quad \le \bigl|\langle (I-f)x_{t_n}, J(x_{t_n}-y)\rangle - \langle (I-f)x', J(x_{t_n}-y)\rangle\bigr| \\
&\qquad + \bigl|\langle (I-f)x', J(x_{t_n}-y)\rangle - \langle (I-f)x', J(x'-y)\rangle\bigr| \\
&\quad \le (1+\rho)\|x_{t_n}-x'\|\,\|x_{t_n}-y\| \\
&\qquad + \bigl|\langle (I-f)x', J(x_{t_n}-y)\rangle - \langle (I-f)x', J(x'-y)\rangle\bigr| \to 0 \quad (n\to\infty).
\end{aligned} \tag{23}$$

Letting $t = t_n \to 0$ ($n \to \infty$) in (22) and combining (23) we have

$$\langle (I-f)x', J(x'-y)\rangle \le 0, \quad \forall y \in \Omega.$$

So x' is the solution of VI (7). By uniqueness, we have $x' = x^*$. Therefore, $x_t \to x^*$ in norm as $t \to 0$. The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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