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Eigenvalues of quasibounded maximal monotone operators

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Abstract

Let X be a real reflexive separable Banach space with dual space X^* and let L be a dense subspace of X. We study a nonlinear eigenvalue problem of the type

$$0 \in Tx + \lambda Cx,$$

where $T : D(T) \subset X \to 2^{X^*}$ is a strongly quasibounded maximal monotone operator and $C : D(C) \subset X \to X^*$ satisfies the condition $(S_+)_{D(C)}$ with $L \subset D(C)$. The method of approach is to use a topological degree theory for $(S_+)_L$-perturbations of strongly quasibounded maximal monotone operators, recently developed by Kartsatos and Quarcoo. Moreover, applying degree theory, a variant of the Fredholm alternative on the surjectivity of the operator $\lambda T + C$ is discussed, where we assume that λ is not an eigenvalue for the pair (T, C), T and C are positively homogeneous, and C satisfies the condition $(S_+)_L$.

1 Introduction and preliminaries

A systematic theory of compact operators emerged from the theory of integral equations of the form

$$Tx + \lambda x = y, \quad \text{where } Tx(t) = \int_a^b k\big(t, s, x(s)\big)\, ds.$$

Here, $\lambda \in \mathbb{R}$ is a parameter, y and k are given functions, and x is the unknown function. Such equations play a role in the theory of differential equations. The study goes back to Krasnosel'skii [1]. Moreover, the eigenvalue problem of the form

$$Tx + \lambda Cx = 0$$

could be solved with the Galerkin method, where C is continuous, bounded, and of type (S); see, *e.g.*, [2].

From now on, we concentrate on the class of maximal monotone operators, as a generalization of linear self-adjoint operators. The theory of nonlinear maximal monotone operators started with a pioneer work of Minty [3] and has been extensively developed, with applications to evolution equations and to variational inequalities of elliptic and parabolic type; see [4, 5]. The eigenvalue problem for various types of nonlinear operators was investigated in [6–10]. As a key tool, topological degree theory was made frequent use of; for instance, the Leray-Schauder degree and the Kartsatos-Skrypnik degree; see [11–15].

Let X be a real reflexive Banach space with dual space X^*. We consider a nonlinear eigenvalue problem of the form

$$0 \in Tx + \lambda Cx, \tag{E}$$

where $T : D(T) \subset X \to 2^{X^*}$ is a maximal monotone multi-valued operator and $C : D(C) \subset X \to X^*$ is a single-valued operator. In the case where the operator C or the resolvents of T are compact, it was studied in [6, 7, 10] by using the Leray-Schauder degree for compact operators. When the operator C is densely defined and quasibounded and satisfies the condition $(\tilde{S}_+)$, Kartsatos and Skrypnik [9] solved the above problem (E) via the topological degree for these operators given in [13].

We are now focused on the quasiboundedness of the operator T instead of that of the operator C. Actually, a strongly quasibounded operator due to Browder and Hess [16] may not necessarily be bounded. One more thing to be considered is the condition $(S_+)_L$, where L is a dense subspace of X with $L \subset D(C)$. In fact, the condition $(S_+)_{0,L}$ was first introduced in [12] and the structure of the class $(S_+)_L$ or $(S_+)_{D(C)}$ was discussed in [17], as a natural extension of the class (S_+); see [4, 14].

In the present paper, the first goal is to study the above eigenvalue problem (E) for strongly quasibounded maximal monotone operators, provided that the operator C satisfies the condition $(S_+)_{D(C)}$. In addition, we assume the following property $(\mathcal{P})$: For $\epsilon > 0$, there exists a $\lambda > 0$ such that the inclusion

$$0 \in Tx + \lambda Cx + \varepsilon Jx$$

has no solution in $D(T) \cap D(C) \cap \Omega$, where Ω is a bounded open set in X and J is a normalized duality operator. This property is closely related to the use of a topological tool for finding the eigensolution on the boundary of Ω; see [9, 10]. To solve the above problem (E), we thus use the degree theory for densely defined $(S_+)_L$-perturbations of maximal monotone operators introduced by Kartsatos and Quarcoo in [18]. Roughly speaking, the degree function is based on the Kartsatos-Skrypnik degree [8] of the densely defined operators $T_t + C$, which is constant for all small values of t, where T_t is the approximant introduced by Brézis *et al.* [19]. Such an approach was first used by Browder in [11]. The second goal is to establish a variant of a Fredholm alternative result on the surjectivity for the operator $\lambda T + C$, where $\lambda \geq 1$ is not an eigenvalue for the pair (T, C) and the operator C satisfies the condition $(S_+)_L$; see [9, 20].

This paper is organized as follows: In Section 2, we give some eigenvalue results for strongly quasibounded maximal monotone operators by applying the Kartsatos-Quarcoo degree theory. Section 3 contains a version of the Fredholm alternative for positively homogeneous operators, with a regularization method by means of a duality operator J_φ.

Let X be a real Banach space, X^* its dual space with the usual dual pairing $\langle \cdot, \cdot \rangle$, and Ω a nonempty subset of X. Let $\overline{\Omega}$, $\operatorname{int}\Omega$, and $\partial\Omega$ denote the closure, the interior, and the boundary of Ω in X, respectively. The symbol $\to$ $(\rightharpoonup)$ stands for strong (weak) convergence. An operator $A : \Omega \to X^*$ is said to be *bounded* if A maps bounded subsets of Ω into bounded subsets of X^*. A is said to be *demicontinuous* if, for every $x_0 \in \Omega$ and for every sequence $\{x_n\}$ in Ω with $x_n \to x_0$, we have $Ax_n \rightharpoonup Ax_0$.

An operator $T : D(T) \subset X \to 2^{X^*}$ is said to be *monotone* if

$$\langle u^* - v^*, x - y \rangle \geq 0 \quad \text{for every } x, y \in D(T) \text{ and every } u^* \in Tx, v^* \in Ty,$$

where $D(T) = \{x \in X : Tx \neq \emptyset\}$ denotes the *effective domain* of T.

The operator T is said to be *maximal monotone* if it is monotone and it follows from $(x, u^*) \in X \times X^*$ and

$$\langle u^* - v^*, x - y \rangle \geq 0 \quad \text{for every } y \in D(T) \text{ and every } v^* \in Ty$$

that $x \in D(T)$ and $u^* \in Tx$.

An operator $T : D(T) \subset X \to 2^{X^*}$ is said to be *strongly quasibounded* if for every $S > 0$ there exists a constant $K(S) > 0$ such that for all $x \in D(T)$ with

$$\|x\| \leq S \quad \text{and} \quad \langle u^*, x \rangle \leq S,$$

where $u^* \in Tx$, we have $\|u^*\| \leq K(S)$.

We say that $T : D(T) \subset X \to 2^{X^*}$ satisfies the condition (S_q) on a set $M \subset D(T)$ if for every sequence $\{x_n\}$ in M with $x_n \rightharpoonup x_0$ and every sequence $\{u_n^*\}$ with $u_n^* \to u^*$ where $u_n^* \in Tx_n$, we have $x_n \to x_0$.

We say that $T : D(T) \subset X \to X^*$ satisfies the condition (S_+) on a set $M \subset D(T)$ if for every sequence $\{x_n\}$ in M with

$$x_n \rightharpoonup x_0 \quad \text{and} \quad \limsup_{n \to \infty} \langle Tx_n, x_n - x_0 \rangle \leq 0,$$

we have $x_n \to x_0$.

Throughout this paper, X will always be an infinite-dimensional real reflexive separable Banach space which has been renormed so that X and its dual X^* are locally uniformly convex.

An operator $J_\varphi : X \to X^*$ is said to be a *duality operator* if

$$\langle J_\varphi x, x \rangle = \varphi(\|x\|)\|x\| \quad \text{and} \quad \|J_\varphi x\| = \varphi(\|x\|) \quad \text{for } x \in X,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. When φ is the identity map I, $J := J_I$ is called a *normalized duality operator*.

It is described in [21] that J_φ is continuous, bounded, surjective, strictly monotone, maximal monotone, and that it satisfies the condition (S_+) on X.

The following properties as regards maximal monotone operators will often be used, taken from [19, Lemma 1.3], [13, Lemma 3.1], [22, Lemma 1], and [18, Lemma D] in this order.

Lemma 1.1 *Let $T : D(T) \subset X \to 2^{X^*}$ be a maximal monotone operator. Then the following statements hold*:

(a) *For each $t \in (0, \infty)$, the operator $T_t \equiv (T^{-1} + tJ^{-1})^{-1} : X \to X^*$ is bounded, demicontinuous, and maximal monotone.*

(b) *If, in addition, $0 \in D(T)$ and $0 \in T(0)$, then the operator $(0, \infty) \times X \to X^*$, $(t, x) \mapsto T_t x$ is continuous on $(0, \infty) \times X$.*

Lemma 1.2 *Let $T : D(T) \subset X \to 2^{X^*}$ and $S : D(S) \subset X \to X^*$ be two maximal monotone operators with $0 \in D(T) \cap D(S)$ and $0 \in T(0) \cap S(0)$ such that $T + S$ is maximal monotone. Assume that there is a sequence $\{t_n\}$ in $(0,\infty)$ with $t_n \downarrow 0$ and a sequence $\{x_n\}$ in $D(S)$ such that $x_n \rightharpoonup x_0 \in X$ and $T_{t_n}x_n + w_n^* \rightharpoonup y_0^* \in X^*$, where $w_n^* \in Sx_n$. Then the following hold:*

(a) *The inequality $\liminf_{n\to\infty}\langle T_{t_n}x_n + w_n^*, x_n - x_0\rangle \geq 0$ is true.*
(b) *If $\lim_{n\to\infty}\langle T_{t_n}x_n + w_n^*, x_n - x_0\rangle = 0$, then $x_0 \in D(T + S)$ and $y_0^* \in (T + S)x_0$.*

Lemma 1.3 *Let $T : D(T) \subset X \to 2^{X^*}$ be a strongly quasibounded maximal monotone operator such that $0 \in D(T)$ and $0 \in T(0)$. If $\{t_n\}$ is a sequence in $(0,\infty)$ and $\{x_n\}$ is a sequence in X such that*

$$\|x_n\| \leq S \quad \textit{and} \quad \langle T_{t_n}x_n, x_n\rangle \leq S_1,$$

where S, S_1 are positive constants, then the sequence $\{T_{t_n}x_n\}$ is bounded in X^*.

Let L be a dense subspace of X and let $\mathcal{F}(L)$ denote the class of all finite-dimensional subspaces of L. Let $\{F_n\}$ be a sequence in the class $\mathcal{F}(L)$ such that for each $n \in \mathbb{N}$

$$F_n \subset F_{n+1}, \qquad \dim F_n = n, \quad \text{and} \quad \overline{\bigcup_{n\in\mathbb{N}} F_n} = X. \tag{1.1}$$

Set $L\{F_n\} := \bigcup_{n\in\mathbb{N}} F_n$.

Definition 1.4 Let $C : D(C) \subset X \to X^*$ be a single-valued operator with $L \subset D(C)$. We say that C satisfies the condition $(S_+)_{0,L}$ if for every sequence $\{F_n\}$ in $\mathcal{F}(L)$ satisfying equation (1.1) and for every sequence $\{x_n\}$ in L with

$$x_n \rightharpoonup x_0, \qquad \limsup_{n\to\infty}\langle Cx_n, x_n\rangle \leq 0, \quad \text{and} \quad \lim_{n\to\infty}\langle Cx_n, y\rangle = 0$$

for every $y \in L\{F_n\}$, we have $x_n \to x_0, x_0 \in D(C)$, and $Cx_0 = 0$.

We say that C satisfies the condition $(S_+)_L$ if the operator $C_h : D(C) \to X^*$, defined by $C_h x := Cx - h$, satisfies the condition $(S_+)_{0,L}$ for every $h \in X^*$.

We say that the operator C satisfies the condition $(S_+)_{0,D(C)}$ if it satisfies the condition $(S_+)_{0,L}$ with '$\{x_n\} \subset L$' replaced by '$\{x_n\} \subset D(C)$'. We say that C satisfies the condition $(S_+)_{D(C)}$ if the operator C_h satisfies the condition $(S_+)_{0,D(C)}$ for every $h \in X^*$.

It is obvious from Definition 1.4 that if the operator C satisfies the condition $(S_+)_{D(C)}$, then C satisfies the condition $(S_+)_L$. However, the converse is not true in general, as we see in Example 3.2 of [17].

2 The existence of eigenvalues

In this section, we deal with some eigenvalue results for strongly quasibounded maximal monotone operators in reflexive separable Banach spaces, based on a topological degree theory for $(S_+)_L$-perturbations of maximal monotone operators due to Kartsatos and Quarcoo [18].

We establish the existence of an eigenvalue concerning $(S_+)_{D(C)}$-perturbations of strongly quasibounded maximal monotone operators.

Theorem 2.1 *Let Ω be a bounded open set in X with $0 \in \Omega$ and let L be a dense subspace of X. Suppose that $T : D(T) \subset X \to 2^{X^*}$ is a multi-valued operator and $C : D(C) \subset X \to X^*$ is a single-valued operator with $L \subset D(C)$ such that*

(t1) *T is maximal monotone and strongly quasibounded with $0 \in D(T)$ and $0 \in T(0)$,*

(c1) *C satisfies the condition $(S_+)_{D(C)}$,*

(c2) *for every $F \in \mathcal{F}(L)$ and $v \in L$, the function $c(F,v) : F \to \mathbb{R}$, defined by $c(F,v)(x) = \langle Cx, v\rangle$, is continuous on F, and*

(c3) *there exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that*

$$\langle Cx, x\rangle \geq -\psi\big(\|x\|\big) \quad \text{for all } x \in D(C).$$

Let Λ and ε_0 be two given positive numbers.

(a) *For a given $\varepsilon > 0$, assume the following property $(\mathcal{P})$:*
There exists a $\lambda \in (0,\Lambda]$ such that the inclusion

$$0 \in Tx + \lambda Cx + \varepsilon Jx$$

has no solution in $D(T + C) \cap \Omega$.
Then there exists a $(\lambda_\varepsilon, x_\varepsilon) \in (0,\Lambda] \times (D(T + C) \cap \partial\Omega)$ such that

$$0 \in Tx_\varepsilon + \lambda_\varepsilon Cx_\varepsilon + \varepsilon Jx_\varepsilon.$$

Here, $D(T + C)$ denotes the intersection of $D(T)$ and $D(C)$.

(b) *If property $(\mathcal{P})$ is fulfilled for every $\varepsilon \in (0,\varepsilon_0]$, T satisfies the condition (S_q) on $D(T) \cap \partial\Omega$, $0 \notin T(D(T) \cap \partial\Omega)$, and the set $C(D(C) \cap \partial\Omega)$ is bounded, then the inclusion*

$$0 \in Tx + \lambda Cx$$

has a solution (λ_0, x_0) in $(0,\Lambda] \times (D(T + C) \cap \partial\Omega)$.

Proof (a) Assume that the conclusion of (a) is not true. Then for every $\lambda \in (0,\Lambda]$, the following boundary condition holds:

$$0 \notin Tx + \lambda Cx + \varepsilon Jx \quad \text{for all } x \in D(T + C) \cap \partial\Omega. \tag{2.1}$$

Considering a multi-valued map H given by

$$H(s,x) := Tx + s\Lambda Cx + \varepsilon Jx \quad \text{for } s \in [0,1],$$

the inclusion $0 \in H(s,x)$ has no solution x in $D(T + C) \cap \partial\Omega$ for all $s \in [0,1]$. Actually, this holds for $s = 0$, in view of the injectivity of the operator $T + \varepsilon J$ with $0 \in (T + \varepsilon J)(D(T) \cap \Omega)$.

Now we consider a single-valued map H_1 given by

$$H_1(t,s,x) := T_t x + s\Lambda Cx + \varepsilon Jx \quad \text{for } t \in (0,\infty) \text{ and } s \in [0,1].$$

We will first show that there exists a positive number t_0 such that the equation

$$H_1(t,s,x) = 0 \tag{2.2}$$

has no solution x in $D(C) \cap \partial\Omega$ for all $t \in (0,t_0]$ and all $s \in [0,1]$. For $s = 0$, assertion (2.2) is obvious because $(T_t + \varepsilon J)x = 0$ implies $x = 0$. Assume that assertion (2.2) does not hold for any $s \in (0,1]$. Then there exist sequences $\{t_n\}$ in $(0,\infty)$, $\{s_n\}$ in $(0,1]$, and $\{x_n\}$ in $D(C) \cap \partial\Omega$ such that $t_n \downarrow 0$, $s_n \to s_0$, $x_n \rightharpoonup x_0$, $Jx_n \rightharpoonup j^*$, and

$$T_{t_n}x_n + s_n\Lambda Cx_n + \varepsilon Jx_n = 0, \tag{2.3}$$

where $s_0 \in [0,1]$, $x_0 \in X$, and $j^* \in X^*$. Let S be a positive upper bound for the bounded sequence $\{\|x_n\|\}$. Note that $s_0 \in (0,1]$. Indeed, if $s_0 = 0$, then we have by the monotonicity of T_{t_n} with $T_{t_n}(0) = 0$, equation (2.3), and (c3)

$$\begin{aligned}\varepsilon\|x_n\|^2 &\le \varepsilon\langle Jx_n, x_n\rangle + \langle T_{t_n}x_n, x_n\rangle = -s_n\Lambda\langle Cx_n, x_n\rangle \\ &\le s_n\Lambda\psi\bigl(\|x_n\|\bigr) \le s_n\Lambda\psi(S)\end{aligned}$$

and so $x_n \to 0 \in \Omega$; but $x_n \in \partial\Omega$, which is a contradiction. Since we have the inequality

$$\langle T_{t_n}x_n, x_n\rangle = -s_n\Lambda\langle Cx_n, x_n\rangle - \varepsilon\langle Jx_n, x_n\rangle \le \Lambda\psi(S),$$

Lemma 1.3 implies that the sequence $\{T_{t_n}x_n\}$ is bounded in the reflexive Banach space X^*. Passing to a subsequence, if necessary, we may suppose that $T_{t_n}x_n \rightharpoonup v^*$ for some $v^* \in X^*$. Set

$$u^* := \frac{1}{s_0\Lambda}\bigl(v^* + \varepsilon j^*\bigr).$$

By equation (2.3), we have $Cx_n \rightharpoonup -u^*$ and hence

$$\lim_{n\to\infty}\bigl\langle Cx_n + u^*, y\bigr\rangle = 0 \quad \text{for every } y \in L\{F_n\}. \tag{2.4}$$

Recall that if two operators $A_1 : D(A_1) \subset X \to 2^{X^*}$ and $A_2 : D(A_2) \subset X \to 2^{X^*}$ are maximal monotone and $D(A_1) \cap \operatorname{int} D(A_2) \neq \emptyset$, then the sum $A_1 + A_2 : D(A_1) \cap D(A_2) \to 2^{X^*}$ is also maximal monotone; see [5, Theorem 32.I]. Since $T + \varepsilon J$ is thus maximal monotone and $T_{t_n}x_n + \varepsilon Jx_n \rightharpoonup v^* + \varepsilon j^*$, Lemma 1.2(a) says that

$$\liminf_{n\to\infty}\langle T_{t_n}x_n + \varepsilon Jx_n, x_n - x_0\rangle \ge 0. \tag{2.5}$$

From equations (2.3), (2.5), and the equality

$$\begin{aligned}\bigl\langle Cx_n + u^*, x_n\bigr\rangle &= \left\langle Cx_n + \frac{1}{s_n\Lambda}(T_{t_n}x_n + \varepsilon Jx_n), x_n\right\rangle - \left\langle \frac{1}{s_n\Lambda}(T_{t_n}x_n + \varepsilon Jx_n), x_n - x_0\right\rangle \\ &\quad - \left\langle \frac{1}{s_n\Lambda}(T_{t_n}x_n + \varepsilon Jx_n), x_0\right\rangle + \bigl\langle u^*, x_n\bigr\rangle\end{aligned}$$

it follows that

$$\limsup_{n\to\infty}\langle Cx_n + u^*, x_n\rangle \le -\liminf_{n\to\infty}\frac{1}{s_n\Lambda}\langle T_{t_n}x_n + \varepsilon Jx_n, x_n - x_0\rangle \le 0. \tag{2.6}$$

Since the operator C satisfies the condition $(S_+)_{D(C)}$, we find from equations (2.4) and (2.6) that $x_n \to x_0 \in D(C)$ and $Cx_0 + u^* = 0$. Since $\lim_{n\to\infty}\langle T_{t_n}x_n, x_n - x_0\rangle = 0$, Lemma 1.2(b) tells us that $x_0 \in D(T)$ and $v^* \in Tx_0$. From $Jx_n \rightharpoonup Jx_0 = j^*$, we get

$$v^* + s_0\Lambda Cx_0 + \varepsilon j^* = 0 \quad \text{or} \quad 0 \in Tx_0 + s_0\Lambda Cx_0 + \varepsilon Jx_0,$$

which contradicts our boundary condition equation (2.1). Consequently, we have proven our first assertion: that there exists a number $t_0 > 0$ such that

$$H_1(t,s,x) \neq 0 \quad \text{for any } (t,s) \in (0,t_0] \times [0,1] \text{ and all } x \in D(C) \cap \partial\Omega.$$

In the next step, we want to show that for each fixed $t \in (0,t_0]$, the degree $d(H_1(t,s,\cdot),\Omega,0)$ is independent of $s \in [0,1]$, where d denotes the Kartsatos-Skrypnik degree from [12]. Fix $t \in (0,t_0]$. For $s \in [0,1]$, let $A_s : D(A_s) \subset X \to X^*$ be defined by

$$A_s x := H_1(t,s,x) = T_t x + s\Lambda Cx + \varepsilon Jx,$$

where $D(A_s) = X$ for $s = 0$ and $D(A_s) = D(C)$ for $s \in (0,1]$. First of all, for every finite-dimensional space $F \subset L\{F_j\}$ and every $v \in L\{F_j\}$, the function $\tilde{a}(F,v) : F \times [0,1] \to \mathbb{R}$, defined by $\tilde{a}(F,v)(x,s) = \langle A_s x, v\rangle$, is continuous on $F \times [0,1]$ because the operators T_t and J are continuous and C satisfies the condition (c2). To show that the family $\{A_s\}$ satisfies the condition $(S_+)^{(s)}_{0,L}$, we assume that $\{s_n\}$ is a sequence in $[0,1]$ and $\{x_n\}$ is a sequence in $L\{F_n\}$ such that $s_n \to s_0$, $x_n \rightharpoonup x_0$, and

$$\limsup_{n\to\infty}\langle A_{s_n}x_n, x_n\rangle \le 0 \quad \text{and} \quad \lim_{n\to\infty}\langle A_{s_n}x_n, y\rangle = 0 \tag{2.7}$$

for every $y \in L\{F_n\}$, where $s_0 \in [0,1]$ and $x_0 \in X$. By Lemma 1.1(a), the sequence $\{T_t x_n\}$ is bounded in X^*. So we may suppose without loss of generality that $T_t x_n \rightharpoonup v^*$ and $Jx_n \rightharpoonup j^*$ for some $v^*, j^* \in X^*$. There are two cases to consider. If $s_0 = 0$, then we have

$$\varepsilon\|x_n\|^2 \le \varepsilon\langle Jx_n, x_n\rangle + \langle T_t x_n, x_n\rangle \le \langle A_{s_n}x_n, x_n\rangle + s_n\Lambda\psi(S),$$

which implies along with equation (2.7)

$$\varepsilon\limsup_{n\to\infty}\|x_n\|^2 \le \lim_{n\to\infty} s_n\Lambda\psi(S) = 0,$$

where S is an upper bound for the sequence $\{\|x_n\|\}$. Hence it follows that $x_n \to 0$, $x_0 = 0 \in X = D(A_{s_0})$, and $A_{s_0}x_0 = 0$. Now let $s_0 \in (0,1]$. We may suppose that $s_n > 0$ for all $n \in \mathbb{N}$. Set

$\tilde{s}_n := 1/(s_n\Lambda)$ and $\tilde{s} := 1/(s_0\Lambda)$. The relation (2.7) can be expressed in the form

$$\begin{aligned} &\limsup_{n\to\infty}\langle Cx_n + \tilde{s}_n(T_t + \varepsilon J)x_n, x_n\rangle \le 0, \\ &\lim_{n\to\infty}\langle Cx_n + \tilde{s}_n(T_t + \varepsilon J)x_n, y\rangle = 0 \quad \text{for every } y \in L\{F_n\}. \end{aligned} \tag{2.8}$$

From the second part of equation (2.8), it is obvious that

$$\lim_{n\to\infty}\langle Cx_n + \tilde{s}(v^* + \varepsilon j^*), y\rangle = 0 \quad \text{for every } y \in L\{F_n\}. \tag{2.9}$$

By the monotonicity of the operator $T_t + \varepsilon J$, we have

$$\liminf_{n\to\infty}\langle (T_t + \varepsilon J)x_n, x_n - x_0\rangle \ge \liminf_{n\to\infty}\langle (T_t + \varepsilon J)x_0, x_n - x_0\rangle = 0. \tag{2.10}$$

Hence it follows from the first part of equation (2.8) and from equation (2.10) that

$$\begin{aligned} \limsup_{n\to\infty}\langle Cx_n + \tilde{s}(v^* + \varepsilon j^*), x_n\rangle &\le -\liminf_{n\to\infty}\tilde{s}_n\langle (T_t + \varepsilon J)x_n, x_n - x_0\rangle \\ &\le 0. \end{aligned} \tag{2.11}$$

Since the operator C satisfies the condition $(S_+)_L$, we find from equations (2.9) and (2.11) that

$$x_n \to x_0, \quad x_0 \in D(C) = D(A_{s_0}) \quad \text{and} \quad Cx_0 + \tilde{s}(v^* + \varepsilon j^*) = 0.$$

By the demicontinuity of the operators T_t and J, we have

$$T_t x_n \rightharpoonup T_t x_0 = v^* \quad \text{and} \quad Jx_n \rightharpoonup Jx_0 = j^*$$

and hence

$$A_{s_0}x_0 = T_t x_0 + s_0\Lambda Cx_0 + \varepsilon Jx_0 = 0.$$

Consequently, the family $\{A_s\}$ satisfies the condition $(S_+)^{(s)}_{0,L}$, as required.

Since $A_s(x) \not\ni 0$ for all $(s,x) \in [0,1] \times (D(A_s) \cap \partial\Omega)$, we see, in view of Theorem A of [18], that the degree $d(A_s, \Omega, 0)$ is independent of the choice of $s \in [0,1]$. Until now, we have shown that for each fixed $t \in (0, t_0]$, the degree $d(H_1(t,s,\cdot), \Omega, 0)$ is constant for all $s \in [0,1]$. Notice that $T + \varepsilon J$ is maximal monotone and strongly quasibounded, $0 \in (T + \varepsilon J)(0)$, and

$$H(s,x) = (T + \varepsilon J)x + s\Lambda Cx \not\ni 0 \quad \text{for all } s \in [0,1] \text{ and all } x \in D(T + C) \cap \partial\Omega.$$

Combining this with our first assertion above, Theorem 2 of [18] says that for each fixed $s \in [0,1]$, the degree $d(T_t + s\Lambda C + \varepsilon J, \Omega, 0)$ is constant for all $t \in (0, t_0]$. If deg denotes the degree introduced in [18], then for every $s \in [0,1]$, we have

$$\deg(T + s\Lambda C + \varepsilon J, \Omega, 0) = \mathrm{d}(T_t + s\Lambda C + \varepsilon J, \Omega, 0) \quad \text{for } t \in (0, t_0]$$

and hence

$$\begin{aligned}\deg(T + s\Lambda C + \varepsilon J, \Omega, 0) &= \mathrm{d}(A_s, \Omega, 0) = \mathrm{d}(A_0, \Omega, 0)\\ &= \mathrm{d}(T_t + \varepsilon J, \Omega, 0) = 1,\end{aligned}$$

where the last equality follows from Theorem 3 in [23]. Thus, for all $s \in (0,1]$, the inclusion

$$0 \in Tx + s\Lambda Cx + \varepsilon Jx$$

has a solution in $D(T + C) \cap \Omega$, which contradicts property $(\mathcal{P})$. We conclude that statement (a) is true.

(b) Let $\{\varepsilon_n\}$ be a sequence in $(0, \varepsilon_0]$ such that $\varepsilon_n \to 0$. According to statement (a), there exists a sequence $\{(\lambda_{\varepsilon_n}, x_{\varepsilon_n})\}$ in $(0, \Lambda] \times (D(T + C) \cap \partial\Omega)$ such that

$$u^*_{\varepsilon_n} + \lambda_{\varepsilon_n} Cx_{\varepsilon_n} + \varepsilon_n Jx_{\varepsilon_n} = 0,$$

where $u^*_{\varepsilon_n} \in Tx_{\varepsilon_n}$. If we set $\lambda_n := \lambda_{\varepsilon_n}$, $x_n := x_{\varepsilon_n}$, and $u^*_n := u^*_{\varepsilon_n}$, it can be rewritten in the form

$$u^*_n + \lambda_n Cx_n + \varepsilon_n Jx_n = 0. \tag{2.12}$$

Notice that the sequence $\{u^*_n\}$ is bounded in X^*. This follows from the strong quasiboundedness of the operator T together with the inequality

$$\langle u^*_n, x_n\rangle = -\lambda_n \langle Cx_n, x_n\rangle - \varepsilon_n \langle Jx_n, x_n\rangle \le \Lambda\psi(S),$$

where S is an upper bound for the sequence $\{\|x_n\|\}$. From equation (2.12), $\{\lambda_n Cx_n\}$ is bounded in X^*. Without loss of generality, we may suppose that

$$\lambda_n \to \lambda_0, \qquad x_n \rightharpoonup x_0, \quad \text{and} \quad u^*_n \rightharpoonup u^*_0, \tag{2.13}$$

where $\lambda_0 \in [0, \Lambda]$, $x_0 \in X$, and $u^*_0 \in X^*$. Note that the limit λ_0 belongs to $(0, \Lambda]$. In fact, if $\lambda_0 = 0$, then the boundedness of the set $C(D(C) \cap \partial\Omega)$ implies that $\lambda_n Cx_n \to 0$ and so by equation (2.12) $u^*_n \to 0$. Since the maximal monotone operator T satisfies the condition (S_q) on $D(T) \cap \partial\Omega$, we find from equation (2.13) and Lemma 1.2(b) that $x_n \to x_0 \in \partial\Omega$, $x_0 \in D(T)$, and $0 \in Tx_0$, which contradicts the hypothesis that $0 \notin T(D(T) \cap \partial\Omega)$. As $Cx_n \rightharpoonup (-1/\lambda_0)u^*_0$, we have

$$\lim_{n\to\infty}\left\langle Cx_n + \frac{1}{\lambda_0}u^*_0, y\right\rangle = 0 \quad \text{for every } y \in L\{F_n\}. \tag{2.14}$$

From equation (2.12) it follows that

$$\begin{aligned}\limsup_{n\to\infty}\left\langle Cx_n + \frac{1}{\lambda_0}u^*_0, x_n\right\rangle &\le -\frac{1}{\lambda_0}\liminf_{n\to\infty}\langle u^*_n + \varepsilon_n Jx_n, x_n - x_0\rangle\\ &\le 0,\end{aligned} \tag{2.15}$$

where the last inequality follows from Lemma 1.2(a). Since the operator C satisfies the condition $(S_+)_{D(C)}$, we obtain from equations (2.14) and (2.15) $x_n \to x_0 \in D(C)$ and $\lambda_0 Cx_0 +$

$u_0^* = 0$. By the maximal monotonicity of the operator T, we have $x_0 \in D(T)$ and $u_0^* \in Tx_0$. We conclude that

$$0 \in Tx_0 + \lambda_0 Cx_0 \quad \text{and} \quad x_0 \in D(T+C) \cap \partial\Omega.$$

This completes the proof. □

Remark 2.2 (a) In Theorem 2.1, it is inevitable that the set $C(D(C) \cap \partial\Omega)$ is assumed to be bounded because it does not hold in general that if $\lambda_n \to 0$ then $\lambda_n Cx_n \to 0$.

(b) When C is quasibounded and satisfies the condition $(\tilde{S}_+)$, it was studied in [9, Theorem 4] by using Kartsatos-Skrypnik degree theory for $(\tilde{S}_+)$-perturbations of maximal monotone operators developed in [13]. For the case where C is generalized pseudomonotone in place of the condition $(\tilde{S}_+)$, we refer to [20, Theorem 2.1].

From Theorem 2.1, we get the following eigenvalue result in the case when the operator C satisfies the condition (S_+).

Corollary 2.3 *Let T, Ω, L, Λ, ε_0 be as in Theorem 2.1. Suppose that $C : X \to X^*$ is a strongly quasibounded demicontinuous operator such that*

(c1′) *C satisfies the condition (S_+) on X,*

(c2) *for every $F \in \mathcal{F}(L)$ and $v \in L$, the function $c(F,v) : F \to \mathbb{R}$, defined by $c(F,v)(x) = \langle Cx, v\rangle$, is continuous on F, and*

(c3) *there exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that*

$$\langle Cx, x\rangle \geq -\psi\left(\|x\|\right) \quad \text{for all } x \in X.$$

Then the following statements hold:

(a) *If property $(\mathcal{P})$ is fulfilled for a given $\varepsilon > 0$, then there exists a $(\lambda_\varepsilon, x_\varepsilon) \in (0,\Lambda] \times (D(T) \cap \partial\Omega)$ such that $0 \in Tx_\varepsilon + \lambda_\varepsilon Cx_\varepsilon + \varepsilon Jx_\varepsilon$.*

(b) *If property $(\mathcal{P})$ is fulfilled for every $\varepsilon \in (0,\varepsilon_0]$, T satisfies the condition (S_q) on $D(T) \cap \partial\Omega$ and $0 \notin T(D(T) \cap \partial\Omega)$, then the inclusion $0 \in Tx + \lambda Cx$ has a solution in $(0,\Lambda] \times (D(T) \cap \partial\Omega)$.*

Proof Statement (a) follows immediately from Theorem 2.1 if we only show that the operator C satisfies the condition $(S_+)_{D(C)}$ with $D(C) = X$. To do this, let $h \in X^*$ be given and suppose that $\{x_n\}$ is any sequence in X such that

$$x_n \rightharpoonup x_0, \qquad \limsup_{n\to\infty}\langle Cx_n - h, x_n\rangle \leq 0, \quad \text{and} \quad \lim_{n\to\infty}\langle Cx_n - h, y\rangle = 0 \tag{2.16}$$

for every $y \in L\{F_n\}$. Then $\{\langle Cx_n, x_n\rangle\}$ is obviously bounded from above. By the strong quasiboundedness of the operator C, the sequence $\{Cx_n\}$ is bounded in X^*. Since $L\{F_n\}$ is dense in the reflexive Banach space X, it follows from the third one of equation (2.16) that $Cx_n \rightharpoonup h$. Hence we obtain from the first and second one of equation (2.16)

$$\begin{aligned}
&\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \\
&\quad \leq \limsup_{n\to\infty}\langle Cx_n - h, x_n\rangle - \lim_{n\to\infty}\langle Cx_n - h, x_0\rangle + \lim_{n\to\infty}\langle h, x_n - x_0\rangle \leq 0.
\end{aligned}$$

Since C satisfies the condition (S_+) on X and is demicontinuous, we have

$$x_n \to x_0 \in X \quad \text{and} \quad Cx_0 - h = 0.$$

Thus, the operator C satisfies the condition $(S_+)_{D(C)}$ with $D(C) = X$.

(b) Let $\{\varepsilon_n\}$ be a sequence in $(0, \varepsilon_0]$ such that $\varepsilon_n \to 0$. In view of (a), there exists a sequence $\{(\lambda_n, x_n)\}$ in $(0, \Lambda] \times (D(T) \cap \partial\Omega)$ such that

$$u_n^* + \lambda_n Cx_n + \varepsilon_n Jx_n = 0, \tag{2.17}$$

where $u_n^* \in Tx_n$. Notice that the sequence $\{Cx_n\}$ is bounded in X^* and so is $\{u_n^*\}$. This follows from the strong quasiboundedness of the operator C and the inequality

$$\langle Cx_n, x_n\rangle = -\frac{1}{\lambda_n}\langle u_n^*, x_n\rangle - \frac{\varepsilon_n}{\lambda_n}\langle Jx_n, x_n\rangle \leq 0.$$

We may suppose that $\lambda_n \to \lambda_0$, $x_n \rightharpoonup x_0$, and $u_n^* \rightharpoonup u_0^*$, where $\lambda_0 \in [0, \Lambda]$, $x_0 \in X$, and $u_0^* \in X^*$. Note that λ_0 belongs to $(0, \Lambda]$. Indeed, if $\lambda_0 = 0$, then we have by the boundedness of $\{Cx_n\}$ and equation (2.17) $u_n^* \to 0$ and hence by the condition (S_q) $x_n \to x_0 \in D(T)$ and $0 \in Tx_0$, which contradicts the hypothesis $0 \notin T(D(T) \cap \partial\Omega)$. The rest of the proof proceeds analogously as in the proof of Theorem 2.1. □

Remark 2.4 (a) The boundedness assumption on the set $C(D(C) \cap \partial\Omega)$ is unnecessary in Corollary 2.3, provided that the operator C is strongly quasibounded.

(b) An analogous result to Corollary 2.3 can be found in [9, Corollary 1], where the operator C is supposed to be bounded.

We close this section by exhibiting a simple example of operators A satisfying the condition $(S_+)_{D(A)}$.

Let G be a bounded open set in $\mathbb{R}^N$. Let $1 < p < \infty$ and $X = W_0^{1,p}(G)$. Define the two operators $A_1, A_2 : X \to X^*$ by

$$\langle A_1 u, v\rangle = \sum_{i=1}^{N} \int_G \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx,$$

$$\langle A_2 u, v\rangle = \int_G |u|^{p-2} uv\,dx.$$

Then the operator A_1 is clearly bounded and continuous, and it satisfies the condition (S_+) on X. The operator A_2 is compact; see [24, Theorem 2.2] and [5, Proposition 26.10]. In particular, the sum $A := A_1 + A_2$ satisfies the condition $(S_+)_{D(A)}$ with $D(A) = X$.

3 Fredholm alternative

In this section, we present a variant of the Fredholm alternative for strongly quasibounded maximal monotone operators, by applying Kartsatos-Quarcoo degree theory as in Section 2.

Given $\gamma > 0$, an operator $A : D(A) \subset X \to X^*$ is said to be *positively homogeneous* of degree γ on a set $M \subset D(A)$ if $A(rx) = r^\gamma Ax$ for all $x \in M$ and all $r > 0$. For example, the duality operator $J_\varphi : X \to X^*$ is positively homogeneous of degree γ on X if $\varphi(t) = t^\gamma$ for $t \in [0,\infty)$. In addition, the operators A_1 and A_2 given at the end of Section 2 are positively homogeneous of degree $p-1$ on $X = W_0^{1,p}(G)$.

Theorem 3.1 *Let L be a dense subspace of X and let $\lambda, \gamma \in [1,\infty)$ be given. Suppose that $T : D(T) = L \to X^*$ is an operator and $C : D(C) \subset X \to X^*$ is an operator with $L \subset D(C)$ and $C(0) = 0$ such that*

(t1) *T is maximal monotone and strongly quasibounded with $T(0) = 0$,*
(t2) *$\lambda Tx + Cx + \mu J_\varphi x = 0$ implies $x = 0$ for every $\mu \geq 0$, where $\varphi(t) = t^\gamma$,*
(c1) *C satisfies the condition $(S_+)_L$,*
(c2) *for every $F \in \mathcal{F}(L)$ and $v \in L$, the function $c(F,v) : F \to \mathbb{R}$, defined by $c(F,v)(x) = \langle Cx, v\rangle$, is continuous on F, and*
(c3) *there exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that*

$$\langle Cx, x\rangle \geq -\psi\bigl(\|x\|\bigr) \quad \textit{for all } x \in D(C).$$

If the operators T and C are positively homogeneous of degree γ on L, then the operator $\lambda T + C$ is surjective.

Proof Let p^* be an arbitrary but fixed element of X^*. For each fixed $\varepsilon > 0$, consider a family of operators $A_t : D(A_t) \subset X \to X^*$, $t \in [0,1]$ given by

$$A_t(x) := H(t,x) := t\bigl(\lambda Tx + Cx + \varepsilon J_\varphi x - p^*\bigr) + (1-t)\varepsilon J_\varphi x,$$

where $D(A_t) = X$ for $t = 0$ and $D(A_t) = L$ for $t \in (0,1]$. The first aim is to prove that the set of all solutions of the equation $H(t,x) = 0$ is bounded, independent of $t \in [0,1]$. If $t = 0$, then $H(0,x) = \varepsilon J_\varphi x = 0$ implies $x = 0$. It suffices to show that $\{(t,x) \in (0,1] \times L : H(t,x) = 0\}$ is bounded. Assume the contrary; then there exist sequences $\{t_n\}$ in $(0,1]$ and $\{x_n\}$ in L such that $t_n \to t_0 \in [0,1]$, $\|x_n\| \to \infty$, and

$$t_n\bigl(\lambda Tx_n + Cx_n + \varepsilon J_\varphi x_n - p^*\bigr) + (1-t_n)\varepsilon J_\varphi x_n = 0,$$

which can be written as

$$\lambda Tx_n + Cx_n - p^* + \frac{\varepsilon}{t_n} J_\varphi x_n = 0. \tag{3.1}$$

We may suppose that $\|x_n\| \geq 1$ for all $n \in \mathbb{N}$. Since the operators T, C, and J_φ are positively homogeneous of degree γ, it follows from equation (3.1) that

$$\lambda T\left(\frac{x_n}{\|x_n\|}\right) + C\left(\frac{x_n}{\|x_n\|}\right) - \frac{1}{\|x_n\|^\gamma} p^* + \frac{\varepsilon}{t_n} J_\varphi\left(\frac{x_n}{\|x_n\|}\right) = 0.$$

Setting $u_n := x_n/\|x_n\|$ and $q_n := 1/t_n$, we have $\|u_n\| = 1$, $q_n > 0$, and

$$\lambda Tu_n + Cu_n - \frac{1}{\|x_n\|^\gamma} p^* + q_n \varepsilon J_\varphi u_n = 0. \tag{3.2}$$

Then we obtain from equation (3.2) and (c3)

$$\begin{aligned}\lambda\langle Tu_n,u_n\rangle &= -\langle Cu_n,u_n\rangle + \frac{1}{\|x_n\|^\gamma}\langle p^*,u_n\rangle - q_n\varepsilon\langle J_\varphi u_n,u_n\rangle \\ &\le \psi(1) + \|p^*\|.\end{aligned}$$

Hence the strong quasiboundedness of T implies that the sequence $\{Tu_n\}$ is bounded in X^*. There are two cases to consider. If $t_0 = 0$, then $q_n \to \infty$, $\langle J_\varphi u_n, u_n\rangle = 1$, and the monotonicity of T with $T(0) = 0$ implies

$$0 \le \lambda\langle Tu_n,u_n\rangle \le \psi(1) + \|p^*\| - q_n\varepsilon \to -\infty,$$

which is a contradiction. Now let $t_0 > 0$ and set $q_0 := 1/t_0$. Without loss of generality, we may suppose that

$$u_n \rightharpoonup u_0, \qquad Tu_n \rightharpoonup v^*, \quad \text{and} \quad J_\varphi u_n \rightharpoonup j^*,$$

where $u_0 \in X$, $v^* \in X^*$, and $j^* \in X^*$. By equation (3.2), we have $Cu_n \rightharpoonup -\lambda v^* - q_0\varepsilon j^*$ and hence

$$\lim_{n\to\infty}\langle Cu_n + \lambda v^* + q_0\varepsilon j^*, y\rangle = 0 \quad \text{for every } y \in L\{F_n\}. \tag{3.3}$$

Since the operator $\lambda T + q_0\varepsilon J_\varphi$ is maximal monotone, we have

$$\liminf_{n\to\infty}\langle \lambda Tu_n + q_n\varepsilon J_\varphi u_n, u_n - u_0\rangle \ge 0. \tag{3.4}$$

In fact, if equation (3.4) is false, then there is a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that

$$\lim_{n\to\infty}\langle \lambda Tu_n + q_n\varepsilon J_\varphi u_n, u_n - u_0\rangle < 0.$$

Hence it is clear that

$$\limsup_{n\to\infty}\langle \lambda Tu_n + q_n\varepsilon J_\varphi u_n, u_n\rangle < \langle \lambda v^* + q_0\varepsilon j^*, u_0\rangle. \tag{3.5}$$

For every $u \in D(T)$, we have, by the monotonicity of the operator $\lambda T + q_n\varepsilon J_\varphi$,

$$\begin{aligned}&\liminf_{n\to\infty}\langle \lambda Tu_n + q_n\varepsilon J_\varphi u_n, u_n\rangle \\ &\quad\ge \liminf_{n\to\infty}\big[\langle \lambda Tu_n + q_n\varepsilon J_\varphi u_n, u\rangle + \langle \lambda Tu + q_n\varepsilon J_\varphi u, u_n - u\rangle\big] \\ &\quad= \langle \lambda v^* + q_0\varepsilon j^*, u\rangle + \langle \lambda Tu + q_0\varepsilon J_\varphi u, u_0 - u\rangle,\end{aligned}$$

which implies along with equation (3.5)

$$\langle \lambda v^* + q_0\varepsilon j^* - (\lambda Tu + q_0\varepsilon J_\varphi u), u_0 - u\rangle > 0. \tag{3.6}$$

By the maximal monotonicity of $\lambda T + q_0\varepsilon J_\varphi$, we have $u_0 \in D(T)$ and $(\lambda T + q_0\varepsilon J_\varphi)u_0 = \lambda v^* + q_0\varepsilon j^*$. Letting $u = u_0 \in D(T)$ in equation (3.6), we get a contradiction. Thus, equation (3.4) is true.

Furthermore, equation (3.4) implies, because of $(1/\|x_n\|^\gamma)p^* \to 0$, that

$$\liminf_{n\to\infty}\left\langle \lambda Tu_n - \frac{1}{\|x_n\|^\gamma}p^* + q_n\varepsilon J_\varphi u_n, u_n - u_0\right\rangle \geq 0. \tag{3.7}$$

From equations (3.2), (3.7), and the equality

$$\begin{aligned}
&\langle Cu_n + \lambda v^* + q_0\varepsilon j^*, u_n\rangle \\
&\quad = \left\langle Cu_n + \lambda Tu_n - \frac{1}{\|x_n\|^\gamma}p^* + q_n\varepsilon J_\varphi u_n, u_n\right\rangle \\
&\qquad - \left\langle \lambda Tu_n - \frac{1}{\|x_n\|^\gamma}p^* + q_n\varepsilon J_\varphi u_n, u_n - u_0\right\rangle \\
&\qquad - \left\langle \lambda Tu_n - \frac{1}{\|x_n\|^\gamma}p^* + q_n\varepsilon J_\varphi u_n, u_0\right\rangle + \langle \lambda v^* + q_0\varepsilon j^*, u_n\rangle
\end{aligned}$$

it follows that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle Cu_n + \lambda v^* + q_0\varepsilon j^*, u_n\rangle \\
&\quad \leq -\liminf_{n\to\infty}\left\langle \lambda Tu_n - \frac{1}{\|x_n\|^\gamma}p^* + q_n\varepsilon J_\varphi u_n, u_n - u_0\right\rangle \\
&\quad \leq 0.
\end{aligned} \tag{3.8}$$

Since the operator C satisfies the condition $(S_+)_L$, we obtain from equations (3.3) and (3.8)

$$u_n \to u_0, \qquad u_0 \in D(C), \quad \text{and} \quad Cu_0 + \lambda v^* + q_0\varepsilon j^* = 0.$$

Since T is maximal monotone and J_φ is continuous, Lemma 1.2(b) implies that

$$u_0 \in D(T), \qquad Tu_0 = v^*, \quad \text{and} \quad J_\varphi u_0 = j^*.$$

Therefore, we obtain

$$\lambda Tu_0 + Cu_0 + q_0\varepsilon J_\varphi u_0 = 0 \quad \text{and} \quad \|u_0\| = 1,$$

which contradicts hypothesis (t2) with $\mu = q_0\varepsilon$. Thus, we have shown that $\{(t,x) \in [0,1] \times L : H(t,x) = 0\}$ is bounded.

So we can choose an open ball $B_r(0)$ in X of radius $r > 0$ centered at the origin 0 so that

$$\bigl\{x \in L : H(t,x) = 0 \text{ for some } t \in [0,1]\bigr\} \subset B_r(0).$$

This means that $H(t,x) = A_t(x) \neq 0$ for all $(t,x) \in [0,1] \times (D(A_t) \cap \partial B_r(0))$. Note that the operator $\tilde{T}_\varepsilon := \lambda T + \varepsilon J_\varphi$ is maximal monotone, strongly quasibounded, $\tilde{T}_\varepsilon(0) = 0$, and the operator $\tilde{C} := C - p^*$ satisfies the condition $(S_+)_L$ and other conditions with $\tilde{c}(F,v)(x) := \langle \tilde{C}x, v\rangle$

for $x \in F$ and $\langle \tilde{C}x, x\rangle \geq -\tilde{\psi}(\|x\|)$ for $x \in D(\tilde{C})$, where $\tilde{\psi}(t) := (1 + \|p^*\|)\max\{\psi(t), t\}$. Moreover, we know from Section 1 that the operator εJ_φ is continuous, bounded and strictly monotone, and that it satisfies the condition (S_+), and $\langle \varepsilon J_\varphi x, x\rangle = \varepsilon\|x\|^{\gamma+1}$ for $x \in X$.

Using the homotopy invariance property of the degree stated in [18, Theorem 3], we have

$$\deg\bigl(\lambda T + C + \varepsilon J_\varphi - p^*, B_r(0), 0\bigr) = \deg\bigl(\varepsilon J_\varphi, B_r(0), 0\bigr) = 1. \tag{3.9}$$

Applying equation (3.9) with $\varepsilon = 1/n$, there exists a sequence $\{x_n\}$ in L such that

$$\lambda Tx_n + Cx_n + \frac{1}{n}J_\varphi x_n = p^*. \tag{3.10}$$

Next, we show that the sequence $\{x_n\}$ is bounded in X. Indeed, assume on the contrary that there is a subsequence of $\{x_n\}$, denoted by $\{x_n\}$, such that $\|x_n\| \to \infty$. Dividing both sides of equation (3.10) by $\|x_n\|^\gamma$ and setting $u_n := x_n/\|x_n\|$ and $w_n^* := \lambda Tu_n + Cu_n$, we get

$$\lambda Tu_n + Cu_n + \frac{1}{n}J_\varphi u_n = \frac{1}{\|x_n\|^\gamma}p^*$$

and so $w_n^* \to 0$. Since $\lambda\langle Tu_n, u_n\rangle = -\langle Cu_n, u_n\rangle + \langle w_n^*, u_n\rangle \leq \psi(1) + \|w_n^*\|$ for all $n \in \mathbb{N}$, it follows from (t1) that the sequence $\{Tu_n\}$ is bounded in X^*. We may suppose that $u_n \rightharpoonup u_0$ and $Tu_n \rightharpoonup v^*$ for some $u_0 \in X$ and some $v^* \in X^*$. As in the proof of equations (3.3) and (3.8) above, we can show that

$$\limsup_{n\to\infty}\bigl\langle Cu_n + \lambda v^*, u_n\bigr\rangle \leq 0 \quad \text{and} \quad \lim_{n\to\infty}\bigl\langle Cu_n + \lambda v^*, y\bigr\rangle = 0$$

for every $y \in L\{F_n\}$. Since the operator C satisfies the condition $(S_+)_L$, we obtain

$$u_n \to u_0, \qquad u_0 \in D(C), \quad \text{and} \quad Cu_0 + \lambda v^* = 0.$$

By Lemma 1.2(b), we have $u_0 \in D(T)$ and $Tu_0 = v^*$ and hence

$$\lambda Tu_0 + Cu_0 = 0 \quad \text{and} \quad \|u_0\| = 1,$$

which contradicts hypothesis (t2) with $\mu = 0$. Therefore, the sequence $\{x_n\}$ is bounded in X.

Combining this with equation (3.10), we know from (c3) and (t1) that the sequence $\{Tx_n\}$ is also bounded in X^*. Thus we may suppose that $x_n \rightharpoonup x_0$ and $Tx_n \rightharpoonup v_0^*$ for some $x_0 \in X$ and some $v_0^* \in X^*$. From $Cx_n \rightharpoonup -\lambda v_0^* + p^*$ and the maximal monotonicity of the operator T, we get as before

$$\limsup_{n\to\infty}\bigl\langle Cx_n + \lambda v_0^* - p^*, x_n\bigr\rangle \leq 0 \quad \text{and} \quad \lim_{n\to\infty}\bigl\langle Cx_n + \lambda v_0^* - p^*, y\bigr\rangle = 0$$

for every $y \in L\{F_n\}$. Since the operator C satisfies the condition $(S_+)_L$ and T is maximal monotone, we conclude that

$$x_0 \in D(\lambda T + C) \quad \text{and} \quad \lambda Tx_0 + Cx_0 = p^*.$$

As $p^* \in X^*$ was arbitrary, this says that the operator $\lambda T + C$ is surjective. This completes the proof. □

Remark 3.2 An analogous result to Theorem 3.1 was investigated in [20, Theorem 4.1], where the method was to use Kartsatos-Skrypnik degree theory for quasibounded densely defined $(\tilde{S}_+)$-perturbations of maximal monotone operators, developed in [13]; see also [9, Theorem 5].

As a particular case of Theorem 3.1, we have another surjectivity result.

Corollary 3.3 *Let L, T, and C be the same as in Theorem* 3.1, *except that hypothesis* (t2) *is replaced by*

(t2′) $\langle \lambda Tx + Cx, x\rangle \geq 0$ *for all* $x \in L$.

If λ is not an eigenvalue for the pair (T, C), that is, $\lambda Tx + Cx = 0$ implies $x = 0$, then the operator $\lambda T + C$ is surjective.

Proof Noting that

$$\langle \lambda Tx + Cx + \mu J_\varphi x, x\rangle \geq \mu \langle J_\varphi x, x\rangle = \mu \|x\|^{\gamma+1} \geq 0$$

for every $x \in L$ and $\mu > 0$, it is clear that hypothesis (t2) in Theorem 3.1 is satisfied. Apply Theorem 3.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
KI conceived of the study and drafted the manuscript. BI participated in coordination. All authors approved the final manuscript.

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