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On reverse Hilbert-type inequalities

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Abstract

By introducing two pairs of conjugate exponents and estimating the weight coefficients, we establish reverse versions of Hilbert-type inequalities, as described by Jin (J. Math. Anal. Appl. 340:932-942, 2008), and we prove that the constant factors are the best possible. As applications, some particular results are considered.

Keywords: reverse Hilbert-type inequality; weight coefficient; best constant factor

1 Introduction

If both a_n and $b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then we have (see [1])

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left\{\sum_{n=1}^{\infty} a_n^2\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty} b_n^2\right\}^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π has the best possible value. Inequality (1.1) is the well-known Hilbert inequality, introduced in 1925; inequality (1.1) has been generalized by Hardy as follows.

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and both a_n and $b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.2) is the well-known Hardy-Hilbert inequality, which is important in analysis and applications (see [2]). In recent years, many results with generalizations of this type of inequality have been obtained (see [3]).

Under the same conditions as in (1.2), there are some Hilbert-type inequalities that are similar to (1.2), which also have been studied and generalized by some mathematicians.

Recently, by studying a Hilbert-type operator, Jin [4] obtained a new bilinear operator inequality with the norm, and he provided some new Hilbert-type inequalities with the best constant factor. First, we repeat the results of [4].

Definition 1.1 Let $H_{p,q}(r,s)$ be the set of functions $k(x,y)$ satisfying the following conditions.

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, suppose that $k(x,y)$ is continuous in $(0,\infty) \times (0,\infty)$ and satisfies:

(1) $k(x,y) = k(y,x) > 0$, where $x, y \in (0,\infty)$.

(2) For $\varepsilon \geq 0$ and $x > 0$, the function $k(x,t)(\frac{x}{t})^{\frac{1+\varepsilon}{l}}$ $(l = r,s)$ is decreasing in $t \in (0,\infty)$.

For $\varepsilon \geq 0$ small enough, $x > 0$, and $\overline{k}_l(\varepsilon,x)$ can be written as

$$\overline{k}_l(\varepsilon,x) := \int_0^\infty k(x,t)\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{l}} dt \quad (l = r,s),$$

where $\overline{k}_l(\varepsilon,x)$ is independent of x, $\overline{k}_l(0,x) := \int_0^\infty k(x,t)(\frac{x}{t})^{\frac{1}{l}}\,dt = k_p$ $(l = r,s)$, k_p is a positive constant independent of x, and $\overline{k}_l(\varepsilon,x) = k_p(\varepsilon) = k_p + o(1)$ $(\varepsilon \to 0^+)$.

$$\text{(3)} \quad \sum_{m=1}^\infty \frac{1}{m^{1+\varepsilon}} \int_0^1 k(m,t)\left(\frac{m}{t}\right)^{\frac{1+\varepsilon(s/q)}{s}} dt = O(1) \quad \bigl(\varepsilon \to 0^+\bigr),$$

$$\sum_{m=1}^\infty \frac{1}{m^{1+\varepsilon}} \int_0^1 k(m,t)\left(\frac{m}{t}\right)^{\frac{1+\varepsilon(r/p)}{r}} dt = O(1) \quad \bigl(\varepsilon \to 0^+\bigr).$$

We have Jin's result as follows.

Theorem 1.1 *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, *and* $k(x,y) \in H_{p,q}(r,s)$, $a_n, b_n \geq 0$, *such that* $0 < \sum_{n=1}^\infty n^{\frac{p}{r}-1} a_n^p < \infty$ *and* $0 < \sum_{n=1}^\infty n^{\frac{q}{s}-1} b_n^q < \infty$, *then we have*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty k(m,n) a_m b_n < k_r \left\{\sum_{n=1}^\infty n^{\frac{p}{r}-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty n^{\frac{q}{s}-1} b_n^q\right\}^{\frac{1}{q}}, \tag{1.3}$$

$$\sum_{n=1}^\infty n^{\frac{p}{r}-1} \left[\sum_{m=1}^\infty k(m,n) a_m\right]^p < (k_r)^p \sum_{n=1}^\infty n^{\frac{p}{r}-1} a_n^p. \tag{1.4}$$

Here the constant factors k_r *and* $(k_r)^p$ *are the best possible. Inequality* (1.3) *is equivalent to* (1.4).

If $p = r$ and $q = s$ in Theorem 1.1, then Theorem 1.1 reduces to Yang's result [5] as follows.

Theorem 1.2 *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $k(x,y) \in H(p,q)$, *and both* a_n *and* $b_n \geq 0$, *such that* $0 < \sum_{n=1}^\infty a_n^p < \infty$ *and* $0 < \sum_{n=1}^\infty a_n^p < \infty$, *then we have*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty k(m,n) a_n b_n < k_p \left\{\sum_{n=1}^\infty a_n^p\right\} \left\{\sum_{n=1}^\infty b_n^q\right\}^{\frac{1}{q}}, \tag{1.5}$$

$$\sum_{n=1}^\infty \left[\sum_{n=1}^\infty k(m,n) a_m\right]^p < (k_p)^p \sum_{n=1}^\infty a_n^p, \tag{1.6}$$

where the constant factors k_p *and* $(k_p)^p$ *are the best possible. Inequality* (1.3) *is equivalent to* (1.4).

In this paper, by introducing some parameters, we establish a reverse version of the inequality (1.3). As applications, some particular results are considered.

2 Some lemmas

Definition 2.1 Let $H_{p,q}(r,s)$ be the set of functions $k(x,y)$ satisfying the following conditions:

Let $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, suppose that $k(x,y)$ is continuous in $(0,\infty)\times(0,\infty)$ and satisfies:

(1) $k(x,y)=k(y,x)>0$, $x,y\in(0,\infty)$.

(2) For $\varepsilon\geq 0$ and $x>0$, the function $k(x,t)(\frac{x}{t})^{\frac{1+\varepsilon}{l}}$ $(l=r,s)$ is decreasing in $t\in(0,\infty)$.

For $\varepsilon\geq 0$ small enough, for $x>0$, $\overline{k}_l(\varepsilon,x)$ can be described as

$$\overline{k}_l(\varepsilon,x):=\int_0^\infty k(x,t)\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{l}}dt\quad(l=r,s),$$

where $\overline{k}_l(\varepsilon,x)$ is independent of x, and $\overline{k}_l(0,x):=\int_0^\infty k(x,t)(\frac{x}{t})^{\frac{1}{l}}\,dt=k_p$ $(l=r,s)$, k_p is a positive constant independent of x, and $\overline{k}_l(\varepsilon,x)=k_p(\varepsilon)=k_p+o(1)$ $(\varepsilon\to 0^+)$.

(3) There exists a positive constant λ' such that

$$\theta_\lambda(s,m)=\frac{1}{k_r}\int_0^1 k(m,t)\left(\frac{m}{t}\right)^{\frac{1}{s}}dt=O\bigl(1/m^{\lambda'}\bigr)\in(0,1)\quad(m\to\infty),$$
$$\theta_\lambda(r,n)=\frac{1}{k_r}\int_0^1 k(t,n)\left(\frac{n}{t}\right)^{\frac{1}{r}}dt=O\bigl(1/n^{\lambda'}\bigr)\in(0,1)\quad(n\to\infty).$$

Lemma 2.2 *If $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $k(x,y)\in H_{p,q}(r,s)$, and the weight coefficients $w(r,p,m)$ and $w(s,q,n)$ are defined as*

$$\omega(r,p,m)=\sum_{n=1}^\infty k(m,n)\frac{m^{\frac{p-1}{r}}}{n^{\frac{1}{s}}},\tag{2.1}$$

$$\omega(s,q,n)=\sum_{m=1}^\infty k(m,n)\frac{n^{\frac{q-1}{s}}}{m^{\frac{1}{r}}},\tag{2.2}$$

then we have

$$\begin{aligned}&m^{\frac{p}{r}-1}k_r\bigl(1-\theta_\lambda(s,m)\bigr)<\omega(r,p,m)<m^{\frac{p}{r}-1}k_r,\\&n^{\frac{q}{s}-1}k_r\bigl(1-\theta_\lambda(r,n)\bigr)<\omega(s,q,n)<n^{\frac{q}{s}-1}k_r.\end{aligned}\tag{2.3}$$

Proof By the assumption of the lemma, because $k(x,t)(\frac{x}{t})^{\frac{1}{s}}$ $(t\in(0,\infty))$ is decreasing, then we find

$$\begin{aligned}\omega(r,p,m)&=m^{\frac{p}{r}-1}\sum_{n=1}^\infty k(m,n)\left(\frac{m}{n}\right)^{\frac{1}{s}}\\&\leq m^{\frac{p}{r}-1}\int_0^\infty k(m,t)\left(\frac{m}{t}\right)^{\frac{1}{s}}dt\\&=m^{\frac{p}{r}-1}k_r.\end{aligned}$$

However, we find

$$\begin{aligned}\omega(r,p,m) &= m^{\frac{p}{r}-1}\sum_{n=1}^{\infty}k(m,n)\left(\frac{m}{n}\right)^{\frac{1}{s}} \geq m^{\frac{p}{r}-1}\int_{1}^{\infty}k(m,t)\left(\frac{m}{t}\right)^{\frac{1}{s}}\,dt \\ &= m^{\frac{p}{r}-1}\int_{0}^{\infty}k(m,t)\left(\frac{m}{t}\right)^{\frac{1}{s}}\,dt - m^{\frac{p}{r}-1}\int_{0}^{1}k(m,t)\left(\frac{m}{t}\right)^{\frac{1}{s}}\,dt \\ &= m^{\frac{p}{r}-1}k_r - m^{\frac{p}{r}-1}\int_{0}^{1}k(m,t)\left(\frac{m}{t}\right)^{\frac{1}{s}}\,dt \\ &= m^{\frac{p}{r}-1}k_r\left(1-\frac{1}{k_r}\int_{0}^{1}k(m,t)\left(\frac{m}{t}\right)^{\frac{1}{s}}\,dt\right) \\ &= m^{\frac{p}{r}-1}k_r\bigl(1-\theta_\lambda(s,m)\bigr).\end{aligned}$$

It is easy to show that the above inequalities take the form of a strict inequality. Hence, we have $m^{\frac{p}{r}-1}k_r(1-\theta_\lambda(s,m)) < \omega(r,p,m) < m^{\frac{p}{r}-1}k_r$. Similarly, we can obtain $n^{\frac{q}{s}-1}k_r(1-\theta_\lambda(r,n)) < \omega(s,q,n) < n^{\frac{q}{s}-1}k_r$. The lemma is proved. □

Lemma 2.3 *If* $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$, $r > 1$, $\frac{1}{r}+\frac{1}{s}=1$, *and* $k(x,y)\in H_{p,q}(r,s)$, *for* $\varepsilon > 0$ *small enough, we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}k(m,n)m^{-\frac{1}{r}-\frac{\varepsilon}{p}}n^{-\frac{1}{s}-\frac{\varepsilon}{q}} < \bigl(kr+o(1)\bigr)\sum_{1}^{\infty}m^{-1-\varepsilon}\quad\bigl(\varepsilon\to 0^{+}\bigr). \tag{2.4}$$

Proof For $\varepsilon > 0$, by Definition 2.1, we have

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}k(m,n)m^{-\frac{1}{r}-\frac{\varepsilon}{p}}n^{-\frac{1}{s}-\frac{\varepsilon}{q}} \\ &\quad= \sum_{m=1}^{\infty}m^{-1-\varepsilon}\varepsilon\sum_{n=1}^{\infty}k(m,n)\left(\frac{m}{n}\right)^{\frac{1+\varepsilon(s/q)}{s}} \\ &\quad\leq \sum_{m=1}^{\infty}m^{-1-\varepsilon}\int_{0}^{\infty}k(m,t)\left(\frac{m}{t}\right)^{\frac{1+\varepsilon(s/q)}{s}}\,dt \\ &\quad= \sum_{m=1}^{\infty}m^{-1-\varepsilon}kr\left(\frac{\varepsilon s}{q}\right) = \bigl(kr+o(1)\bigr)\sum_{1}^{\infty}m^{-1-\varepsilon}.\end{aligned}$$

The lemma is proved. □

3 Main results

Theorem 3.1 *If* $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$, $r > 1$, $\frac{1}{r}+\frac{1}{s}=1$, *and* $k(x,y)\in H_{p,q}(r,s)$, $a_n, b_n \geq 0$, *such that* $0 < \sum_{n=1}^{\infty}n^{\frac{p}{r}-1}a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty}n^{\frac{q}{s}-1}b_n^q < \infty$, *then we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k(m,n)a_m b_n > k_r\left\{\sum_{n=1}^{\infty}\bigl[1-\theta_\lambda(s,n)\bigr]n^{\frac{p}{r}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{s}-1}b_n^q\right\}^{\frac{1}{q}}, \tag{3.1}$$

$$\sum_{n=1}^{\infty}n^{\frac{p}{r}-1}\left[\sum_{m=1}^{\infty}k(m,n)a_m\right]^p > (k_r)^p\sum_{n=1}^{\infty}\bigl[1-\theta_\lambda(s,n)\bigr]n^{\frac{p}{r}-1}a_n^p, \tag{3.2}$$

where the constant factor k_r *and* $(k_r)^p$ *are the best possible. Inequality* (3.1) *is equivalent to* (3.2).

Proof By Hölder's inequality, we have (see [6])

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k(m,n)a_m b_n &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\left[k(m,n)\right]^{\frac{1}{q}}\frac{m^{\frac{1}{qr}}}{n^{\frac{1}{ps}}}a_m\right\}\left\{\left[k(m,n)\right]^{\frac{1}{q}}\frac{n^{\frac{1}{ps}}}{m^{\frac{1}{qr}}}b_n\right\}\\ &\geq \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}k(m,n)\frac{m^{\frac{p-1}{r}}}{n^{\frac{1}{s}}}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k(m,n)\frac{n^{\frac{q-1}{s}}}{m^{\frac{1}{r}}}b_n^q\right\}^{\frac{1}{q}}\\ &= \left\{\sum_{m=1}^{\infty}\omega(r,p,m)a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega(s,q,n)b_n^q\right\}^{\frac{1}{q}}.\end{aligned}\tag{3.3}$$

Then, by (2.3), in view of $0 < p < 1$ and $q < 0$, we have (3.1).

For $\varepsilon > 0$, setting $\overline{a}_n = n^{-\frac{1}{r}-\frac{\varepsilon}{q}}$ and $\overline{b}_n = n^{-\frac{1}{s}-\frac{\varepsilon}{q}}$, we find

$$\begin{aligned}&\left\{\sum_{n=1}^{\infty}\left[1-\theta_\lambda(s,n)\right]n^{\frac{p}{r}-1}\overline{a}_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{s}-1}\overline{b}_n^q\right\}^{\frac{1}{q}}\\ &\quad= \left\{\sum_{n=1}^{\infty}n^{-1-\varepsilon}-\sum_{n=1}^{\infty}O\left(1/n^{\lambda'}\right)n^{-1-\varepsilon}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{-1-\varepsilon}\right\}^{\frac{1}{q}}\\ &\quad= \sum_{n=1}^{\infty}n^{-1-\varepsilon}\left[1-\left(\sum_{n=1}^{\infty}n^{-1-\varepsilon}\right)^{-1}\sum_{n=1}^{\infty}O\left(1/n^{\lambda'}\right)n^{-1-\varepsilon}\right]^{\frac{1}{p}}.\end{aligned}\tag{3.4}$$

By virtue of (2.4), we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k(m,n)\overline{a}_m\overline{b}_n\\ &\quad= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}k(m,n)m^{-\frac{1}{r}-\frac{\varepsilon}{p}}n^{-\frac{1}{s}-\frac{\varepsilon}{q}} < \left(kr+o(1)\right)\sum_{1}^{\infty}m^{-1-\varepsilon}\quad\left(\varepsilon\to 0^+\right).\end{aligned}\tag{3.5}$$

If the constant factor k_r in (3.1) is not the best possible factor, then there exists a positive number K (with $K > k_r$), such that (3.1) is still valid if the constant factor k_r is replaced by K. In particular, by (3.4) and (3.5), we have

$$\left(kr+o(1)\right)\sum_{1}^{\infty}n^{-1-\varepsilon} > K\sum_{n=1}^{\infty}n^{-1-\varepsilon}\left[1-\left(\sum_{n=1}^{\infty}n^{-1-\varepsilon}\right)^{-1}\sum_{n=1}^{\infty}O\left(1/n^{\lambda'}\right)n^{-1-\varepsilon}\right]^{\frac{1}{p}},$$

that is,

$$\left(kr+o(1)\right) > K\left[1-\left(\sum_{n=1}^{\infty}n^{-1-\varepsilon}\right)^{-1}\sum_{n=1}^{\infty}O\left(1/n^{\lambda'}\right)n^{-1-\varepsilon}\right]^{\frac{1}{p}}.$$

For $\varepsilon \to 0^+$, it follows that $K \leq k_r$, which contradicts the fact that $K > k_r$. Hence, the constant factor k_r in (3.1) is the best possible.

Setting b_n as

$$b_n := n^{\frac{p}{r}-1}\left[\sum_{m=1}^{\infty} k(m,n)a_m\right]^{p-1},$$

by (3.1), we have

$$\begin{aligned}\left\{\sum_{n=1}^{\infty} n^{\frac{q}{s}-1}b_n^q\right\}^p &= \left\{\sum_{n=1}^{\infty} n^{\frac{p}{r}-1}\left[\sum_{m=1}^{\infty} k(m,n)a_m\right]^p\right\}^p \\ &= \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(m,n)a_m b_n\right\}^p \\ &\geq (k_r)^p\left\{\sum_{n=1}^{\infty}\left[1-\theta_\lambda(s,n)\right]n^{\frac{p}{r}-1}a_n^p\right\}\left\{\sum_{n=1}^{\infty} n^{\frac{q}{s}-1}b_n^q\right\}^{p-1}. \end{aligned} \tag{3.6}$$

Hence, we obtain

$$\infty > \sum_{n=1}^{\infty} n^{\frac{p}{r}-1}\left[\sum_{m=1}^{\infty} k(m,n)a_m\right]^p \geq (k_r)^p\sum_{n=1}^{\infty}\left[1-\theta_\lambda(s,n)\right]n^{\frac{p}{r}-1}a_n^p > 0. \tag{3.7}$$

By (3.1), both (3.6) and (3.7) take the form of a strict inequality, and we have (3.2).

However, if (3.2) is valid, by Hölder's inequality, we find

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(m,n)a_m b_n \\ &\quad= \sum_{n=1}^{\infty}\left[n^{\frac{1}{q}-\frac{1}{s}}\sum_{m=1}^{\infty} k(m,n)a_m\right]\left[n^{\frac{1}{s}-\frac{1}{q}}b_n\right] \\ &\quad\geq \left\{\sum_{n=1}^{\infty} n^{\frac{p}{r}-1}\left[\sum_{m=1}^{\infty} k(m,n)a_m\right]^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{\frac{q}{s}-1}b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{3.8}$$

Then, by (3.2), we have (3.1). Hence (3.2) and (3.1) are equivalent.

If the constant factor $(k_r)^p$ in (3.2) is not the best possible, by using (3.8), we find the contradiction that the constant factor k_r in (3.1) is not the best possible. The theorem is completed. □

4 Some particular results

(1) Setting

$$k(x,y) = \frac{(xy)^{\frac{\lambda-1}{2}}}{(x+y)^\lambda}\left(1-2\min\left\{\frac{1}{r},\frac{1}{s}\right\} < \lambda \leq 1+2\min\left\{\frac{1}{r},\frac{1}{s}\right\}\right),$$

for $0 \leq \varepsilon < \min\{p(\frac{\lambda+1}{2}-\frac{1}{r}), q(\frac{\lambda+1}{2}-\frac{1}{s})\}$, and for fixed $x > 0$, we find (see [4])

$$\overline{k}_s(\varepsilon,x) \to B\left(\frac{s(\lambda+1)-2}{2s},\frac{r(\lambda+1)-2}{2r}\right) = k_r \quad \left(\varepsilon \to 0^+\right),$$

and $\overline{k}_s(\varepsilon,x) \to k_r$ ($\varepsilon \to 0^+$);

$$0 < \int_0^1 k(m,t)\left(\frac{m}{t}\right)^{\frac{1}{s}} dt = \int_0^1 \frac{(mt)^{\frac{\lambda-1}{2}}}{(m+t)^{\lambda}}\left(\frac{m}{t}\right)^{\frac{1}{s}} dt$$
$$\le \int_0^1 \frac{(mt)^{\frac{\lambda-1}{2}}}{m^{\lambda}}\left(\frac{m}{t}\right)^{\frac{1}{s}} dt = \frac{1}{(\frac{\lambda-1}{2}+\frac{1}{r})}\frac{1}{m^{\frac{1+\lambda}{2}-\frac{1}{s}}}.$$

Hence, $\theta_\lambda(s,m) = O(m^{\frac{1}{s}-\frac{1+\lambda}{2}})$. Similarly, we obtain $\theta_\lambda(r,n) = O(n^{\frac{1}{r}-\frac{1+\lambda}{2}})$. For $\varepsilon \ge 0$, $1 - 2\min\{\frac{1}{r},\frac{1}{s}\} < \lambda \le 1 + 2\min\{\frac{1}{r},\frac{1}{s}\}$, and fixed $x > 0$, the function

$$k(x,t)\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{l}} = \frac{(xt)^{\frac{\lambda-1}{2}}}{(x+t)^{\lambda}}\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{l}} = \frac{x^{\frac{1+\varepsilon}{l}+\frac{\lambda-1}{2}}}{(x+t)^{\lambda}}t^{\frac{\lambda-1}{2}-\frac{1+\varepsilon}{l}} \quad (l = r,s)$$

is decreasing in $(0,\infty)$. Hence, $k(x,y) \in H_p(r,s)$. By Theorem 3.1, we have the following.

Corollary 4.1 *If* $0 < p < 1$, $1/p + 1/q = 1$, $1/r + 1/s = 1$, $1 - 2\min\{\frac{1}{r},\frac{1}{s}\} < \lambda \le 1 + 2\min\{\frac{1}{r},\frac{1}{s}\}$, *and both* $a_n, b_n \ge 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{\frac{p}{r}-1}a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{\frac{q}{s}-1}b_n^q < \infty$, *then we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}}{(m+n)^{\lambda}}a_m b_n$$
$$> B\left(\frac{s(\lambda+1)-2}{2s},\frac{r(\lambda+1)-2}{2r}\right)\left\{\sum_{n=1}^{\infty}\left[1-\theta_\lambda(s,n)\right]n^{\frac{p}{r}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}b_n^q\right\}^{\frac{1}{q}}, \tag{4.1}$$

$$\sum_{n=1}^{\infty}n^{\frac{p}{r}-1}\left[\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}}{(m+n)^{\lambda}}a_m\right]^p$$
$$> \left[B\left(\frac{s(\lambda+1)-2}{2s},\frac{r(\lambda+1)-2}{2r}\right)\right]^p\sum_{n=1}^{\infty}\left[1-\theta_\lambda(s,n)\right]n^{\frac{p}{r}-1}a_n^p, \tag{4.2}$$

where the constant factors

$$B\left(\frac{s(\lambda+1)-2}{2s},\frac{r(\lambda+1)-2}{2r}\right) \quad \textit{and} \quad \left[B\left(\frac{s(\lambda+1)-2}{2s},\frac{r(\lambda+1)-2}{2r}\right)\right]^p$$

are the best possible. Inequality (4.1) *is equivalent to* (4.2).

In particular, (a) for $r = q$, $s = p$, and $1 - 2\min\{\frac{1}{p},\frac{1}{q}\} < \lambda \le 1 + 2\min\{\frac{1}{p},\frac{1}{q}\}$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}}{(m+n)^{\lambda}}a_m b_n$$
$$> B\left(\frac{p(\lambda+1)-2}{2p},\frac{q(\lambda+1)-2}{2q}\right)\left\{\sum_{n=1}^{\infty}\left[1-\theta_\lambda(p,n)\right]n^{p-2}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q-2}b_n^q\right\}^{\frac{1}{q}}, \tag{4.3}$$

$$\sum_{n=1}^{\infty}n^{p-2}\left[\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}}{(m+n)^{\lambda}}a_m\right]^p$$
$$> \left[B\left(\frac{p(\lambda+1)-2}{2p},\frac{q(\lambda+1)-2}{2q}\right)\right]^p\sum_{n=1}^{\infty}\left[1-\theta_\lambda(p,n)\right]n^{p-2}a_n^p. \tag{4.4}$$

(b) For $r = s = 2$ and $0 < \lambda \leq 2$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}}{(m+n)^{\lambda}}a_m b_n > B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\sum_{n=1}^{\infty}\left[1-\theta_{\lambda}(2,n)\right]n^{\frac{p}{2}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{2}-1}b_n^q\right\}^{\frac{1}{q}}, \quad (4.5)$$

$$\sum_{n=1}^{\infty}n^{\frac{p}{2}-1}\left[\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}}{(m+n)^{\lambda}}a_m\right]^p > \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^p\sum_{n=1}^{\infty}\left[1-\theta_{\lambda}(2,n)\right]n^{\frac{p}{2}-1}a_n^p. \quad (4.6)$$

(2) Let

$$k(x,y) = \frac{(xy)^{\frac{\lambda-1}{2}}}{x^{\lambda}+y^{\lambda}}\left(1-2\min\left\{\frac{1}{r},\frac{1}{s}\right\} < \lambda \leq 1+2\min\left\{\frac{1}{r},\frac{1}{s}\right\}\right).$$

For $0 \leq \varepsilon < \min\{p(\frac{\lambda+1}{2}-\frac{1}{r}), q(\frac{\lambda+1}{2}-\frac{1}{s})\}$ and $x > 0$, we find (see [4])

$$\overline{k}_s(\varepsilon,x) \to \frac{1}{\lambda}B\left(\frac{s(\lambda+1)-2}{2s\lambda},\frac{r(\lambda+1)-2}{2r\lambda}\right) = k_r \quad (\varepsilon \to 0^+),$$

and $\overline{k}_s(\varepsilon,x) \to k_r$ $(\varepsilon \to 0^+)$;

$$\begin{aligned}\int_0^1 k(m,t)\left(\frac{m}{t}\right)^{\frac{1}{s}}dt &= \int_0^1\frac{(mt)^{\frac{\lambda-1}{2}}}{m^{\lambda}+t^{\lambda}}\left(\frac{m}{t}\right)^{\frac{1}{s}}dt\\ &\leq \int_0^1\frac{(mt)^{\frac{\lambda-1}{2}}}{m^{\lambda}}\left(\frac{m}{t}\right)^{\frac{1}{s}}dt = \frac{1}{(\frac{\lambda-1}{2}+\frac{1}{r})}m^{\frac{1}{s}-\frac{1+\lambda}{2}}.\end{aligned}$$

Hence, $\theta_{\lambda}(s,m) = O(m^{\frac{1}{s}-\frac{1+\lambda}{2}})$. Similarly, we can obtain $\theta_{\lambda}(r,n) = O(n^{\frac{1}{r}-\frac{1+\lambda}{2}})$. For $\varepsilon \geq 0$, $1-2\min\{\frac{1}{r},\frac{1}{s}\} < \lambda \leq 1+2\min\{\frac{1}{r},\frac{1}{s}\}$, and $x > 0$, the function

$$k(x,t)\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{l}} = \frac{(xt)^{\frac{\lambda-1}{2}}}{x^{\lambda}+t^{\lambda}}\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{l}} = \frac{x^{\frac{1+\varepsilon}{l}+\frac{\lambda-1}{2}}}{x^{\lambda}+t^{\lambda}}t^{\frac{\lambda-1}{2}-\frac{1+\varepsilon}{l}} \quad (l = r,s)$$

is decreasing in $(0,\infty)$. Hence $k(x,y) \in H_p(r,s)$. By Theorem 3.1, we have the following corollary.

Corollary 4.2 *If* $0 < p < 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $r > 1$, $\frac{1}{r}+\frac{1}{s} = 1$, $1-2\min\{\frac{1}{r},\frac{1}{s}\} < \lambda \leq 1+2\min\{\frac{1}{r},\frac{1}{s}\}$, *and both* $a_n, b_n \geq 0$ *such that* $0 < \sum_{n=1}^{\infty}n^{\frac{p}{r}-1}a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty}n^{\frac{q}{s}-1}b_n^q < \infty$, *then we have*

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}}{m^{\lambda}+n^{\lambda}}a_m b_n\\ &\quad > \frac{1}{\lambda}B\left(\frac{s(\lambda+1)-2}{2s\lambda},\frac{r(\lambda+1)-2}{2r\lambda}\right)\left\{\sum_{n=1}^{\infty}\left[1-\theta_{\lambda}(s,n)\right]n^{\frac{p}{r}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}b_n^q\right\}^{\frac{1}{q}}, \quad (4.7)\end{aligned}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{\frac{p}{r}-1}\left[\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}}{m^{\lambda}+n^{\lambda}}a_m\right]^p\\ &\quad > \left[\frac{1}{\lambda}B\left(\frac{s(\lambda+1)-2}{2s\lambda},\frac{r(\lambda+1)-2}{2r\lambda}\right)\right]^p\sum_{n=1}^{\infty}\left[1-\theta_{\lambda}(s,n)\right]n^{\frac{p}{r}-1}a_n^p, \quad (4.8)\end{aligned}$$

where the constant factors

$$\frac{1}{\lambda}B\left(\frac{s(\lambda+1)-2}{2s\lambda},\frac{r(\lambda+1)-2}{2r\lambda}\right)\quad \textit{and}\quad \left[\frac{1}{\lambda}B\left(\frac{s(\lambda+1)-2}{2s\lambda},\frac{r(\lambda+1)-2}{2r\lambda}\right)\right]^p$$

are the best possible. Inequality (4.7) *is equivalent to* (4.8).

In particular, (a) for $r = q$, $s = p$, and $1-2\min\{\frac{1}{p},\frac{1}{q}\} < \lambda \le 1+2\min\{\frac{1}{p},\frac{1}{q}\}$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}}{m^\lambda+n^\lambda}a_m b_n\\ &\quad > \frac{1}{\lambda}B\left(\frac{p(\lambda+1)-2}{2p\lambda},\frac{q(\lambda+1)-2}{2q\lambda}\right)\\ &\qquad\times\left\{\sum_{n=1}^{\infty}\left[1-\theta_\lambda(p,n)\right]n^{p-2}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q-2}b_n^q\right\}^{\frac{1}{q}},\end{aligned}\tag{4.9}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{p-2}\left[\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}}{(m+n)^\lambda}a_m\right]^p\\ &\quad > \left[\frac{1}{\lambda}B\left(\frac{p(\lambda+1)-2}{2p\lambda},\frac{q(\lambda+1)-2}{2q\lambda}\right)\right]^p\sum_{n=1}^{\infty}\left[1-\theta_\lambda(p,n)\right]n^{p-2}a_n^p.\end{aligned}\tag{4.10}$$

(b) For $r = s = 2$ and $0 < \lambda \le 2$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}}{m^\lambda+n^\lambda}a_m b_n > \frac{\pi}{\lambda}\left\{\sum_{n=1}^{\infty}\left[1-\theta_\lambda(2,n)\right]n^{\frac{p}{2}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{2}-1}b_n^q\right\}^{\frac{1}{q}},\tag{4.11}$$

$$\sum_{n=1}^{\infty}n^{\frac{p}{2}-1}\left[\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}}{m^\lambda+n^\lambda}a_m\right]^p > \left[\frac{\pi}{\lambda}\right]^p\sum_{n=1}^{\infty}\left[1-\theta_\lambda(2,n)\right]n^{\frac{p}{2}-1}a_n^p.\tag{4.12}$$

(3) Let

$$k(x,y)=\frac{(xy)^{\frac{\lambda-1}{2}}}{(\max\{x,y\})^\lambda}\left(1-2\min\left\{\frac{1}{r},\frac{1}{s}\right\}<\lambda\le 1+2\min\left\{\frac{1}{r},\frac{1}{s}\right\}\right),$$

for $0 \le \varepsilon < \min\{p(\frac{\lambda+1}{2}-\frac{1}{r}), q(\frac{\lambda+1}{2}-\frac{1}{s})\}$ and $x > 0$, then we find (see [4])

$$\overline{k}_s(\varepsilon,x)\to\frac{4rs\lambda}{[r(\lambda+1)-2][s(\lambda+1)-2]}=k_r\quad\left(\varepsilon\to 0^+\right),$$

and $\overline{k}_r(\varepsilon,x)\to k_r$ ($\varepsilon\to 0^+$)

$$\begin{aligned}0&<\int_0^1 k(m,t)\left(\frac{m}{t}\right)^{\frac{1}{s}}dt=\int_0^1\frac{(mt)^{\frac{\lambda-1}{2}}}{(\max\{m,t\})^\lambda}\left(\frac{m}{t}\right)^{\frac{1}{s}}dt\\ &=\int_0^1\frac{(mt)^{\frac{\lambda-1}{2}}}{m^\lambda}\left(\frac{m}{t}\right)^{\frac{1}{s}}dt=\frac{1}{(\frac{\lambda-1}{2}+\frac{1}{r})}\frac{1}{m^{\frac{1+\lambda}{2}-\frac{1}{s}}}.\end{aligned}$$

Hence, $\theta_\lambda(s,m) = O(\frac{1}{m^{\frac{1+\lambda}{2}-\frac{1}{s}}})$. Similarly, we can obtain $\theta_\lambda(r,n) = O(\frac{1}{n^{\frac{1+\lambda}{2}-\frac{1}{r}}})$. For $\varepsilon \geq 0$, $1 - 2\min\{\frac{1}{r}, \frac{1}{s}\} < \lambda \leq 1 + 2\min\{\frac{1}{r}, \frac{1}{s}\}$, and $x > 0$, the function

$$k(x,t)\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{l}} = \frac{(mt)^{\frac{\lambda-1}{2}}}{(\max\{m,t\})^\lambda}\left(\frac{x}{t}\right)^{\frac{1+\varepsilon}{l}} = \frac{x^{\frac{1+\varepsilon}{l}+\frac{\lambda-1}{2}}}{(\max\{m,t\})^\lambda} t^{\frac{\lambda-1}{2}-\frac{1+\varepsilon}{l}} \quad (l = r,s)$$

is decreasing in $(0,\infty)$. Hence, $k(x,y) \in H_{p,q}(r,s)$. By Theorem 3.1, we have the following corollary.

Corollary 4.3 *If* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $1 - 2\min\{\frac{1}{r}, \frac{1}{s}\} < \lambda \leq 1 + 2\min\{\frac{1}{r}, \frac{1}{s}\}$, *and both* $a_n, b_n \geq 0$, *such that* $0 < \sum_{n=1}^\infty n^{\frac{p}{r}-1}a_n^p < \infty$ *and* $0 < \sum_{n=1}^\infty n^{\frac{q}{s}-1}b_n^q < \infty$, *then we have*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(mn)^{\frac{\lambda-1}{2}}}{(\max\{m,n\})^\lambda} a_m b_n > \frac{4rs\lambda}{[r(\lambda+1)-2][s(\lambda+1)-2]} \left\{\sum_{n=1}^\infty \bigl[1-\theta_\lambda(s,n)\bigr] n^{\frac{p}{r}-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty n^{\frac{q}{s}-1} b_n^q\right\}^{\frac{1}{q}}, \tag{4.13}$$

$$\sum_{n=1}^\infty n^{\frac{p}{r}-1}\left[\frac{(mn)^{\frac{\lambda-1}{2}}}{(\max\{m,n\})^\lambda} a_m\right]^p > \left(\frac{4rs\lambda}{[r(\lambda+1)-2][s(\lambda+1)-2]}\right)^p \sum_{n=1}^\infty \bigl[1-\theta_\lambda(s,n)\bigr] n^{\frac{p}{r}-1} a_n^p. \tag{4.14}$$

Here the constant factors $\frac{4rs\lambda}{[r(\lambda+1)-2][s(\lambda+1)-2]}$ *and* $(\frac{4rs\lambda}{[r(\lambda+1)-2][s(\lambda+1)-2]})^p$ *are the best possible. Inequality* (4.13) *is equivalent to* (4.14).

In particular, (a) for $r = q$, $s = p$, and $1 - 2\min\{\frac{1}{p}, \frac{1}{q}\} < \lambda \leq 1 + 2\min\{\frac{1}{p}, \frac{1}{q}\}$, we have

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(mn)^{\frac{\lambda-1}{2}}}{(\max\{m,n\})^\lambda} a_m b_n > \frac{4pq\lambda}{[p(\lambda+1)-2][q(\lambda+1)-2]} \left\{\sum_{n=1}^\infty \bigl[1-\theta_\lambda(p,n)\bigr] n^{p-2} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty n^{q-2} b_n^q\right\}^{\frac{1}{q}}, \tag{4.15}$$

$$\sum_{n=1}^\infty n^{p-2}\left[\sum_{m=1}^\infty \frac{(mn)^{\frac{\lambda-1}{2}}}{(\max\{m,n\})^\lambda} a_m\right]^p > \left(\frac{4pq\lambda}{[p(\lambda+1)-2][q(\lambda+1)-2]}\right)^p \sum_{n=1}^\infty \bigl[1-\theta_\lambda(p,n)\bigr] n^{p-2} a_n^p. \tag{4.16}$$

(b) For $r = s = 2$ and $0 < \lambda \leq 2$, we have

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(mn)^{\frac{\lambda-1}{2}}}{(\max\{m,n\})^\lambda} a_m b_n > \frac{4}{\lambda} \left\{\sum_{n=1}^\infty \bigl[1-\theta_\lambda(2,n)\bigr] n^{\frac{p}{2}-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty n^{\frac{q}{2}-1} b_n^q\right\}^{\frac{1}{q}}, \tag{4.17}$$

$$\sum_{n=1}^\infty n^{\frac{p}{2}-1}\left[\sum_{m=1}^\infty \frac{(mn)^{\frac{\lambda-1}{2}}}{(\max\{m,n\})^\lambda} a_m\right]^p > \left(\frac{4}{\lambda}\right)^p \sum_{n=1}^\infty \bigl[1-\theta_\lambda(2,n)\bigr] n^{\frac{p}{2}-1} a_n^p. \tag{4.18}$$

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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