Existence of positive solutions for p-Laplacian impulsive boundary value problems on time scales

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Abstract

In this paper, by using fixed point index theory and a double fixed point theorem, we study the existence of many positive solutions for a class of second-order p-Laplacian boundary value problems with impulse on time scales. An example which supports our theoretical results is also indicated.

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Keywords: multiple positive solutions; double fixed point theorem; fixed point index; time scales; impulsive boundary value problems

1 Introduction

It is well known that the theory of impulsive differential equations has become more important in recent years in some mathematical models of real processes and phenomena. For the introduction of the basic theory of impulsive equations, see [1–4] and the references therein.

The theory of dynamic equations on time scales has been developing rapidly and has received much attention in recent years. The study unifies existing result in differential and finite difference equations and provides powerful new tools for exploring connections between the traditionally separated fields. We refer to the books by Bohner and Peterson [5, 6].

However, there is not so much work on the impulsive boundary value problem with p-Laplacian on time scales except that in [7–11]. The authors studied impulsive dynamic equations on time scales without p-Laplacian in [12–15] (see also references therein), while in [16–18] the authors considered p-Laplacian impulsive differential equations without time scales. The authors considered impulsive boundary value problems without p-Laplacian and time scales in [19, 20].

Motivated by the above results, in this study we consider the following second-order impulsive boundary value problem (BVP) on time scales:

$$-\left[\phi_p\left(u^\Delta(t)\right)\right]^\nabla = f\left(t, u(t)\right), \quad t \in J := [0,1]_{\mathbb{T}}, t \neq t_k, k = 1, 2, \ldots, m, \tag{1.1}$$

$$-\Delta u|_{t=t_k} = I_k\left(u(t_k)\right), \tag{1.2}$$

$$u^\Delta(0) = 0, \qquad \alpha u(1) + \beta u^\Delta(1) = \int_0^1 g(s)u(s)\Delta s, \tag{1.3}$$

where $\mathbb{T}$ is a time scale, $0, 1 \in \mathbb{T}$, $J = [0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T}$, $t_k \in (0,1)_{\mathbb{T}}$, $k = 1,2,\ldots,m$ with $0 < t_1 < t_2 < \cdots < t_m < 1$, $\alpha > \int_0^1 g(s)\Delta s$, $\beta > 0$, $\phi_p(s)$ is a p-Laplacian operator, *i.e.*, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, *i.e.*,

$$\Delta u|_{t=t_k} = u\big(t_k^+\big) - u\big(t_k^-\big),$$

where $u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim_{h \to 0^+} u(t_k - h)$ represent the right-hand limit and left-hand limit, respectively, of $u(t)$ at $t = t_k$. In addition, f, I_k, and g satisfy

(H1) $f \in \mathcal{C}([0,1]_{\mathbb{T}} \times [0,+\infty), [0,+\infty))$;

(H2) $I_k \in \mathcal{C}([0,+\infty), [0,+\infty))$;

(H3) $g \in \mathcal{C}_{rd}([0,1]_{\mathbb{T}}, [0,+\infty))$ is nonnegative.

We remark that by a solution u of (1.1)-(1.3), we mean $u : \mathbb{T} \to \mathbb{R}$ is delta differentiable, $u^\Delta : \mathbb{T}^k \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}^k \cap \mathbb{T}_k$ and $u^{\Delta\nabla} : \mathbb{T}^k \cap \mathbb{T}_k \to \mathbb{R}$ is continuous, and it satisfies the impulsive and boundary value conditions (1.2)-(1.3). If $u^{\Delta\nabla} \le 0$ on $[0,1]_{\mathbb{T}^k \cap \mathbb{T}_k}$, then we say that u is concave on $[0,1]_{\mathbb{T}}$.

By using the fixed point index theory in the cone [21], we get the existence of at least two positive solutions for the impulsive BVP (1.1)-(1.3). Then we generalized this to obtain many positive solutions. By using a double fixed point theorem [22], we also establish the existence of countably many positive solutions for the impulsive BVP (1.1)-(1.3). To the best of our knowledge, no paper has considered the second-order impulsive BVP with integral boundary conditions and one dimensional p-Laplacian on time scales. In fact, our results are also new when $\mathbb{T} = \mathbb{R}$ (the differential case) and $\mathbb{T} = \mathbb{Z}$ (the discrete case). Therefore, the results can be considered as a contribution to this field.

This paper is organized as follows. In Section 2, we provide some necessary background about time scales, the theory of cones in Banach space and some preliminary lemmas. We give and prove our main results in Section 3. Finally, in Section 4, we give an example to demonstrate our results.

2 Preliminaries

In this section, we present some definitions and theorems, which will be needed in the proof of the main results.

Definition 2.1 Let $(\mathbb{B}, \|\cdot\|)$ be a real Banach space. A nonempty closed set $K \subset \mathbb{B}$ is said to be a cone provided that

(i) $ax + by \in K$ for all $x, y \in K$ and $a, b \ge 0$;

(ii) $y, -y \in K$ implies $y = 0$.

Every cone $K \subset \mathbb{B}$ induces an ordering in $\mathbb{B}$ given by $x \le y$ if and only if $y - x \in K$.

Definition 2.2 The map α is said to be a nonnegative continuous concave functional on a cone K of a real Banach space $\mathbb{B}$, provided that $\alpha : K \to [0,\infty)$ is continuous and

$$\alpha\big(tu + (1-t)v\big) \ge t\alpha(u) + (1-t)\alpha(v),$$

for all $u, v \in K$, $0 \le t \le 1$.

Similarly, we say that the map γ is a nonnegative continuous convex functional on a cone K of a real Banach space $\mathbb{B}$ provided that

$$\gamma\big(tu + (1-t)v\big) \leq t\gamma(u) + (1-t)\gamma(v),$$

for all $u, v \in K$, $0 \leq t \leq 1$.

For a nonnegative continuous functional γ on a cone K in a real Banach space $\mathbb{B}$, and for each $d > 0$, we set

$$K(\gamma, d) = \big\{x \in K \mid \gamma(x) < d\big\}.$$

The following fixed point theorems are fundamental and important to the proofs of our main results.

Theorem 2.1 (Double fixed point theorem [22]) *Let K be a cone in a real Banach space $\mathbb{B}$. Let α and γ be increasing, nonnegative, continuous functionals on K, and let θ be a non-negative, continuous functional on K with $\theta(0) = 0$ such that, for some $c > 0$ and $M > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \quad and \quad \|x\| \leq M\gamma(x)$$

for all $x \in \overline{K(\gamma, c)}$. Suppose that there exist positive numbers a and b with $a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x), \quad for\ 0 \leq \lambda \leq 1\ and\ x \in \partial K(\theta, b)$$

and

$$T : \overline{K(\gamma, c)} \to K$$

is a completely continuous operator such that:
  (i) *$\gamma(Tx) > c$, for all $x \in \partial K(\gamma, c)$;*
 (ii) *$\theta(Tx) < b$, for all $x \in \partial K(\theta, b)$;*
(iii) *$K(\alpha, a) \neq \emptyset$, and $\alpha(Tx) > a$, for all $x \in \partial K(\alpha, a)$.*
Then T has at least two fixed points, x_1 and x_2 belonging to $\overline{K(\gamma, c)}$ such that

$$a < \alpha(x_1), \quad with\ \theta(x_1) < b,$$

and

$$b < \theta(x_2), \quad with\ \gamma(x_2) < c.$$

Theorem 2.2 [21] *Let K be a cone in a real Banach space $\mathbb{B}$. Let D be an open bounded subset of $\mathbb{B}$ with $D_K = D \cap K \neq \emptyset$ and $\overline{D_K} \neq K$. Assume that $T : \overline{D_K} \to K$ is completely continuous such that $x \neq Tx$ for $x \in \partial D_K$. Then the following results hold:*
  (i) *If $\|Tx\| \leq x$, $x \in \partial D_K$, then $i_K(T, D_K) = 1$.*
 (ii) *If there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i_K(T, D_K) = 0$.*
(iii) *Let U be open in P such that $\overline{U} \subset D_K$. If $i_K(T, D_K) = 1$ and $i_K(T, U_K) = 0$, then T has a fixed point in $D_K \setminus \overline{U_K}$. The same result holds if $i_K(T, D_K) = 0$ and $i_K(T, U_K) = 1$.*

3 Main results

In this section, by defining an appropriate Banach space and cone, we impose growth conditions on f and I_k which allow us to apply the theorems in Section 2 to establish the existence results of the positive solutions for the BVP (1.1)-(1.3).

Let $J' = [0,1]_{\mathbb{T}} \setminus \{t_1, t_2, \ldots, t_m\}$. We define

$$\mathbb{B} = \big\{u \mid u : [0,1]_{\mathbb{T}} \to \mathbb{R} \text{ is continuous at } t \neq t_k, \text{ left continuous at } t = t_k,$$
$$\text{and there exist } u(t_k^-) \text{ and } u(t_k^+) \text{ with } u(t_k^-) = u(t_k) \text{ for } k = 1, 2, \ldots, m\big\}.$$

Then $\mathbb{B}$ is a real Banach space with the norm $\|u\| = \sup_{t \in [0,1]_{\mathbb{T}}} |u(t)|$. By a solution of (1.1)-(1.3), we mean a function $u \in \mathbb{B} \cap \mathcal{C}^2(J')$ which satisfies (1.1)-(1.3). We define a cone $K \subset \mathbb{B}$ as

$$K = \bigg\{u \in \mathbb{B} : u \text{ is a nonnegative and nonincreasing function, } u^\Delta \text{ is}$$
$$\text{nonincreasing on } J', \alpha u(1) + \beta u^\Delta(1) = \int_0^1 g(s)u(s)\Delta s\bigg\}.$$

We need the following lemmas that will be used to prove our main results.

Lemma 3.1 *Suppose that* (H1)-(H3) *are satisfied. Then* $u \in \mathbb{B} \cap \mathcal{C}^2(J')$ *is a solution of the impulsive boundary value problem* (1.1)-(1.3) *if and only if* $u(t)$ *is a solution of the following integral equation*:

$$u(t) = \frac{1}{\alpha - \int_0^1 g(s)\Delta s} \int_0^1 g(s)\bigg(\int_s^1 \phi_q\bigg(\int_0^\tau f(r, u(r))\nabla r\bigg)\Delta\tau\bigg)\Delta s$$
$$+ \frac{1}{\alpha - \int_0^1 g(s)\Delta s}\bigg[\beta\phi_q\bigg(\int_0^1 f(s, u(s))\nabla s\bigg)$$
$$+ \int_0^1 g(s)\bigg(\sum_{s \leq t_k} I_k(u(t_k))\bigg)\Delta s\bigg] + \int_t^1 \phi_q\bigg(\int_0^s f(r, u(r))\nabla r\bigg)\Delta s$$
$$+ \sum_{t \leq t_k} I_k(u(t_k)), \tag{3.1}$$

and $u(t) \geq 0$.

Proof Integrating (1.1) from 0 to t, one has

$$-\phi_p(u^\Delta(t)) + \phi_p(u^\Delta(0)) = \int_0^t f(s, u(s))\nabla s.$$

By the boundary condition (1.3), we have

$$u^\Delta(t) = -\phi_q\bigg(\int_0^t f(s, u(s))\nabla s\bigg).$$

Integrating the dynamic equation above from t to 1, we get

$$u(t) = u(1) + \int_t^1 \phi_q \left(\int_0^s f(r, u(r)) \nabla r \right) \Delta s + \sum_{t \leq t_k} I_k(u(t_k)). \tag{3.2}$$

Applying the second boundary condition, one has

$$u(1) = \frac{1}{\alpha - \int_0^1 g(s) \Delta s} \int_0^1 g(s) \left(\int_s^1 \phi_q \left(\int_0^\tau f(r, u(r)) \nabla r \right) \Delta \tau \right) \Delta s$$

$$+ \frac{1}{\alpha - \int_0^1 g(s) \Delta s} \left[\beta \phi_q \left(\int_0^1 f(s, u(s)) \nabla s \right) + \int_0^1 g(s) \sum_{t \leq t_k} I_k(u(t_k)) \Delta s \right]. \tag{3.3}$$

Therefore, by (3.2) and (3.3), we have

$$u(t) = \frac{1}{\alpha - \int_0^1 g(s) \Delta s} \int_0^1 g(s) \left(\int_s^1 \phi_q \left(\int_0^\tau f(r, u(r)) \nabla r \right) \Delta \tau \right) \Delta s$$

$$+ \frac{1}{\alpha - \int_0^1 g(s) \Delta s} \left[\beta \phi_q \left(\int_0^1 f(s, u(s)) \nabla s \right) \right.$$

$$+ \int_0^1 g(s) \left(\sum_{s \leq t_k} I_k(u(t_k)) \right) \Delta s \right]$$

$$+ \int_t^1 \phi_q \left(\int_0^s f(r, u(r)) \nabla r \right) \Delta s + \sum_{t \leq t_k} I_k(u(t_k)).$$

Then the proof of the sufficient part is complete.

Conversely, let u be as in (3.1). Taking the delta derivative of $u(t)$ gives

$$u^\Delta(t) = -\phi_q \left(\int_0^t f(s, u(s)) \nabla s \right), \quad i.e., \quad \phi_p(u^\Delta(t)) = -\int_0^t f(s, u(s)) \nabla s. \tag{3.4}$$

So $u^\Delta(0) = 0$. Also, it is easy to see that $u(t)$ satisfies (1.2) and (1.3). Furthermore, from (H1)-(H3), and (3.1), it is clear that $u(t) \geq 0$. So, the proof of lemma is completed. □

Lemma 3.2 *If $u \in K$, then $\min_{t \in [0,1]_{\mathbb{T}}} u(t) \geq \gamma \|u\|$, where $\gamma = \frac{\int_0^1 g(s)(1-s) \Delta s}{\alpha - \int_0^1 g(s) s \Delta s}$.*

Proof For $u \in K$, nonnegative and nonincreasing, we have

$$\|u\| = u(0), \qquad \min_{t \in [0,1]_{\mathbb{T}}} u(t) = u(1).$$

On the other hand, $u^\Delta(t)$ is nonincreasing on J'. So, for every $t \in [0,1]_{\mathbb{T}}$, we have

$$\frac{u(t) - u(1)}{1 - t} \geq \frac{u(0) - u(1)}{1},$$

i.e., $u(t) \geq (1 - t)u(0) + tu(1)$.

Therefore,

$$\int_0^1 g(s) u(s) \Delta s \geq \int_0^1 g(s)(1 - s) u(0) \Delta s + \int_0^1 g(s) s u(1) \Delta s.$$

This, together with $\alpha u(1) + \beta u^\Delta(1) = \int_0^1 g(s)u(s)\Delta s$, implies that

$$u(1) \geq \frac{\int_0^1 g(s)(1-s)\Delta s}{\alpha - \int_0^1 g(s)s\Delta s} u(0).$$

So, the proof of the lemma is completed. □

Define $T : K \to \mathbb{B}$ by

$$
\begin{aligned}
(Tu)(t) = {} & \frac{1}{\alpha - \int_0^1 g(s)\Delta s} \int_0^1 g(s) \left(\int_s^1 \phi_q \left(\int_0^\tau f(r, u(r)) \nabla r \right) \Delta \tau \right) \Delta s \\
& + \frac{1}{\alpha - \int_0^1 g(s)\Delta s} \left[\beta \phi_q \left(\int_0^1 f(s, u(s)) \nabla s \right) \right. \\
& \left. + \int_0^1 g(s) \left(\sum_{s \leq t_k} I_k(u(t_k)) \right) \Delta s \right] \\
& + \int_t^1 \phi_q \left(\int_0^s f(r, u(r)) \nabla r \right) \Delta s + \sum_{t \leq t_k} I_k(u(t_k)).
\end{aligned}
\tag{3.5}
$$

From (3.5) and Lemma 3.1, it is easy to obtain the following result.

Lemma 3.3 *Assume that* (H1)-(H3) *hold. Then* $T : K \to K$ *is completely continuous.*

Now we consider the existence of at least two positive solutions for the impulsive boundary value problem (1.1)-(1.3) by the fixed point theorem in [22]. Let us define the increasing, nonnegative, continuous functionals γ, β, and α on K by

$$\gamma(u) = \min_{t \in [0,\xi]_\mathbb{T}} u(t) = u(\xi),$$

$$\beta(u) = \max_{t \in [\xi,\eta]_\mathbb{T}} u(t) = u(\xi),$$

$$\alpha(u) = \max_{t \in [0,\eta]_\mathbb{T}} u(t) = u(0).$$

It is obvious that for each $u \in K$,

$$\gamma(u) \leq \beta(u) \leq \alpha(u).$$

In addition, by Lemma 3.2, for each $u \in K$,

$$\|u\| \leq \frac{1}{\gamma} \min_{t \in [0,1]_\mathbb{T}} u(t) \leq \frac{1}{\gamma} \min_{t \in [0,\xi]_\mathbb{T}} u(t) = \frac{1}{\gamma} \gamma(u).$$

Thus,

$$\|u\| \leq \frac{1}{\gamma} \gamma(u), \quad \forall u \in K.$$

For the convenience, we denote

$$A = (1 - \xi)\phi_q(\xi),$$

$$B = \frac{(m+1)\alpha + \beta}{\alpha - \int_0^1 g(s)\Delta s} - \xi.$$

Theorem 3.1 *Suppose the assumptions of* (H1)-(H3) *are satisfied. Let there exist positive numbers $a < b < c$ such that*

$$0 < a < \frac{A}{B}b < \frac{A\gamma}{B}c,$$

and assume that f satisfies the following conditions:

(H4) $f(t, u) > \phi_p(\frac{c}{A})$, *for all* $(t, u) \in [0, \xi]_{\mathbb{T}} \times [c, \frac{1}{\gamma}c]$,

(H5) $f(t, u) < \phi_p(\frac{b}{B})$, $\sup\{I_j(u)\} < \frac{b}{B}$, $j = 1, 2, \dots, m$, *for all* $(t, u) \in [0, 1]_{\mathbb{T}} \times [0, \frac{1}{\gamma}b]$,

(H6) $f(t, u) > \phi_p(\frac{a}{A})$, *for all* $(t, u) \in [0, 1]_{\mathbb{T}} \times [0, a]$.

Then the boundary value problem (1.1)-(1.3) *has at least two positive solutions u_1 and u_2 satisfying*

$$a < \alpha(u_1) \quad \text{with } \beta(u_1) < b,$$

and

$$b < \beta(u_2) \quad \text{with } \gamma(u_2) < c.$$

Proof We define the completely continuous operator T by (3.5). Thus, it is easy to check that $T : \overline{K(\gamma, c)} \to K$.

We now show that all the conditions of Theorem 2.1 are satisfied. In order to show that condition (i) of Theorem 2.1, we choose $u \in \partial K(\gamma, c)$. Then $\gamma(u) = \min_{t \in [0, \xi]_{\mathbb{T}}} u(t) = u(\xi) = c$, this implies that $c \le u(t)$ for $t \in [0, \xi]_{\mathbb{T}}$. We recall that $\|u\| \le \frac{1}{\gamma}\gamma(u) = \frac{1}{\gamma}c$. So, we have

$$c \le u(t) \le \frac{1}{\gamma}c, \quad t \in [0, \xi]_{\mathbb{T}}.$$

Then assumption (H4) implies

$$f(t, u) > \phi_p\left(\frac{c}{a}\right), \quad \text{for all } (t, u) \in [0, \xi]_{\mathbb{T}} \times \left[c, \frac{1}{\gamma}c\right].$$

Therefore,

$$
\begin{aligned}
\gamma(Tu) &= \min_{t \in [0, \xi]_{\mathbb{T}}} (Tu)(t) = (Tu)(\xi) \\
&= \frac{1}{\alpha - \int_0^1 g(s)\Delta s}\left[\int_0^1 g(s)\left(\int_s^1 \phi_q\left(\int_0^\tau f(r, u(r))\nabla r\right)\Delta\tau\right)\Delta s \right. \\
&\quad \left. + \beta\phi_q\left(\int_0^1 f(s, u(s))\nabla s\right)\right] + \int_\xi^1 \phi_q\left(\int_0^s f(r, u(r))\nabla r\right)\Delta s \\
&\quad + \frac{1}{\alpha - \int_0^1 g(s)\Delta s}\int_0^1 g(s)\left(\sum_{s \le t_k} I_k(u(t_k))\right)\Delta s + \sum_{t \le t_k} I_k(u(t_k))
\end{aligned}
$$

$$\geq \int_\xi^1 \phi_q\left(\int_0^s f(r,u(r))\nabla r\right)\Delta s \geq \int_\xi^1 \phi_q\left(\int_0^\xi f(r,u(r))\nabla r\right)\Delta s$$

$$= (1-\xi)\phi_q\left(\int_0^\xi f(r,u(r))\nabla r\right)$$

$$> (1-\xi)\phi_q(\xi)\frac{c}{A} = c.$$

Hence, condition (i) is satisfied.

Secondly, we show that (ii) of Theorem 2.1 is satisfied. For this, we take $u \in \partial K(\beta, b)$. Then $\beta(u) = \max_{t\in[\xi,\eta]_{\mathbb{T}}} u(t) = u(\xi) = b$, this means $0 \leq u(t) \leq b$, for all $t \in [\xi,1]_{\mathbb{T}}$. Noticing that $\|u\| \leq \frac{1}{\gamma}\gamma(u) = \frac{1}{\gamma}\beta(u) = \frac{1}{\gamma}b$, we get

$$0 \leq u(t) \leq \frac{1}{\gamma}b, \quad \text{for } t \in [0,1]_{\mathbb{T}}.$$

Then assumption (H5) implies

$$f(t,u) < \phi_p\left(\frac{b}{B}\right), \qquad \sup\{I_j(u)\} < \frac{b}{B}.$$

Therefore

$$\beta(Tu) = \max_{t\in[\xi,\eta]_{\mathbb{T}}}(Tu)(t) = (Tu)(\xi)$$

$$= \frac{1}{\alpha - \int_0^1 g(s)\Delta s}\left[\int_0^1 g(s)\left(\int_s^1 \phi_q\left(\int_0^\tau f(r,u(r))\nabla r\right)\Delta\tau\right)\Delta s\right.$$

$$\left. + \beta\phi_q\left(\int_0^1 f(s,u(s))\nabla s\right)\right] + \int_\xi^1 \phi_q\left(\int_0^s f(r,u(r))\nabla r\right)\Delta s$$

$$+ \frac{1}{\alpha - \int_0^1 g(s)\Delta s}\int_0^1 g(s)\left(\sum_{s\leq t_k} I_k(u(t_k))\right)\Delta s + \sum_{t\leq t_k} I_k(u(t_k))$$

$$\leq \frac{1}{\alpha - \int_0^1 g(s)\Delta s}\left[\int_0^1 g(s)\left(\int_0^1 \phi_q\left(\int_0^1 f(r,u(r))\nabla r\right)\Delta\tau\right)\Delta s\right.$$

$$\left. + \beta\phi_q\left(\int_0^1 f(s,u(s))\nabla s\right)\right] + \int_\xi^1 \phi_q\left(\int_0^1 f(r,u(r))\nabla r\right)\Delta s$$

$$+ \frac{1}{\alpha - \int_0^1 g(s)\Delta s}\int_0^1 g(s)\left(\sum_{k=1}^m I_k(u(t_k))\right)\Delta s + \sum_{k=1}^m I_k(u(t_k))$$

$$= \left[\frac{\alpha+\beta}{\alpha - \int_0^1 g(s)\Delta s} - \xi\right]\phi_q\left(\int_0^1 f(r,u(r))\nabla r\right) + \frac{\alpha}{\alpha - \int_0^1 g(s)\Delta s}\sum_{k=1}^m I_k(u(t_k))$$

$$< \left[\frac{\alpha+\beta}{\alpha - \int_0^1 g(s)\Delta s} - \xi\right]\frac{b}{B} + \frac{\alpha}{\alpha - \int_0^1 g(s)\Delta s}m\frac{b}{B}$$

$$= \frac{b}{B}\left[\frac{(m+1)\alpha+\beta}{\alpha - \int_0^1 g(s)\Delta s} - \xi\right]$$

$$= b.$$

So, we get $\beta(Tu) < b$. Hence, condition (ii) is satisfied.

Finally, we show that condition (iii) of Theorem 2.1 is satisfied. We note that $u(t) = \frac{2}{3}a$, $t \in [0,1]_{\mathbb{T}}$ is a member of $K(\alpha, a)$, and so $K(\alpha, a) \neq \emptyset$.

Now, let $u \in \partial K(\alpha, a)$. Then $\alpha(u) = \max_{t \in [0,\eta]_{\mathbb{T}}} u(t) = u(0) = a$. This implies $0 \leq u(t) \leq a$, $t \in [0,1]_{\mathbb{T}}$. Then we get

$$0 \leq u(t) \leq a, \quad t \in [0,1]_{\mathbb{T}}.$$

By assumption (H7),

$$f(t, u) > \phi_p\left(\frac{a}{A}\right).$$

Then

$$
\begin{aligned}
\alpha(Tu) &= \max_{t \in [0,\eta]_{\mathbb{T}}} (Tu)(t) = (Tu)(0) \\
&\geq \int_0^1 \phi_q\left(\int_0^s f(r, u(r))\nabla r\right)\Delta s \geq \int_\xi^1 \phi_q\left(\int_0^s f(r, u(r))\nabla r\right)\Delta s \\
&\geq (1-\xi)\phi_q\left(\int_0^\xi f(r, u(r))\nabla r\right) \\
&> (1-\xi)\frac{a}{A}\phi_q\left(\int_0^\xi \nabla r\right) \\
&= (1-\xi)\frac{a}{A}\phi_q(\xi) = a.
\end{aligned}
$$

So, we get $\alpha(Tu) > a$. Thus, (iii) of Theorem 2.1 is satisfied. Hence, the impulsive boundary value problem (1.1)-(1.3) has at least two positive solutions u_1 and u_2 satisfying

$$a < \alpha(u_1) \quad \text{with } \beta(u_1) < b,$$

and

$$b < \beta(u_2) \quad \text{with } \gamma(u_2) < c.$$

The proof is complete. □

Next we consider the existence of at least two positive solutions for the BVP (1.1)-(1.3) by the fixed point theorem in [21].

We define

$$K_\rho = \left\{u \in K : \|u\| < \rho\right\},$$

$$\Omega_\rho = \left\{u \in K : \min_{t \in [0,1]_{\mathbb{T}}} u(t) < \gamma\rho\right\} = \left\{u \in K : \gamma\|u\| \leq \min_{t \in [0,1]_{\mathbb{T}}} u(t) < \gamma\rho\right\}.$$

The following results are proved in [21] (Lemma 2.5, p.693).

Lemma 3.4 Ω_ρ *has the following properties*:

(a) Ω_ρ *is open relative to* K.

(b) $K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$.

(c) $u \in \Omega_\rho$ *if and only if* $\min_{t \in [0,1]_\mathbb{T}} u(t) = \gamma\rho$.

(d) *If* $u \in \Omega_\rho$, *then* $\gamma\rho \leq u(t) \leq \rho$ *for* $t \in [0,1]_\mathbb{T}$.

Now for convenience we introduce the following notations. Let

$$f_{\gamma\rho}^\rho = \min\left\{ \min_{t \in [0,1]_\mathbb{T}} \frac{f(t,u)}{\phi_p(\rho)} : u \in [\gamma\rho,\rho] \right\},$$

$$f_0^\rho = \max\left\{ \max_{t \in [0,1]_\mathbb{T}} \frac{f(t,u)}{\phi_p(\rho)} : u \in [0,\rho] \right\},$$

$$I_0^\rho(k) = \max\{ I_k(u) : u \in [0,\rho] \},$$

$$\frac{1}{l} = \frac{(m+1)\alpha + \beta}{\alpha - \int_0^1 g(s)\Delta s}, \qquad \frac{1}{L} = \frac{\beta}{\alpha - \int_0^1 g(s)\Delta s}.$$

Theorem 3.2 *Suppose* (H1) *and* (H3) *hold*.

(H7) *There exist* $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ *with* $\rho_1 < \gamma\rho_2$ *and* $\rho_2 < \rho_3$ *such that*

$$f_0^{\rho_1} < \phi_p(l), \qquad I_0^{\rho_1}(k) < l\rho_1, \qquad f_{\gamma\rho_2}^{\rho_2} > \phi_p(L), \qquad f_0^{\rho_3} < \phi_p(l), \qquad I_0^{\rho_3}(k) < l\rho_3.$$

Then problem (1.1)-(1.3) *has at least two positive solutions* u_1, u_2 *with*
$u_1 \in \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$, $u_2 \in K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$.

(H8) *There exist* $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ *with* $\rho_1 < \rho_2 < \gamma\rho_3 < \rho_3$ *such that*

$$f_{\gamma\rho_1}^{\rho_1} > \phi_p(L), \qquad f_0^{\rho_2} < \phi_p(l), \qquad I_0^{\rho_2}(k) < l\rho_2, \qquad f_{\gamma\rho_3}^{\rho_3} > \phi_p(L).$$

Then problem (1.1)-(1.3) *has at least two positive solutions* u_1, u_2 *with*
$u_1 \in K_{\rho_2} \setminus \overline{\Omega}_{\rho_1}$, $u_2 \in \Omega_{\rho_3} \setminus \overline{K}_{\rho_2}$.

Proof We only consider condition (H7). If (H8) holds, then the proof is similar to that of the case when (H7) holds. By Lemma 3.3, we know that the operator $T : K \to K$ is completely continuous.

First, we show that $i_K(T, K_{\rho_1}) = 1$. In fact, by (3.5), $f_0^{\rho_1} < \phi_p(l)$, and $I_0^{\rho_1}(k) < l\rho_1$, we have, for $u \in \partial K_{\rho_1}$,

$$\begin{aligned}
(Tu)(t) &= \frac{1}{\alpha - \int_0^1 g(s)\Delta s} \int_0^1 g(s)\left(\int_s^1 \phi_q\left(\int_0^\tau f(r,u(r))\nabla r \right)\Delta\tau \right)\Delta s \\
&\quad + \frac{1}{\alpha - \int_0^1 g(s)\Delta s}\left[\beta\phi_q\left(\int_0^1 f(s,u(s))\nabla s \right) \right. \\
&\quad \left. + \int_0^1 g(s)\left(\sum_{s \leq t_k} I_k(u(t_k)) \right)\Delta s \right] + \int_t^1 \phi_q\left(\int_0^s f(r,u(r))\nabla r \right)\Delta s \\
&\quad + \sum_{t \leq t_k} I_k(u(t_k))
\end{aligned}$$

$$\leq \frac{1}{\alpha - \int_0^1 g(s)\Delta s} \int_0^1 g(s) \left(\int_0^1 \phi_q \left(\int_0^\tau f(r, u(r)) \nabla r \right) \Delta \tau \right) \Delta s$$

$$+ \frac{1}{\alpha - \int_0^1 g(s)\Delta s} \left[\beta \phi_q \left(\int_0^1 f(s, u(s)) \nabla s \right) \right.$$

$$+ \int_0^1 g(s) \left(\sum_{k=1}^m I_k(u(t_k)) \right) \Delta s \right] + \int_0^1 \phi_q \left(\int_0^s f(r, u(r)) \nabla r \right) \Delta s$$

$$+ \sum_{k=1}^m I_k(u(t_k))$$

$$= \frac{\alpha}{\alpha - \int_0^1 g(s)\Delta s} \left[\int_0^1 \phi_q \left(\int_0^s f(r, u(r)) \nabla r \right) \Delta s + \sum_{k=1}^m I_k(u(t_k)) \right]$$

$$+ \frac{\beta}{\alpha - \int_0^1 g(s)\Delta s} \phi_q \left(\int_0^1 f(s, u(s)) \nabla s \right)$$

$$< \frac{\beta + \alpha}{\alpha - \int_0^1 g(s)\Delta s} \phi_q \left(\int_0^1 f(s, u(s)) \nabla s \right) + \frac{\alpha \sum_{k=1}^m I_k(u(t_k))}{\alpha - \int_0^1 g(s)\Delta s}$$

$$< \frac{\beta + \alpha}{\alpha - \int_0^1 g(s)\Delta s} l\rho_1 + \frac{\alpha}{\alpha - \int_0^1 g(s)\Delta s} l\rho_1 m$$

$$= l\rho_1 \left[\frac{\beta + (m+1)\alpha}{\alpha - \int_0^1 g(s)\Delta s} \right] = \rho_1,$$

i.e., $\|Tu\| < \|u\|$ for $u \in \partial K_{\rho_1}$. By (i) of Theorem 2.2, we obtain $i_K(T, K_{\rho_1}) = 1$.

Secondly, we show that $i_K(T, \Omega_{\rho_2}) = 0$. Let $e(t) \equiv 1$. Then $e \in \partial K_1$. We claim that

$$u \neq Tu + Be, \quad u \in \partial\Omega_{\rho_2}, B > 0.$$

Suppose that there exist $u_0 \in \partial\Omega_{\rho_2}$ and $B_0 > 0$ such that

$$u_0 = Tu_0 + B_0 e. \tag{3.6}$$

Then Lemma 3.1, Lemma 3.2, and (3.6) imply that for $t \in [0, 1]$

$$u_0 = Tu_0 + B_0 e \geq \gamma \|Tu_0\| + B_0$$

$$\geq \gamma \frac{\beta}{\alpha - \int_0^1 g(s)\Delta s} \phi_q \left(\int_0^1 f(s, u(s)) \nabla s \right) + B_0$$

$$> \gamma \frac{\beta}{\alpha - \int_0^1 g(s)\Delta s} \phi_q \left(\int_0^1 \phi_p(L\rho_2) \nabla s \right) + B_0$$

$$= \gamma \frac{\beta}{\alpha - \int_0^1 g(s)\Delta s} L\rho_2 + B_0 = \gamma\rho_2 + B_0,$$

i.e. $\gamma\rho_2 > \gamma\rho_2 + B_0$, which is a contradiction. Hence by (ii) of Theorem 2.2, it follows that $i_K(T, \Omega_{\rho_2}) = 0$.

Finally, similar to the proof of $i_K(T, K_{\rho_1}) = 1$, we can prove that $i_K(T, K_{\rho_3}) = 1$. Since $\rho_1 < \gamma\rho_2$ and Lemma 3.4(b), we have $\overline{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. Similarly to $\rho_2 < \rho_3$ and Lemma 3.4(b),

we have $\overline{\Omega}_{\rho_2} \subset K_{\rho_2} \subset K_{\rho_3}$. Therefore (iii) of Theorem 2.2 implies that BVP (1.1)-(1.3) has at least two positive solutions u_1, u_2 with $u_1 \in \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$, $u_2 \in K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$. □

Theorem 3.2 can be generalized to obtain many solutions.

Theorem 3.3 *Suppose* (H1)-(H3) *hold. Then we have the following assertions.*

(H9) *There exists* $\{\rho_i\}_{i=1}^{2m_0+1} \subset (0,\infty)$ *with* $\rho_1 < \gamma\rho_2 < \rho_2 < \rho_3 < \gamma\rho_4 < \cdots < \gamma\rho_{2m_0} < \rho_{2m_0} < \rho_{2m_0+1}$ *such that*

$$f_0^{\rho_{2m-1}} < \phi_p(l), \qquad I_0^{\rho_{2m-1}} < l\rho_{2m-1} \quad (m = 1, 2, \ldots, m_0, m_0 + 1),$$
$$f_{\gamma\rho_{2m}}^{\rho_{2m}} > \phi_p(L) \quad (m = 1, 2, \ldots, m_0).$$

Then problem (1.1)-(1.3) *has at least* $2m_0$ *solutions in* K.

(H10) *There exists* $\{\rho_i\}_{i=1}^{2m_0} \subset (0,\infty)$ *with* $\rho_1 < \gamma\rho_2 < \rho_2 < \rho_3 < \gamma\rho_4 < \cdots < \gamma\rho_{2m_0} < \rho_{2m_0}$ *such that*

$$f_0^{\rho_{2m-1}} < \phi_p(l), \qquad I_0^{\rho_{2m-1}} < l\rho_{2m-1}, \qquad f_{\gamma\rho_{2m}}^{\rho_{2m}} > \phi_p(L) \quad (m = 1, 2, \ldots, m_0).$$

Then problem (1.1)-(1.3) *has at least* $2m_0 - 1$ *solutions in* K.

Theorem 3.4 *Suppose* (H1)-(H3) *hold. Then we have the following assertions.*

(H11) *There exists* $\{\rho_i\}_{i=1}^{2m_0+1} \subset (0,\infty)$ *with* $\rho_1 < \rho_2 < \gamma\rho_3 < \rho_3 < \cdots < \rho_{2m_0} < \gamma\rho_{2m_0+1} < \rho_{2m_0+1}$ *such that*

$$f_{\gamma\rho_{2m-1}}^{\rho_{2m-1}} > \phi_p(L) \quad (m = 1, 2, \ldots, m_0, m_0 + 1), \qquad f_0^{\rho_{2m}} < \phi_p(l),$$
$$I_0^{\rho_{2m}} < l\rho_{2m} \quad (m = 1, 2, \ldots, m_0).$$

Then problem (1.1)-(1.3) *has at least* $2m_0$ *solutions in* K.

(H12) *There exists* $\{\rho_i\}_{i=1}^{2m_0} \subset (0,\infty)$ *with* $\rho_1 < \rho_2 < \gamma\rho_3 < \rho_3 < \cdots < \gamma\rho_{2m_0-1} < \rho_{2m_0-1} < \rho_{2m_0}$ *such that*

$$f_{\gamma\rho_{2m-1}}^{\rho_{2m-1}} > \phi_p(L), \qquad f_0^{\rho_{2m}} < \phi_p(l), \qquad I_0^{\rho_{2m}} < l\rho_{2m} \quad (m = 1, 2, \ldots, m_0).$$

Then problem (1.1)-(1.3) *has at least* $2m_0 - 1$ *solutions in* K.

4 Example

Example 4.1 In BVP (1.1)-(1.3), suppose that $m = 1, p = 3, t_1 = \frac{1}{3}, g(s) = s, \alpha = \frac{3}{2}, \beta = \frac{1}{2}$, and $\mathbb{T} = \{\frac{n+1}{4} : n \in \mathbb{N}\} \cup [2,5]$, *i.e.*,

$$-\left[\phi_3\left(u^\Delta(t)\right)\right]^\nabla = f\left(t, u(t)\right), \quad t \in [0,1]_\mathbb{T} \setminus \left\{\frac{1}{3}\right\}, \tag{4.1}$$

$$-\left(u\left(\frac{1}{3}^+\right) - u\left(\frac{1}{3}^-\right)\right) = I_1\left(u\left(\frac{1}{3}\right)\right), \tag{4.2}$$

$$u^\Delta(0) = 0, \qquad \frac{3}{2}u(1) + \frac{1}{2}u^\Delta(1) = \int_0^1 su(s)\Delta s, \tag{4.3}$$

where

$$f(t,u) = \begin{cases} \frac{1}{25}(-u+2), & u \in [0,2]; \\ 2{,}838(u-2), & u \in (2,5]; \\ 8{,}514, & u \in (5,\infty); \end{cases}$$

$$I_1(u) = \frac{u}{8}.$$

Taking $\rho_1 = 2$, $\rho_2 = 41$, $\rho_3 = 200$, we have $l = \frac{15}{28}$, $L = \frac{9}{4}$, $\gamma = \frac{5}{41}$. We obtain

$$2 = \rho_1 < \gamma\rho_2 = 5 \quad \text{and} \quad 41 = \rho_2 < \rho_3 = 200.$$

Now, we show that (H7) is satisfied:

$$f_0^2 = \frac{1}{50} < \phi_3\left(\frac{15}{28}\right) = \frac{225}{784}, \qquad I_0^2 = \frac{1}{4} < l\rho_1 = \frac{15}{14},$$

$$f_5^{41} = \frac{8{,}514}{1{,}681} > \phi_3\left(\frac{9}{4}\right) = \frac{81}{16},$$

$$f_0^{200} = \frac{4{,}257}{20{,}000} < \phi_3\left(\frac{15}{28}\right) = \frac{225}{784} \quad \text{and} \quad I_0^{200} = 25 < l\rho_3 = \frac{750}{7}.$$

Then all conditions of Theorem 3.2 hold. Hence, we find that the BVP (4.1)-(4.3) has at least two positive solutions.

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