Zhang and Shi *Journal of Inequalities and Applications* 2014, 2014:191
http://www.journalofinequalitiesandapplications.com/content/2014/1/191

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Two double inequalities for *k*-gamma and *k*-Riemann zeta functions

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Abstract

By using methods in the theory of majorization, a double inequality for the gamma function is extended to the *k*-gamma function and the *k*-Riemann zeta function.

MSC: 33B15; 26D07; 26B25

Keywords: majorization; Schur convexity; *k*-gamma function; *k*-Riemann zeta function; Apéry's constant; log-convexity

1 Introduction

The Euler gamma function $\Gamma(x)$ is defined [1] for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt. \tag{1}$$

In 2005, by using a geometrical method, Alsina and Tomás [2] proved the following double inequality:

$$\frac{1}{n!} \le \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \le 1, \quad x \in [0,1], n \in \mathbb{N}. \tag{2}$$

In 2009, Nguyen and Ngo [3] obtained the following generalization of the double inequality (2):

$$\frac{\prod_{i=1}^n \Gamma(1+\alpha_i)}{\Gamma(\beta + \sum_{i=1}^n \alpha_i)} \le \frac{\prod_{i=1}^n \Gamma(1+\alpha_i x)}{\Gamma(\beta + (\sum_{i=1}^n \alpha_i)x)} \le \frac{1}{\Gamma(\beta)}, \tag{3}$$

where $x \in [0,1]$, $\beta \ge 1$, $\alpha_i > 0$, $n \in \mathbb{N}$.

For $k > 0$, the function Γ_k is defined [4] by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad x \in \mathbb{C}\backslash k\mathbb{Z}^-, \tag{4}$$

where $(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k)$.

The above definition is a generalization of the gamma function. For $x \in \mathbb{C}$ with $\mathrm{Re}(x) > 0$, the function $\Gamma_k(x)$ is given by the integral [4]

$$\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{k}} t^{x-1}\, dt. \tag{5}$$

It satisfies the following properties [4–6]:

(i) $\Gamma_k(k) = 1$;

(ii) $\Gamma_1(x) = \Gamma(x)$.

For $k > 0$, the k-Riemann zeta function is defined [5] by the integral

$$\zeta_k(x) = \frac{1}{\Gamma_k(x)} \int_0^\infty \frac{t^{x-k}}{e^t - 1}\, dt, \quad x > k. \tag{6}$$

Note that when k tends to 1 we obtain the known Riemann zeta function $\zeta(x)$.

In this note, by using methods on the theory of majorization, we extended the double inequality (3) to the function $\Gamma_k(x)$ and the k-Riemann zeta function, namely, we established the following theorems.

Theorem 1

$$\frac{\prod_{i=1}^n \Gamma_k(1+\alpha_i)}{\Gamma_k(\beta + \sum_{i=1}^n \alpha_i)} \le \frac{\prod_{i=1}^n \Gamma_k(1+\alpha_i x)}{\Gamma_k(\beta + (\sum_{i=1}^n \alpha_i)x)} \le \frac{1}{\Gamma_k(\beta)}, \tag{7}$$

where $x \in [0,1]$, $\beta \ge 1$, $\alpha_i > 0$, $i = 1, \dots, n$, $n \in \mathbb{N}$.

Theorem 2

$$\begin{aligned} &\frac{\prod_{i=1}^n \zeta_k(k+1+\alpha_i)\Gamma_k(k+1+\alpha_i)}{\zeta_k(\beta + k + \sum_{i=1}^n \alpha_i)\Gamma_k(\beta + k + \sum_{i=1}^n \alpha_i)} \\ &\quad \le \frac{\prod_{i=1}^n \zeta_k(k+1+\alpha_i x)\Gamma_k(k+1+\alpha_i x)}{\zeta_k(\beta + k + (\sum_{i=1}^n \alpha_i)x)\Gamma_k(\beta + k + (\sum_{i=1}^n \alpha_i)x)} \\ &\quad \le \frac{(\pi^2/6)^n}{\zeta_k(\beta+k)\Gamma_k(\beta+k)}, \end{aligned} \tag{8}$$

where $x \in [0,1]$, $\beta \ge 1$, $\alpha_i > 0$, $i = 1, \dots, n$, $n \in \mathbb{N}$.

Substituting $k = 1$ and $\alpha_i = 1$ ($i = 1, \dots, n$) into (8) and taking into account that $\Gamma(3) = 2$ and $\zeta(2) = \pi^2/6$, we obtain the following.

Corollary 1

$$\frac{(2\zeta(3))^n}{\zeta(1+\beta+n)\Gamma(1+\beta+n)} \le \frac{(\zeta(2+x)\Gamma(2+x))^n}{\zeta(1+\beta+nx)\Gamma(1+\beta+nx)} \le \frac{(\zeta(2))^n}{\zeta(1+\beta)\Gamma(1+\beta)}, \tag{9}$$

where $x \in [0,1]$, $\beta \ge 1$, $n \in \mathbb{N}$.

Remark 1 $\zeta(3)$ is Apéry's constant [7].

2 Definitions and lemmas

We need the following definitions and auxiliary lemmas.

Definition 1 [8, 9] Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$.

(i) We say $\mathbf{y}$ majorizes $\mathbf{x}$ ($\mathbf{x}$ is said to be majorized by $\mathbf{y}$), denoted by $\mathbf{x} \prec \mathbf{y}$, if $\sum_{i=1}^k x_{[i]} \le \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge \cdots \ge y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

(ii) Let $\Omega \subseteq \mathbb{R}^n$, a function $\varphi : \Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. A function φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

Definition 2 [8, 9] Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $\alpha \in [0,1]$.

(i) A set $\Omega \subseteq \mathbb{R}^n$ is said to be a convex set if $\mathbf{x}, \mathbf{y} \in \Omega$ implies $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} = (\alpha x_1 + (1-\alpha)y_1, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

(ii) Let $\Omega \subseteq \mathbb{R}^n$ be a convex set. A function φ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi\big(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\big) \leq \alpha\varphi(\mathbf{x}) + (1-\alpha)\varphi(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \Omega$. A function φ is said to be a concave function on Ω if and only if $-\varphi$ is a convex function on Ω.

(iii) Let $\Omega \subseteq \mathbb{R}^n$. A function $\varphi : \Omega \to \mathbb{R}$ is said to be a log-convex function on Ω if the function $\log\varphi$ is convex.

Lemma 1 [8, p.186] *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $x_1 \geq x_2 \geq \cdots \geq x_n$, and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. If for some k, $1 \leq k < n$, $x_i \leq y_i$, $i = 1, \ldots, k$, $x_i \geq y_i$ for $i = k+1, \ldots, n$, then $\mathbf{x} \prec \mathbf{y}$.*

Lemma 2 *Let f, g be a continuous nonnegative functions defined on an interval $[a,b] \subset \mathbb{R}$. Then*

$$I(x) = \int_a^b g(t)\big(f(t)\big)^x \, dt$$

is log-convex on $[0,+\infty)$.

Proof Let $\alpha, \beta \geq 0$, $0 < s < 1$ by the Hölder integral inequality [10, p.140], we have

$$\begin{aligned} I\big(s\alpha + (1-s)\beta\big) &= \int_a^b g(t)\big(f(t)\big)^{s\alpha+(1-s)\beta} \, dt = \int_a^b \big(g(t)\big(f(t)\big)^{\alpha}\big)^s \big(g(t)\big(f(t)\big)^{\beta}\big)^{1-s} \, dt \\ &\leq \left(\int_a^b g(t)\big(f(t)\big)^{\alpha} \, dt\right)^s \left(\int_a^b g(t)\big(f(t)\big)^{\beta} \, dt\right)^{1-s} = \big(I(\alpha)\big)^s \big(I(\beta)\big)^{1-s}, \end{aligned}$$

i.e.

$$\log I\big(s\alpha + (1-s)\beta\big) \leq s \log I(\alpha) + (1-s) \log I(\beta),$$

this means that $I(x)$ is log-convex on $[0,+\infty)$. □

Remark 2 When $b = +\infty$, the results of Lemma 2 presented previously hold true.

Lemma 3 [8, p.105] *Let g be a continuous nonnegative function defined on an interval $I \subseteq \mathbb{R}$. Then*

$$\varphi(\mathbf{x}) = \prod_{i=1}^n g(x_i), \quad \mathbf{x} \in I^n,$$

is Schur-convex on I^n if and only if $\log g$ is convex on I.

Lemma 4 *Let*

$$\mathbf{u} = (u_1, \ldots, u_n, u_{n+1}) = \left(\beta + \left(\sum_{i=1}^{n} \alpha_i\right)x - 1, \alpha_1, \ldots, \alpha_n\right) \tag{10}$$

and

$$\mathbf{v} = (v_1, \ldots, v_n, v_{n+1}) = \left(\beta + \sum_{i=1}^{n} \alpha_i - 1, \alpha_1 x, \ldots, \alpha_n x\right), \tag{11}$$

where $x \in [0,1]$, $\beta \geq 1$, $\alpha_i > 0$, $i = 1, \ldots, n$, $n \in \mathbb{N}$. *Then* $\mathbf{u} \prec \mathbf{v}$.

Proof It is clear that $\sum_{i=1}^{n+1} u_i = \sum_{i=1}^{n+1} v_i$.

Without loss of generality, we may assume that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. So $v_1 \geq \cdots \geq v_{n+1}$. The following discussion is divided into two cases:

Case 1. $\beta + (\sum_{i=1}^{n} \alpha_i)x - 1 \geq \alpha_1$. Notice that $x \in [0,1]$, and $\alpha_i > 0$, $i = 1, \ldots, n$, and we have

$$u_1 = \beta + \left(\sum_{i=1}^{n} \alpha_i\right)x - 1 \leq \beta + \sum_{i=1}^{n} \alpha_i - 1 = v_1$$

and

$$u_i = \alpha_{i-1} \geq \alpha_{i-1}x = v_i, \quad i = 2, \ldots, n+1.$$

Hence from Lemma 1, it follows that $\mathbf{u} \prec \mathbf{v}$.

Case 2. $\beta + (\sum_{i=1}^{n} \alpha_i)x - 1 < \alpha_1$. Let $u_{[1]} \geq \cdots \geq u_{[n+1]}$ denote the components of $\mathbf{u}$ in a decreasing order. There exist $k \in \{2, 3, \ldots, n\}$ such that

$$\alpha_1 \geq \cdots \geq \alpha_{k-1} \geq \beta + \left(\sum_{i=1}^{n} \alpha_i\right)x - 1 \geq \alpha_{k+1} \geq \cdots \geq \alpha_n.$$

Notice that $\beta - 1 \geq 0$, $x \in [0,1]$, and $\alpha_i > 0$, and if $1 \leq m \leq k-1$, then

$$\sum_{i=1}^{m} u_{[i]} = \sum_{i=1}^{m} \alpha_i \leq \beta + \sum_{i=1}^{n} \alpha_i - 1 \leq \sum_{i=1}^{m} v_i.$$

If $n \geq m > k-1$, then

$$\begin{aligned}
\sum_{i=1}^{m} u_{[i]} &= \beta + \left(\sum_{i=1}^{n} \alpha_i\right)x - 1 + \sum_{i=1}^{k-1} \alpha_i + \sum_{i=k+1}^{m} \alpha_i \quad \left(\text{If } m = k, \text{ let } \sum_{i=k+1}^{m} \alpha_i = 0\right) \\
&= \beta + \left(\left(\sum_{i=1}^{m-1} \alpha_i\right)x + \alpha_m x + \left(\sum_{i=m+1}^{n} \alpha_i\right)x\right) - 1 + \sum_{i=1}^{k-1} \alpha_i + \sum_{i=k+1}^{m} \alpha_i \\
&= \beta + \left(\sum_{i=1}^{m-1} \alpha_i\right)x - 1 + \left(\sum_{i=1}^{k-1} \alpha_i + \alpha_m x + \sum_{i=k+1}^{m} \alpha_i + \left(\sum_{i=m+1}^{n} \alpha_i\right)x\right)
\end{aligned}$$

$$\begin{aligned}&\le \beta + \left(\sum_{i=1}^{m-1}\alpha_i\right)x - 1 + \sum_{i=1}^{n}\alpha_i\\&= \sum_{i=1}^{m} v_i.\end{aligned}$$

Hence from Definition 1(i), it follows that $\mathbf{u} \prec \mathbf{v}$. □

Lemma 5 *Let*

$$\mathbf{w} = (w_1, \ldots, w_n, w_{n+1}) = (\beta - 1, \alpha_1 x, \ldots, \alpha_n x) \tag{12}$$

and

$$\mathbf{z} = (z_1, \ldots, z_n, z_{n+1}) = \left(\beta + \left(\sum_{i=1}^{n}\alpha_i\right)x - 1, \underbrace{0, \ldots, 0}_{n}\right), \tag{13}$$

where $x \in [0,1]$, $\beta \ge 1$, $\alpha_i > 0$, $i = 1, \ldots, n$, $n \in \mathbb{N}$. *Then* $\mathbf{w} \prec \mathbf{z}$.

Proof It is clear that $\sum_{i=1}^{n+1} w_i = \sum_{i=1}^{n+1} z_i$.

The following discussion is divided into two cases:

Case 1. $\beta - 1 \ge \alpha_1 x$. Notice that $x \in [0,1]$ and $\alpha_i > 0$, $i = 1, \ldots, n$, we have

$$w_1 = \beta - 1 \le \beta + \left(\sum_{i=1}^{n}\alpha_i\right)x - 1 = z_1$$

and

$$w_i = \alpha_{i-1}x \ge 0 = z_i, \quad i = 2, \ldots, n+1.$$

Hence from the Lemma 1, it follows that $\mathbf{w} \prec \mathbf{z}$.

Case 2. $\beta - 1 < \alpha_1 x$. Let $w_{[1]} \ge \cdots \ge w_{[n+1]}$ denote the components of $\mathbf{w}$ in a decreasing order. There exist $k \in k = 2, \ldots, n$ such that

$$\alpha_1 x \ge \cdots \ge \alpha_{k-1}x \ge \beta - 1 \ge \alpha_{k+1}x \ge \cdots \ge \alpha_n x.$$

Now notice that $\beta - 1 \ge 0$, $x \in [0,1]$ and $\alpha_i > 0$, we have

$$\begin{aligned}&w_{[1]} = \alpha_1 x \le \beta + \left(\sum_{i=1}^{n}\alpha_i\right)x - 1 = z_1,\\&w_{[i]} = \alpha_i x \ge 0 = z_i, \quad i = 2, \ldots, k-1,\\&w_{[k]} = \beta - 1 \ge 0 = z_k\end{aligned}$$

and

$$w_{[i]} = \alpha_{i-1}x \ge 0 = z_i, \quad i = k+1, \ldots, n+1.$$

Hence from the Lemma 1, it follows that $\mathbf{w} \prec \mathbf{z}$. □

The Schur-convexity described the ordering of majorization, the order-preserving functions were first comprehensively studied by Issai Schur in 1923. It has important applications in analytic inequalities, combinatorial optimization, special functions, probabilistic, statistical, and so on. See [8, 11–13].

3 Proof of main result

Proof of Theorem 1 Taking $g(t) = e^{-\frac{t^k}{k}}, f(t) = t, a = 0, b = +\infty$, then

$$I(x) = \int_a^b g(t)\big(f(t)\big)^x \, dt = \int_0^{+\infty} e^{-\frac{t^k}{k}} t^x \, dt = \Gamma_k(x+1). \tag{14}$$

By Lemma 2, $I(x)$ is log-convex on $[0, +\infty)$, and then from Lemma 3, $\varphi(\mathbf{x}) = \prod_{i=1}^{n+1} I(x_i)$ is Schur-convex on $[0, +\infty)^{n+1}$. Combining Lemma 4 and Lemma 5, respectively, we have

$$\varphi(\mathbf{u}) \leq \varphi(\mathbf{v})$$

and

$$\varphi(\mathbf{w}) \leq \varphi(\mathbf{z}),$$

i.e.

$$\Gamma_k\left(\beta + \left(\sum_{i=1}^n \alpha_i\right)x\right) \prod_{i=1}^n \Gamma_k(1+\alpha_i) \leq \Gamma_k\left(\beta + \sum_{i=1}^n \alpha_i\right) \prod_{i=1}^n \Gamma_k(1+\alpha_i x) \tag{15}$$

and

$$\Gamma_k(\beta) \prod_{i=1}^n \Gamma_k(1+\alpha_i x) \leq \Gamma_k\left(\beta + \left(\sum_{i=1}^n \alpha_i\right)x\right). \tag{16}$$

Thus, we have proved the double inequality (7).

The proof of Theorem 1 is completed. □

Proof of Theorem 2 Let

$$\xi_k(x) = \int_0^\infty \frac{t^{x-k}}{e^t - 1} \, dt, \quad x > k,$$

i.e.

$$\xi_k(x) = \zeta_k(x)\Gamma_k(x).$$

Taking $g(t) = \frac{t}{e^t-1}, f(t) = t, a = 0, b = +\infty$, then

$$J(x) = \int_a^b g(t)\big(f(t)\big)^x \, dt = \int_0^{+\infty} \frac{t^{x+1}}{e^t - 1} \, dt = \xi_k(x+k+1). \tag{17}$$

By Lemma 2, $J(x)$ is log-convex on $[0,+\infty)$, and then from Lemma 3, $\psi(\mathbf{x}) = \prod_{i=1}^{n+1} J(x_i)$ is Schur-convex on $[0,+\infty)^{n+1}$. Combining Lemma 4 and Lemma 5, respectively, we have

$$\psi(\mathbf{u}) \le \psi(\mathbf{v})$$

and

$$\psi(\mathbf{w}) \le \psi(\mathbf{z}),$$

i.e.

$$\begin{aligned}&\xi_k\left(\beta + k + \left(\sum_{i=1}^{n}\alpha_i\right)x\right)\prod_{i=1}^{n}\xi_k(k+1+\alpha_i)\\&\quad\le \xi_k\left(\beta + k + \sum_{i=1}^{n}\alpha_i\right)\prod_{i=1}^{n}\xi_k(k+1+\alpha_i x)\end{aligned}$$

and

$$\begin{aligned}&\xi_k(\beta + k)\prod_{i=1}^{n}\xi_k(k+1+\alpha_i x)\\&\quad\le \xi_k\left(\beta + k + \left(\sum_{i=1}^{n}\alpha_i\right)x\right)\left(\frac{\pi^2}{6}\right)^n,\end{aligned}$$

notice that $\xi_k(k+1) = \frac{\pi^2}{6}$.

Further, we have

$$\begin{aligned}&\zeta_k\left(\beta + k + \left(\sum_{i=1}^{n}\alpha_i\right)x\right)\Gamma_k\left(\beta + k + \left(\sum_{i=1}^{n}\alpha_i\right)x\right)\prod_{i=1}^{n}\zeta_k(k+1+\alpha_i)\Gamma_k(k+1+\alpha_i)\\&\quad\le \zeta_k\left(\beta + k + \sum_{i=1}^{n}\alpha_i\right)\Gamma_k\left(\beta + k + \sum_{i=1}^{n}\alpha_i\right)\prod_{i=1}^{n}\zeta_k(k+1+\alpha_i x)\Gamma_k(k+1+\alpha_i x)\end{aligned} \tag{18}$$

and

$$\begin{aligned}&\zeta_k(\beta + k)\Gamma_k(\beta + k)\prod_{i=1}^{n}\zeta_k(k+1+\alpha_i x)\Gamma_k(k+1+\alpha_i x)\\&\quad\le \zeta_k\left(\beta + k + \left(\sum_{i=1}^{n}\alpha_i\right)x\right)\Gamma_k\left(\beta + k + \left(\sum_{i=1}^{n}\alpha_i\right)x\right)\left(\frac{\pi^2}{6}\right)^n.\end{aligned} \tag{19}$$

Rearranging (18) and (19) gives the double inequality (8).

The proof of Theorem 2 is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

The main idea of this paper was proposed by JZ and H-NS. This work was carried out in collaboration between both authors. They read and approved the final manuscript.

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Acknowledgements

The work was supported by the National Natural Science Foundation of China (Grant No. 11101034) and the Funding Project for Academic Human Resources Development in Institutions of Higher Learning under the Jurisdiction of Beijing Municipality (PHR (IHLB)) (PHR201108407).

Received: 5 January 2014 Accepted: 29 April 2014 Published: 13 May 2014

References

1. Andrews, GE, Askey, R, Roy, R: Special Functions. Cambridge University Press, Cambridge (1999)
2. Alsina, C, Tomás, MS: A geometrical proof of a new inequality for the gamma function. J. Inequal. Pure Appl. Math. **6**(2), Article 48 (2005)
3. Nguyen, VV, Ngo, PNN: An inequality for the gamma function. Int. Math. Forum **4**(28), 1379-1382 (2009)
4. Díaz, R, Pariguan, E: On hypergeometric functions and Pochhammer *k*-symbol. Divulg. Mat. **15**(2), 179-192 (2007)
5. Kokologiannaki, CG, Krasniqi, V: Some properties of the *k*-gamma function. Matematiche **68**(1), 13-22 (2013)
6. Díaz, R, Teruel, C: q,k-Generalized gamma and beta functions. J. Nonlinear Math. Phys. **12**(1), 118-134 (2005)
7. van der Poorten, A: A proof that Euler missed. Math. Intell. **1**(4), 195-203 (1979)
8. Marshall, AW, Olkin, I, Arnold, BC: Inequalities: Theory of Majorization and Its Application, 2nd edn. Springer, New York (2011)
9. Wang, BY: Foundations of Majorization Inequalities. Beijing Normal University Press, Beijing (1990) (in Chinese)
10. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Cambridge University Press, London (1934)
11. Marshall, AW, Olkin, I: Schur-convexity, gamma functions, and moments. Int. Ser. Numer. Math. **157**, 245-250 (2009)
12. Merkle, M: On log-convexity of a ratio of gamma functions. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **8**, 114-119 (1997)
13. Merkle, M: Conditions for convexity of a derivative and some applications to the gamma function. Aequ. Math. **55**, 273-280 (1998)

10.1186/1029-242X-2014-191
Cite this article as: Zhang and Shi: **Two double inequalities for *k*-gamma and *k*-Riemann zeta functions.** *Journal of Inequalities and Applications* 2014, 2014:191