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Weighted boundedness of a multilinear operator associated to a singular integral operator with general kernels

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Abstract

In this paper, we establish the weighted sharp maximal function inequalities for a multilinear operator associated to a singular integral operator with general kernels. As an application, we obtain the boundedness of the operator on weighted Lebesgue and Morrey spaces.

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1 Introduction and preliminaries

As the development of singular integral operators (see [1–3]), their commutators and multilinear operators have been well studied. In [4–6], the authors prove that the commutators generated by singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [7]) proves a similar result when singular integral operators are replaced by fractional integral operators. In [8, 9], the boundedness for the commutators generated by singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)$ $(1 < p < \infty)$ spaces is obtained. In [10, 11], the boundedness for the commutators generated by singular integral operators and weighted *BMO* and Lipschitz functions on $L^p(R^n)$ $(1 < p < \infty)$ spaces is obtained. In [12], some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by *BMO* and Lipschitz functions is obtained (see [8, 12, 13]). Motivated by these, in this paper, we study the multilinear operator generated by the singular integral operator with general kernel and the weighted Lipschitz and *BMO* functions.

First, let us introduce some notations. Throughout this paper, Q denotes a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left| f(y) - f_Q \right| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)\, dx$. It is well known that (see [1, 2])

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q \left| f(y) - c \right| dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy.$$

For $\eta > 0$, let $M^{\#}_{\eta}(f)(x) = M^{\#}(|f|^{\eta})^{1/\eta}(x)$ and $M_{\eta}(f)(x) = M(|f|^{\eta})^{1/\eta}(x)$.

For $0 < \eta < n$, $1 \le p < \infty$ and the non-negative weight function w, set

$$M_{\eta,p,w}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{w(Q)^{1-p\eta/n}} \int_Q |f(y)|^p w(y)\,dy \right)^{1/p}.$$

We write $M_{\eta,p,w}(f) = M_{p,w}(f)$ if $\eta = 0$.

The A_p weight is defined by (see [1]), for $1 < p < \infty$,

$$A_p = \left\{ w \in L^1_{\mathrm{loc}}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\,dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}\,dx \right)^{p-1} < \infty \right\}$$

and

$$A_1 = \left\{ w \in L^p_{\mathrm{loc}}(R^n) : M(w)(x) \le Cw(x), \text{a.e.} \right\}.$$

Given a non-negative weight function w, for $1 \le p < \infty$, the weighted Lebesgue space $L^p(R^n, w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x)\,dx \right)^{1/p} < \infty.$$

For $0 < \beta < 1$ and the non-negative weight function w, the weighted Lipschitz space $Lip_{\beta}(w)$ is the space of functions b such that

$$\|b\|_{Lip_{\beta}(w)} = \sup_Q \frac{1}{w(Q)^{\beta/n}} \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p}\,dy \right)^{1/p} < \infty,$$

and the weighted BMO space $BMO(w)$ is the space of functions b such that

$$\|b\|_{BMO(w)} = \sup_Q \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p}\,dy \right)^{1/p} < \infty.$$

Remark (1) It has been known that (see [11, 14]), for $b \in Lip_{\beta}(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^kQ}| \le Ck\|b\|_{Lip_{\beta}(w)} w(x) w(2^kQ)^{\beta/n}.$$

(2) It has been known that (see [2, 14]), for $b \in BMO(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^kQ}| \le Ck\|b\|_{BMO(w)} w(x).$$

(3) Let $b \in Lip_{\beta}(w)$ or $b \in BMO(w)$ and $w \in A_1$. By [14], we know that spaces $Lip_{\beta}(w)$ or $BMO(w)$ coincide and the norms $\|b\|_{Lip_{\beta}(w)}$ or $\|b\|_{BMO(w)}$ are equivalent with respect to different values $1 \le p < \infty$.

Definition 1 Let φ be a positive, increasing function on R^+, and there exists a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \quad \text{for } t \ge 0.$$

Let w be a non-negative weight function on R^n and f be a locally integrable function on R^n. Set, for $1 \le p < \infty$,

$$\|f\|_{L^{p,\varphi}(w)} = \sup_{x\in R^n, d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} \left|f(y)\right|^p w(y)\,dy \right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized weighted Morrey space is defined by

$$L^{p,\varphi}\left(R^n, w\right) = \left\{f \in L^1_{\text{loc}}\left(R^n\right) : \|f\|_{L^{p,\varphi}(w)} < \infty\right\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n, w) = L^{p,\delta}(R^n, w)$, which is the classical Morrey space (see [15, 16]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n, w) = L^p(R^n, w)$, which is the weighted Lebesgue space (see [1]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [17–20]).

In this paper, we study some singular integral operators as follows (see [12]).

Definition 2 Let $T : S \to S'$ be a linear operator such that T is bounded on $L^2(R^n)$ and has a kernel K, that is, there exists a locally integrable function $K(x,y)$ on $R^n \times R^n \setminus \{(x,y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x,y)f(y)\,dy$$

for every bounded and compactly supported function f, where K satisfies

$$\left|K(x,y)\right| \le C|x-y|^{-n},$$
$$\int_{2|y-z|<|x-y|} \left(\left|K(x,y) - K(x,z)\right| + \left|K(y,x) - K(z,x)\right|\right) dx \le C,$$

and there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \ge 1$,

$$\left(\int_{2^k|z-y|\le|x-y|<2^{k+1}|z-y|} \left(\left|K(x,y) - K(x,z)\right| + \left|K(y,x) - K(z,x)\right|\right)^q dy\right)^{1/q} \le C_k\left(2^k|z-y|\right)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$. Moreover, let m be a positive integer and b be a function on R^n. Set

$$R_{m+1}(b;x,y) = b(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha b(y)(x-y)^\alpha.$$

The multilinear operator related to the operator T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y) f(y)\,dy.$$

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 1 (see [1–3, 5, 6]) and that the commutator $[b,T](f) = bT(f) - T(bf)$ is a particular operator of the multilinear operator T^b if $m = 0$. The multilinear operator T^b is a non-trivial generalization of the commutator. It is well known that commutators and multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [21–23]). The main purpose of this paper is to prove the sharp maximal inequalities for the multilinear operator T^b. As application, we obtain the weighted L^p-norm inequality and Morrey space boundedness for the multilinear operator T^b.

2 Theorems and lemmas

We shall prove the following theorems.

Theorem 1 *Let T be a singular integral operator as in Definition* 2, *the sequence* $\{kC_k\} \in l^1$, $w \in A_1$, $0 < \eta < 1$, $q' < r < \infty$ *and* $D^\alpha b \in BMO(w)$ *for all* α *with* $|\alpha| = m$. *Then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M_\eta^{\#}\big(T^b(f)\big)(\tilde{x}) \le C \sum_{|\alpha|=m} \big\|D^\alpha b\big\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

Theorem 2 *Let T be a singular integral operator as in Definition* 2, *the sequence* $\{kC_k\} \in l^1$, $w \in A_1$, $0 < \eta < 1$, $q' < r < \infty$, $0 < \beta < 1$ *and* $D^\alpha b \in Lip_\beta(w)$ *for all* α *with* $|\alpha| = m$. *Then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M_\eta^{\#}\big(T^b(f)\big)(\tilde{x}) \le C \sum_{|\alpha|=m} \big\|D^\alpha b\big\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).$$

Theorem 3 *Let T be a singular integral operator as in Definition* 2, *the sequence* $\{kC_k\} \in l^1$, $w \in A_1$, $q' < u < \infty$ *and* $D^\alpha b \in BMO(w)$ *for all* α *with* $|\alpha| = m$. *Then* T^b *is bounded from* $L^u(R^n,w)$ *to* $L^u(R^n,w^{1-u})$.

Theorem 4 *Let T be a singular integral operator as in Definition* 2, *the sequence* $\{kC_k\} \in l^1$, $w \in A_1$, $q' < u < \infty$, $0 < D < 2^n$ *and* $D^\alpha b \in BMO(w)$ *for all* α *with* $|\alpha| = m$. *Then* T^b *is bounded from* $L^{u,\varphi}(R^n,w)$ *to* $L^{u,\varphi}(R^n,w^{1-u})$.

Theorem 5 *Let T be a singular integral operator as in Definition* 2, *the sequence* $\{kC_k\} \in l^1$, $w \in A_1$, $0 < \beta < 1$, $q' < u < n/\beta$, $1/v = 1/u - \beta/n$ *and* $D^\alpha b \in Lip_\beta(w)$ *for all* α *with* $|\alpha| = m$. *Then* T^b *is bounded from* $L^u(R^n,w)$ *to* $L^v(R^n,w^{1-v})$.

Theorem 6 *Let T be a singular integral operator as in Definition* 2, *the sequence* $\{kC_k\} \in l^1$, $w \in A_1$, $0 < \beta < 1$, $0 < D < 2^n$, $q' < u < n/\beta$, $1/v = 1/u - \beta/n$ *and* $D^\alpha b \in Lip_\beta(w)$ *for all* α *with* $|\alpha| = m$. *Then* T^b *is bounded from* $L^{u,\varphi}(R^n,w)$ *to* $L^{v,\varphi}(R^n,w^{1-v})$.

To prove the theorems, we need the following lemmas.

Lemma 1 (see [1, p.485]) *Let $0 < u < v < \infty$ and for any function $f \geq 0$, we define that, for $1/r = 1/u - 1/v$,*

$$\|f\|_{WL^v} = \sup_{\lambda>0} \lambda \left|\left\{x \in R^n : f(x) > \lambda\right\}\right|^{1/v}, \qquad N_{u,v}(f) = \sup_Q \|f\chi_Q\|_{L^u}/\|\chi_Q\|_{L^r},$$

where the sup *is taken for all measurable sets Q with $0 < |Q| < \infty$. Then*

$$\|f\|_{WL^v} \leq N_{u,v}(f) \leq \left(v/(v-u)\right)^{1/u} \|f\|_{WL^v}.$$

Lemma 2 (see [12]) *Let T be a singular integral operator as in Definition 2, the sequence $\{C_k\} \in l^1$. Then T is bounded on $L^p(R^n, w)$ for $w \in A_p$ with $1 < p < \infty$, and weak (L^1, L^1) bounded.*

Lemma 3 (see [1, 7]) *Let $0 \leq \eta < n$, $1 \leq s < u < n/\eta$, $1/v = 1/u - \eta/n$ and $w \in A_1$. Then*

$$\left\|M_{\eta,s,w}(f)\right\|_{L^v(w)} \leq C\|f\|_{L^u(w)}.$$

Lemma 4 (see [1]) *Let $0 < p, \eta < \infty$ and $w \in \bigcup_{1\leq r<\infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,*

$$\int_{R^n} M_\eta(f)(x)^p w(x)\,dx \leq C \int_{R^n} M_\eta^\#(f)(x)^p w(x)\,dx.$$

Lemma 5 (see [17, 20]) *Let $0 < p < \infty$, $0 < \eta < \infty$, $0 < D < 2^n$ and $w \in A_1$. Then, for any smooth function f for which the left-hand side is finite,*

$$\left\|M_\eta(f)\right\|_{L^{p,\varphi}(w)} \leq C\left\|M_\eta^\#(f)\right\|_{L^{p,\varphi}(w)}.$$

Lemma 6 (see [17, 20]) *Let $0 \leq \eta < n$, $0 < D < 2^n$, $1 \leq s < u < n/\eta$, $1/v = 1/u - \eta/n$ and $w \in A_1$. Then*

$$\left\|M_{\eta,s,w}(f)\right\|_{L^{v,\varphi}(w)} \leq C\|f\|_{L^{u,\varphi}(w)}.$$

Lemma 7 (see [22]) *Let b be a function on R^n and $D^\alpha A \in L^u(R^n)$ for all α with $|\alpha| = m$ and any $u > n$. Then*

$$\left|R_m(b;x,y)\right| \leq C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} \left|D^\alpha b(z)\right|^u dz\right)^{1/u},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

3 Proofs of theorems

Proof of Theorem 1 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 that the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right|^\eta dx\right)^{1/\eta} \leq C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}(x) = b(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha b)_{\tilde{Q}} x^\alpha$, then $R_m(b; x, y) = R_m(\tilde{b}; x, y)$ and $D^\alpha \tilde{b} = D^\alpha b - (D^\alpha b)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} T^b(f)(x) &= \int_{R^n} \frac{R_m(\tilde{b}; x, y)}{|x-y|^m} K(x,y) f_1(y)\, dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{(x-y)^\alpha D^\alpha \tilde{b}(y)}{|x-y|^m} K(x,y) f_1(y)\, dy \\ &\quad + \int_{R^n} \frac{R_{m+1}(\tilde{b}; x, y)}{|x-y|^m} K(x,y) f_2(y)\, dy \\ &= T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x-\cdot|^m} f_1\right) - T\left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x-\cdot)^\alpha D^\alpha \tilde{b}}{|x-\cdot|^m} f_1\right) + T^{\tilde{b}}(f_2)(x), \end{aligned}$$

then

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - T^{\tilde{b}}(f_2)(x_0)\right|^\eta dx\right)^{1/\eta} \\ &\quad\le C\left(\frac{1}{|Q|}\int_Q \left|T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x-\cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} \\ &\qquad + C\left(\frac{1}{|Q|}\int_Q \left|T\left(\sum_{|\alpha|=m} \frac{(x-\cdot)^\alpha D^\alpha \tilde{b}}{|x-\cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} \\ &\qquad + C\left(\frac{1}{|Q|}\int_Q \left|T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)\right|^\eta dx\right)^{1/\eta} \\ &\quad= I_1 + I_2 + I_3. \end{aligned}$$

For I_1, noting that $w \in A_1$, w satisfies the reverse of Hölder's inequality,

$$\left(\frac{1}{|Q|}\int_Q w(x)^{p_0}\, dx\right)^{1/p_0} \le \frac{C}{|Q|}\int_Q w(x)\, dx$$

for all cube Q and some $1 < p_0 < \infty$ (see [1]). We take $u = rp_0/(r + p_0 - 1)$ in Lemma 7 and have $1 < u < r$ and $p_0 = u(r-1)/(r-u)$. Then by Lemma 7 and Hölder's inequality, we get

$$\begin{aligned} &\left|R_m(\tilde{b}; x, y)\right| \\ &\quad\le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} \left|D^\alpha \tilde{b}(z)\right|^u dz\right)^{1/u} \\ &\quad\le C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/u} \left(\int_{\tilde{Q}(x,y)} \left|D^\alpha \tilde{b}(z)\right|^u w(z)^{u(1-r)/r} w(z)^{u(r-1)/r}\, dz\right)^{1/u} \\ &\quad\le C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/u} \left(\int_{\tilde{Q}(x,y)} \left|D^\alpha \tilde{b}(z)\right|^r w(z)^{1-r}\, dz\right)^{1/r} \\ &\qquad\times \left(\int_{\tilde{Q}(x,y)} w(z)^{u(r-1)/(r-u)}\, dz\right)^{(r-u)/ru} \end{aligned}$$

$$\begin{aligned}
&\le C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/u} \|D^\alpha b\|_{BMO(w)} w(\tilde{Q})^{1/r} |\tilde{Q}|^{(r-u)/ru} \\
&\quad \times \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} w(z)^{p_0}\, dz \right)^{(r-u)/ru} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} |\tilde{Q}|^{-1/u} w(\tilde{Q})^{1/r} |\tilde{Q}|^{1/u-1/r} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} w(z)\, dz \right)^{(r-1)/r} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} |\tilde{Q}|^{-1/u} w(\tilde{Q})^{1/r} |\tilde{Q}|^{1/u-1/r} w(\tilde{Q})^{1-1/r} |\tilde{Q}|^{1/r-1} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \frac{w(\tilde{Q})}{|\tilde{Q}|} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}).
\end{aligned}$$

Thus, by the L^s-boundedness of T (see Lemma 2) for $1 < s < r$ and $w \in A_1 \subseteq A_{r/s}$, we obtain

$$\begin{aligned}
I_1 &\le \frac{C}{|Q|} \int_Q \left| T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x-\cdot|^m} f_1 \right) \right| dx \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \left(\frac{1}{|Q|} \int_{R^n} |T(f_1)(x)|^s\, dx \right)^{1/s} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{R^n} |f_1(x)|^s\, dx \right)^{1/s} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{\tilde{Q}} |f(x)|^s w(x)^{s/r} w(x)^{-s/r}\, dx \right)^{1/s} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{\tilde{Q}} |f(x)|^r w(x)\, dx \right)^{1/r} \left(\int_{\tilde{Q}} w(x)^{-s/(r-s)}\, dx \right)^{(r-s)/rs} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} w(\tilde{Q})^{1/r} \left(\frac{1}{w(\tilde{Q})} \int_{\tilde{Q}} |f(x)|^r w(x)\, dx \right)^{1/r} \\
&\quad \times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{-s/(r-s)}\, dx \right)^{(r-s)/rs} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)\, dx \right)^{1/r} |\tilde{Q}|^{1/s} w(\tilde{Q})^{-1/r} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
\end{aligned}$$

For I_2, by the weak (L^1, L^1) boundedness of T (see Lemma 2) and Kolmogoro's inequality (see Lemma 1), we obtain

$$\begin{aligned}
I_2 &\le C \sum_{|\alpha|=m} \left(\frac{1}{|Q|} \int_Q \left| T\left(D^\alpha \tilde{b} f_1\right)(x) \right|^\eta dx \right)^{1/\eta} \\
&\le C \sum_{|\alpha|=m} \frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}} \frac{\|T(D^\alpha \tilde{b} f_1)\chi_Q\|_{L^\eta}}{\|\chi_Q\|_{L^{\eta/(1-\eta)}}} \\
&\le C \sum_{|\alpha|=m} \frac{1}{|Q|} \left\| T\left(D^\alpha \tilde{b} f_1\right) \right\|_{WL^1}
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{|\alpha|=m}\frac{1}{|Q|}\int_{R^n}\left|D^\alpha\tilde{b}(x)f_1(x)\right|dx \\
&\le C\sum_{|\alpha|=m}\frac{1}{|Q|}\int_{\tilde{Q}}\left|D^\alpha b(x)-\left(D^\alpha b\right)_{\tilde{Q}}\right|w(x)^{-1/r}\left|f(x)\right|w(x)^{1/r}\,dx \\
&\le C\sum_{|\alpha|=m}\frac{1}{|Q|}\left(\int_{\tilde{Q}}\left|\left(D^\alpha b(x)-\left(D^\alpha b\right)_{\tilde{Q}}\right)\right|^{r'}w(x)^{1-r'}\,dx\right)^{1/r'}\left(\int_{\tilde{Q}}\left|f(x)\right|^{r}w(x)\,dx\right)^{1/r} \\
&\le C\sum_{|\alpha|=m}\frac{1}{|Q|}\left\|D^\alpha b\right\|_{BMO(w)}w(\tilde{Q})^{1/r'}w(\tilde{Q})^{1/r}\left(\frac{1}{w(\tilde{Q})}\int_{\tilde{Q}}\left|f(x)\right|^{r}w(x)\,dx\right)^{1/r} \\
&\le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(w)}\frac{w(\tilde{Q})}{|\tilde{Q}|}M_{r,w}(f)(\tilde{x}) \\
&\le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(w)}w(\tilde{x})M_{r,w}(f)(\tilde{x}).
\end{aligned}$$

For I_3, noting that $|x-y|\approx|x_0-y|$ for $x\in Q$ and $y\in R^n\setminus Q$, we write

$$\begin{aligned}
&\left|T^{\tilde{b}}(f_2)(x)-T^{\tilde{b}}(f_2)(x_0)\right| \\
&\quad\le\int_{R^n}\left|R_m(\tilde{b};x,y)-R_m(\tilde{b};x_0,y)\right|\frac{|K(x,y)|}{|x-y|^m}\left|f_2(y)\right|dy \\
&\qquad+\int_{R^n}\left|\frac{K(x,y)}{|x-y|^m}-\frac{K(x_0,y)}{|x_0-y|^m}\right|\left|R_m(\tilde{b};x_0,y)\right|\left|f_2(y)\right|dy \\
&\qquad+\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left|\frac{K(x,y)(x-y)^\alpha}{|x-y|^m}-\frac{K(x_0,y)(x_0-y)^\alpha}{|x_0-y|^m}\right|\left|D^\alpha\tilde{b}(y)\right|\left|f_2(y)\right|dy \\
&\quad=I_3^{(1)}(x)+I_3^{(2)}(x)+I_3^{(3)}(x).
\end{aligned}$$

For $I_3^{(1)}(x)$, by the formula (see [22])

$$R_m(\tilde{b};x,y)-R_m(\tilde{b};x_0,y)=\sum_{|\gamma|<m}\frac{1}{\gamma!}R_{m-|\gamma|}\left(D^\gamma\tilde{b};x,x_0\right)(x-y)^\gamma$$

and Lemma 7, we have, similar to the proof of I_1,

$$\left|R_m(\tilde{b};x,y)-R_m(\tilde{b};x_0,y)\right|\le C\sum_{|\gamma|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\gamma|}|x-y|^{|\gamma|}\left\|D^\alpha b\right\|_{BMO(w)}w(\tilde{x})$$

and

$$\left|R_m(\tilde{b};x_0,y)\right|\le C\sum_{|\alpha|=m}|x-x_0|^m\left\|D^\alpha b\right\|_{BMO(w)}w(\tilde{x}).$$

Thus, by $w\in A_1\subseteq A_r$, we get

$$\begin{aligned}
I_3^{(1)}(x)&\le\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus2^k\tilde{Q}}\left|R_m(\tilde{b};x,y)-R_m(\tilde{b};x_0,y)\right|\frac{|K(x,y)|}{|x-y|^m}\left|f(y)\right|dy \\
&\le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(w)}w(\tilde{x})\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}\left|f(y)\right|dy
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})\sum_{k=1}^{\infty}\frac{d}{(2^kd)^{n+1}}\int_{2^k\tilde{Q}}|f(y)|w(y)^{1/r}w(y)^{-1/r}\,dy\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})\sum_{k=1}^{\infty}\frac{d}{(2^kd)^{n+1}}\left(\int_{2^k\tilde{Q}}|f(y)|^r w(y)\,dy\right)^{1/r}\\
&\quad\times\left(\int_{2^k\tilde{Q}}w(y)^{-1/(r-1)}\,dy\right)^{(r-1)/r}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})\sum_{k=1}^{\infty}\frac{d}{(2^kd)^{n+1}}w(2^k\tilde{Q})^{1/r}\\
&\quad\times\left(\frac{1}{w(2^k\tilde{Q})}\int_{2^k\tilde{Q}}|f(y)|^r w(y)\,dx\right)^{1/r}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-1/(r-1)}\,dy\right)^{(r-1)/r}\\
&\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)\,dy\right)^{1/r}|2^k\tilde{Q}|w(2^k\tilde{Q})^{-1/r}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})M_{r,w}(f)(\tilde{x})\sum_{k=1}^{\infty}2^{-k}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})M_{r,w}(f)(\tilde{x}).
\end{aligned}
$$

For $I_3^{(2)}(x)$, we take $1 < p < \infty$ such that $1/p + 1/q + 1/r = 1$. Recalling $r > q'$ and $w \in A_1 \subseteq A_{r/p+1}$, we get

$$
\begin{aligned}
I_3^{(2)}(x) &\le \sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x,y)-K(x_0,y)|\frac{|R_m(\tilde{b};x_0,y)|}{|x-y|^m}|f(y)|w(y)^{1/r}w(y)^{-1/r}\,dy\\
&\quad+\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left|\frac{1}{|x-y|^m}-\frac{1}{|x_0-y|^m}\right||K(x_0,y)||R_m(\tilde{b};x_0,y)||f(y)|\,dy\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})\sum_{k=0}^{\infty}\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x,y)-K(x_0,y)|^q\,dy\right)^{1/q}\\
&\quad\times\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|f(y)|^r w(y)\,dy\right)^{1/r}\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}w(y)^{-p/r}\,dy\right)^{1/p}\\
&\quad+C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|w(y)^{1/r}w(y)^{-1/r}\,dy\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})\sum_{k=1}^{\infty}C_k|2^kQ|^{-1/q'}w(2^k\tilde{Q})^{1/r}\\
&\quad\times\left(\frac{1}{w(2^k\tilde{Q})}\int_{2^k\tilde{Q}}|f(y)|^r w(y)\,dy\right)^{1/r}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-p/r}\,dy\right)^{1/p}\\
&\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)\,dy\right)^{1/r}|2^k\tilde{Q}|^{1/r+1/p}w(2^k\tilde{Q})^{-1/r}\\
&\quad+C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})\sum_{k=1}^{\infty}\frac{d}{(2^kd)^{n+1}}\\
&\quad\times\left(\int_{2^k\tilde{Q}}|f(y)|^r w(y)\,dy\right)^{1/r}\left(\int_{2^k\tilde{Q}}w(y)^{-1/(r-1)}\,dy\right)^{(r-1)/r}
\end{aligned}
$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=1}^{\infty} (C_k + 2^{-k}) \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y)\, dy \right)^{1/r}$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

Similarly, we have, for $r < p_1' < \infty$, $1 < s_1, s_2 < \infty$ with $1/p_1 + 1/q + 1/r + 1/s_1 = 1$ and $1/q + 1/r + 1/s_2 = 1$,

$$\begin{aligned}
I_3^{(3)}(x) \leq{}& \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} |K(x,y) - K(x_0,y)| \frac{|(x-y)^\alpha|}{|x-y|^m} \\
&\times |D^\alpha b(y) - (D^\alpha b)_{2^{k+1}\tilde{Q}}| w(y)^{(1-p_1)/p_1} |f(y)| w(y)^{1/r} w(y)^{(p_1-1)/p_1 - 1/r}\, dy \\
&+ \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} |K(x,y) - K(x_0,y)| \frac{|(x-y)^\alpha|}{|x-y|^m} \\
&\times |(D^\alpha b)_{2^{k+1}\tilde{Q}} - (D^\alpha b)_{\tilde{Q}}| |f(y)| w(y)^{1/r} w(y)^{-1/r}\, dy \\
&+ \sum_{|\alpha|=m} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left| \frac{(x-y)^\alpha}{|x-y|^m} - \frac{(x_0-y)^\alpha}{|x_0-y|^m} \right| |K(x_0,y)| |f(y)| |D^\alpha \tilde{b}(y)|\, dy \\
\leq{}& C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} |K(x,y) - K(x_0,y)|^q\, dy \right)^{1/q} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^r w(y)\, dy \right)^{1/r} \\
&\times \left(\int_{2^{k+1}\tilde{Q}} |D^\alpha b(y) - (D^\alpha b)_{2^{k+1}\tilde{Q}}|^{p_1} w(y)^{1-p_1}\, dy \right)^{1/p_1} \\
&\times \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-(1/r - 1/p_1')s_1}\, dy \right)^{1/s_1} \\
&+ C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} |K(x,y) - K(x_0,y)|^q\, dy \right)^{1/q} \\
&\times \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^r w(y)\, dy \right)^{1/r} \\
&\times |(D^\alpha b)_{2^{k+1}\tilde{Q}} - (D^\alpha b)_{\tilde{Q}}| \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s_2/r}\, dy \right)^{1/s_2} \\
&+ C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1}} \\
&\times |f(y)| |D^\alpha b(y) - (D^\alpha b)_{2^{k+1}\tilde{Q}}| w(y)^{1/r} w(y)^{-1/r}\, dy \\
&+ C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1}} \\
&\times |f(y)| |(D^\alpha b)_{2^{k+1}\tilde{Q}} - (D^\alpha b)_{\tilde{Q}}| w(y)^{1/r} w(y)^{-1/r}\, dy \\
\leq{}& C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} C_k |2^k Q|^{-1/q'} \|D^\alpha b\|_{BMO(w)} w(2^k\tilde{Q})^{1/p_1} w(2^k\tilde{Q})^{1/r} \\
&\times \left(\frac{1}{w(2^k\tilde{Q})} \int_{2^k\tilde{Q}} |f(y)|^r w(y)\, dy \right)^{1/r}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)\,dy\right)^{1/r-1/p_1'}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)^{-(1/r-1/p_1')s_1}\,dy\right)^{1/s_1}\\
&\times \left|2^k\tilde{Q}\right|^{1/s_1+1/r-1/p_1'} w\left(2^k\tilde{Q}\right)^{-1/r+1/p_1'}\\
&+ C\sum_{|\alpha|=m}\sum_{k=1}^{\infty} C_k\left|2^kQ\right|^{-1/q'} k\left\|D^\alpha b\right\|_{BMO(w)} w(\tilde{x}) w\left(2^k\tilde{Q}\right)^{1/r}\\
&\times \left(\frac{1}{w(2^k\tilde{Q})}\int_{2^k\tilde{Q}}\left|f(y)\right|^r w(y)\,dy\right)^{1/r}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)\,dy\right)^{1/r}\\
&\times \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)^{-s_2/r}\,dy\right)^{1/s_2}\left|2^k\tilde{Q}\right|^{1/s_2+1/r} w\left(2^k\tilde{Q}\right)^{-1/r}\\
&+ C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}\frac{d}{(2^kd)^{n+1}}\left(\int_{2^k\tilde{Q}}\left|D^\alpha b(y)-\left(D^\alpha b\right)_{2^k\tilde{Q}}\right|^{r'} w(y)^{1-r'}\,dy\right)^{1/r'}\\
&\times \left(\int_{2^{k+1}\tilde{Q}}\left|f(y)\right|^r w(y)\,dy\right)^{1/r}\\
&+ C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}\frac{d}{(2^kd)^{n+1}} k\left\|D^\alpha b\right\|_{BMO(w)} w(\tilde{x})\left(\int_{2^k\tilde{Q}} w(y)^{-r'/r}\,dy\right)^{1/r'}\\
&\times \left(\int_{2^{k+1}\tilde{Q}}\left|f(y)\right|^r w(y)\,dy\right)^{1/r}\\
\le{}& \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(w)}\sum_{k=1}^{\infty} C_k\frac{w(2^k\tilde{Q})}{|2^k\tilde{Q}|}\left(\frac{1}{w(2^k\tilde{Q})}\int_{2^k\tilde{Q}}\left|f(y)\right|^r w(y)\,dy\right)^{1/r}\\
&+ \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(w)} w(\tilde{x})\sum_{k=1}^{\infty} kC_k\left(\frac{1}{w(2^k\tilde{Q})}\int_{2^k\tilde{Q}}\left|f(y)\right|^r w(y)\,dy\right)^{1/r}\\
&+ C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(w)}\sum_{k=1}^{\infty} 2^{-k}\frac{w(2^k\tilde{Q})}{|2^k\tilde{Q}|}\left(\frac{1}{w(2^k\tilde{Q})}\int_{2^k\tilde{Q}}\left|f(y)\right|^r w(y)\,dx\right)^{1/r}\\
&+ C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(w)} w(\tilde{x})\sum_{k=1}^{\infty} k2^{-k}\left(\frac{1}{w(2^k\tilde{Q})}\int_{2^k\tilde{Q}}\left|f(y)\right|^r w(y)\,dx\right)^{1/r}\\
\le{}& C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
\end{aligned}
$$

Thus

$$
I_3 \le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
$$

These complete the proof of Theorem 1. □

Proof of Theorem 2 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 that the following inequality holds:

$$
\left(\frac{1}{|Q|}\int_Q\left|T^b(f)(x)-C_0\right|^\eta\,dx\right)^{1/\eta} \le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).
$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - T^{\tilde{b}}(f_2)(x_0)\right|^\eta dx\right)^{1/\eta} \\
&\quad\le C\left(\frac{1}{|Q|}\int_Q \left|T\left(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m}f_1\right)\right|^\eta dx\right)^{1/\eta} \\
&\qquad + C\left(\frac{1}{|Q|}\int_Q \left|T\left(\sum_{|\alpha|=m}\frac{(x-\cdot)^\alpha D^\alpha\tilde{b}}{|x-\cdot|^m}f_1\right)\right|^\eta dx\right)^{1/\eta} \\
&\qquad + C\left(\frac{1}{|Q|}\int_Q \left|T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)\right|^\eta dx\right)^{1/\eta} \\
&\quad= J_1 + J_2 + J_3.
\end{aligned}$$

For J_1 and J_2, by using the same argument as in the proof of Theorem 1, we get

$$\begin{aligned}
&\left|R_m(\tilde{b};x,y)\right| \\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}|\tilde{Q}|^{-1/q}\left(\int_{\tilde{Q}(x,y)}\left|D^\alpha\tilde{b}(z)\right|^q w(z)^{q(1-r)/r}w(z)^{q(r-1)/r}\,dz\right)^{1/q} \\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}|\tilde{Q}|^{-1/q}\left(\int_{\tilde{Q}(x,y)}\left|D^\alpha\tilde{b}(z)\right|^r w(z)^{1-r}\,dz\right)^{1/r} \\
&\qquad\times\left(\int_{\tilde{Q}(x,y)}w(z)^{q(r-1)/(r-q)}\,dz\right)^{(r-q)/rq} \\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}|\tilde{Q}|^{-1/q}\left\|D^\alpha b\right\|_{Lip_\beta(w)}w(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{(r-q)/rq} \\
&\qquad\times\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}w(z)^{p_0}\,dz\right)^{(r-q)/rq} \\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{Lip_\beta(w)}|\tilde{Q}|^{-1/q}w(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{1/q-1/r} \\
&\qquad\times\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}w(z)\,dz\right)^{(r-1)/r} \\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{Lip_\beta(w)}|\tilde{Q}|^{-1/q}w(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{1/q-1/r}w(\tilde{Q})^{1-1/r}|\tilde{Q}|^{1/r-1} \\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{Lip_\beta(w)}\frac{w(\tilde{Q})^{\beta/n+1}}{|\tilde{Q}|} \\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{Lip_\beta(w)}w(\tilde{Q})^{\beta/n}w(\tilde{x}).
\end{aligned}$$

Thus

$$\begin{aligned}
J_1 &\le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{Lip_\beta(w)}w(\tilde{Q})^{\beta/n}w(\tilde{x})|Q|^{-1/s}\left(\int_{R^n}\left|f_1(x)\right|^s dx\right)^{1/s} \\
&\le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{Lip_\beta(w)}w(\tilde{Q})^{\beta/n}w(\tilde{x})|Q|^{-1/s}
\end{aligned}$$

$$\times \left(\int_{\tilde{Q}} \left|f(x)\right|^r w(x)\,dx\right)^{1/r} \left(\int_{\tilde{Q}} w(x)^{-s/(r-s)}\,dx\right)^{(r-s)/rs}$$

$$\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{Lip_\beta(w)} w(\tilde{x}) |\tilde{Q}|^{-1/s} w(\tilde{Q})^{1/r} \left(\frac{1}{w(\tilde{Q})^{1-r\beta/n}} \int_{\tilde{Q}} \left|f(x)\right|^r w(x)\,dx\right)^{1/r}$$

$$\times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{-s/(r-s)}\,dx\right)^{(r-s)/rs} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)\,dx\right)^{1/r} |\tilde{Q}|^{1/s} w(\tilde{Q})^{-1/r}$$

$$\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}),$$

$$J_2 \le C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{\tilde{Q}} \left|D^\alpha b(x) - \left(D^\alpha b\right)_{\tilde{Q}}\right| w(x)^{-1/r} \left|f(x)\right| w(x)^{1/r}\,dx$$

$$\le C \sum_{|\alpha|=m} \frac{1}{|Q|} \left(\int_{\tilde{Q}} \left|\left(D^\alpha b(x) - \left(D^\alpha b\right)_{\tilde{Q}}\right)\right|^{r'} w(x)^{1-r'}\,dx\right)^{1/r'} \left(\int_{\tilde{Q}} \left|f(x)\right|^r w(x)\,dx\right)^{1/r}$$

$$\le C \sum_{|\alpha|=m} \frac{1}{|Q|} \left\|D^\alpha b\right\|_{Lip_\beta(w)} w(\tilde{Q})^{\beta/n+1/r'} w(\tilde{Q})^{1/r-\beta/n} \left(\frac{1}{w(\tilde{Q})^{1-r\beta/n}} \int_{\tilde{Q}} \left|f(x)\right|^r w(x)\,dx\right)^{1/r}$$

$$\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{Lip_\beta(w)} \frac{w(\tilde{Q})}{|\tilde{Q}|} M_{\beta,r,w}(f)(\tilde{x})$$

$$\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).$$

For J_3, we have

$$\left|R_m(\tilde{b};x,y) - R_m(\tilde{b};x_0,y)\right|$$
$$\le C \sum_{|\gamma|<m} \sum_{|\alpha|=m} |x-x_0|^{m-|\gamma|} |x-y|^{|\gamma|} \left\|D^\alpha b\right\|_{Lip_\beta(w)} w(\tilde{x}) w\left(2^k \tilde{Q}\right)$$

and

$$\left|R_m(\tilde{b};x,y)\right| \le C \sum_{|\alpha|=m} |x-x_0|^m \left\|D^\alpha b\right\|_{Lip_\beta(w)} w(\tilde{x}) w\left(2^k \tilde{Q}\right).$$

Thus, for $1 < p < \infty$ with $1/p + 1/q + 1/r = 1$ and $r < p_1' < \infty$, $1 < s_1, s_2 < \infty$ with $1/p_1 + 1/q + 1/r + 1/s_1 = 1$ and $1/q + 1/r + 1/s_2 = 1$, we obtain

$$\left|T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)\right|$$
$$\le \int_{R^n} \left|R_m(\tilde{b};x,y) - R_m(\tilde{b};x_0,y)\right| \frac{|K(x,y)|}{|x-y|^m} \left|f_2(y)\right|\,dy$$
$$+ \int_{R^n} \left|\frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m}\right| \left|R_m(\tilde{b};x_0,y)\right| \left|f_2(y)\right|\,dy$$
$$+ \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left|\frac{K(x,y)(x-y)^\alpha}{|x-y|^m} - \frac{K(x_0,y)(x_0-y)^\alpha}{|x_0-y|^m}\right| \left|D^\alpha \tilde{b}(y)\right| \left|f_2(y)\right|\,dy$$
$$\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=0}^{\infty} w\left(2^{k+1}\tilde{Q}\right)^{\beta/n} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1}} \left|f(y)\right|\,dy$$

$$
\begin{aligned}
&+ C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x})\sum_{k=0}^{\infty} w\big(2^{k+1}\tilde{Q}\big)^{\beta/n}\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\big|K(x,y)-K(x_0,y)\big|^q\,dy\right)^{1/q}\\
&\times\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\big|f(y)\big|^r w(y)\,dy\right)^{1/r}\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} w(y)^{-p/r}\,dy\right)^{1/p}\\
&+ C\sum_{|\alpha|=m}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}\big|D^\alpha b(y)-(D^\alpha b)_{2^{k+1}\tilde{Q}}\big|\big|f(y)\big|\,dy\\
&+ C\sum_{|\alpha|=m}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}\big|(D^\alpha b)_{2^{k+1}\tilde{Q}}-(D^\alpha b)_{\tilde{Q}}\big|\big|f(y)\big|\,dy\\
&+ C\sum_{|\alpha|=m}\sum_{k=0}^{\infty}\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\big|K(x,y)-K(x_0,y)\big|^q\,dy\right)^{1/q}\left(\int_{2^{k+1}\tilde{Q}}\big|f(y)\big|^r w(y)\,dy\right)^{1/r}\\
&\times\left(\int_{2^{k+1}\tilde{Q}}\big|D^\alpha b(y)-(D^\alpha b)_{2^{k+1}\tilde{Q}}\big|^{p_1} w(y)^{1-p_1}\,dy\right)^{1/p_1}\left(\int_{2^{k+1}\tilde{Q}} w(y)^{-(1/r-1/p_1')s_1}\,dy\right)^{1/s_1}\\
&+ C\sum_{|\alpha|=m}\sum_{k=0}^{\infty}\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\big|K(x,y)-K(x_0,y)\big|^q\,dy\right)^{1/q}\left(\int_{2^{k+1}\tilde{Q}}\big|f(y)\big|^r w(y)\,dy\right)^{1/r}\\
&\times\big|(D^\alpha b)_{2^{k+1}\tilde{Q}}-(D^\alpha b)_{\tilde{Q}}\big|\left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s_2/r}\,dy\right)^{1/s_2}\\
\le{}& C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x})\sum_{k=1}^{\infty} k\frac{d}{(2^k d)^{n+1}} w\big(2^k\tilde{Q}\big)^{\beta/n}\left(\int_{2^k\tilde{Q}}\big|f(y)\big|^r w(y)\,dx\right)^{1/r}\\
&\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)^{-1/(r-1)}\,dy\right)^{(r-1)/r}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)\,dy\right)^{1/r}\big|2^k\tilde{Q}\big| w\big(2^k\tilde{Q}\big)^{-1/r}\\
&+ C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x})\sum_{k=1}^{\infty} C_k\big|2^k Q\big|^{-1/q'} w\big(2^k\tilde{Q}\big)^{\beta/n}\left(\int_{2^k\tilde{Q}}\big|f(y)\big|^r w(y)\,dx\right)^{1/r}\\
&\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)^{-p/r}\,dy\right)^{1/p}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)\,dy\right)^{1/r}\big|2^k\tilde{Q}\big|^{1/p+1/r} w\big(2^k\tilde{Q}\big)^{-1/r}\\
&+ C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}\frac{d}{(2^k d)^{n+1}}\left(\int_{2^k\tilde{Q}}\big|D^\alpha b(y)-(D^\alpha b)_{2^k\tilde{Q}}\big|^{r'} w(y)^{1-r'}\,dy\right)^{1/r'}\\
&\times\left(\int_{2^k\tilde{Q}}\big|f(y)\big|^r w(y)\,dy\right)^{1/r}\\
&+ C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x})\sum_{k=1}^{\infty} C_k\big|2^k Q\big|^{-1/q'} w\big(2^k\tilde{Q}\big)^{\beta/n}\left(\int_{2^k\tilde{Q}}\big|f(y)\big|^r w(y)\,dx\right)^{1/r}\\
&\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)^{-(1/r-1/p_1')s_1}\,dy\right)^{1/s_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)\,dy\right)^{1/r-1/p_1'}\\
&\times\big|2^k\tilde{Q}\big|^{1/s_1+1/r-1/p_1'} w\big(2^k\tilde{Q}\big)^{-1/r+1/p_1'}\\
&+ C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x})\sum_{k=1}^{\infty} kC_k\big|2^k Q\big|^{-1/q'} w\big(2^k\tilde{Q}\big)^{\beta/n}\left(\int_{2^k\tilde{Q}}\big|f(y)\big|^r w(y)\,dy\right)^{1/r}\\
&\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)\,dy\right)^{1/r}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)^{-s_2/r}\,dy\right)^{1/s_2}\big|2^k\tilde{Q}\big|^{1/s_2+1/r} w\big(2^k\tilde{Q}\big)^{-1/r}
\end{aligned}
$$

$$\begin{aligned}
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}w(\tilde{x})\sum_{k=1}^\infty k\left(C_k+2^{-k}\right)\\
&\quad\times\left(\frac{1}{w(2^k\tilde{Q})^{1-r\beta/n}}\int_{2^k\tilde{Q}}\left|f(y)\right|^r w(y)\,dx\right)^{1/r}\\
&\quad+C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}\sum_{k=1}^\infty k\left(C_k+2^{-k}\right)\frac{w(2^k\tilde{Q})}{|2^k\tilde{Q}|}\\
&\quad\times\left(\frac{1}{w(2^k\tilde{Q})^{1-r\beta/n}}\int_{2^k\tilde{Q}}\left|f(y)\right|^r w(y)\,dx\right)^{1/r}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}w(\tilde{x})M_{\beta,r,w}(f)(\tilde{x}).
\end{aligned}$$

This completes the proof of Theorem 2. □

Proof of Theorem 3 Choose $1<r<u$ in Theorem 1 and notice $w^{1-u}\in A_1$, then we have, by Lemmas 3 and 4,

$$\begin{aligned}
\left\|T^b(f)\right\|_{L^u(w^{1-u})} &\le \left\|M_\eta\left(T^b(f)\right)\right\|_{L^u(w^{1-u})}\le C\left\|M_\eta^\#\left(T^b(f)\right)\right\|_{L^u(w^{1-u})}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}\left\|wM_{r,w}(f)\right\|_{L^u(w^{1-u})}\\
&= C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}\left\|M_{r,w}(f)\right\|_{L^u(w)}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}\|f\|_{L^u(w)}.
\end{aligned}$$

This completes the proof of Theorem 3. □

Proof of Theorem 4 Choose $1<r<u$ in Theorem 1 and notice $w^{1-u}\in A_1$, then we have, by Lemmas 5 and 6,

$$\begin{aligned}
\left\|T^b(f)\right\|_{L^{u,\varphi}(w^{1-u})} &\le \left\|M_\eta\left(T^b(f)\right)\right\|_{L^{u,\varphi}(w^{1-u})}\le C\left\|M_\eta^\#\left(T^b(f)\right)\right\|_{L^{u,\varphi}(w^{1-u})}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}\left\|wM_{r,w}(f)\right\|_{L^{u,\varphi}(w^{1-u})}\\
&= C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}\left\|M_{r,w}(f)\right\|_{L^{u,\varphi}(w)}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}\|f\|_{L^{u,\varphi}(w)}.
\end{aligned}$$

This completes the proof of Theorem 4. □

Proof of Theorem 5 Choose $1<r<u$ in Theorem 2 and notice $w^{1-v}\in A_1$, then we have, by Lemmas 3 and 4,

$$\begin{aligned}
\left\|T^b(f)\right\|_{L^v(w^{1-v})} &\le \left\|M_\eta\left(T^b(f)\right)\right\|_{L^v(w^{1-v})}\le C\left\|M_\eta^\#\left(T^b(f)\right)\right\|_{L^v(w^{1-v})}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}\left\|wM_{\beta,r,w}(f)\right\|_{L^v(w^{1-v})}
\end{aligned}$$

$$= C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|M_{\beta,r,w}(f)\|_{L^v(w)}$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|f\|_{L^u(w)}.$$

This completes the proof of Theorem 5. □

Proof of Theorem 6 Choose $1 < r < u$ in Theorem 2 and notice $w^{1-v} \in A_1$, then we have, by Lemmas 5 and 6,

$$\|T^b(f)\|_{L^{v,\varphi}(w^{1-v})} \leq \|M_\eta(T^b(f))\|_{L^{v,\varphi}(w^{1-v})} \leq C\|M_\eta^\#(T^b(f))\|_{L^{v,\varphi}(w^{1-v})}$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|wM_{\beta,r,w}(f)\|_{L^{v,\varphi}(w^{1-v})}$$
$$= C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|M_{\beta,r,w}(f)\|_{L^{v,\varphi}(w)}$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|f\|_{L^{u,\varphi}(w)}.$$

This completes the proof of Theorem 6. □

Competing interests
The author declares that they have no competing interests.

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