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Geometric properties of certain analytic functions associated with generalized fractional integral operators

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Abstract

Let $\mathcal{A}$ be the class of normalized analytic functions in the unit disk $\mathcal{U}$ and define the class $\mathcal{P}(\beta) = \{f \in \mathcal{A} : \exists\varphi \in \mathbb{R}$ such that $\operatorname{Re}[e^{i\varphi}(f'(z) - \beta)] > 0, z \in \mathcal{U}\}$. In this paper we find conditions on the number β and the non-negative weight function $\lambda(t)$ such that the integral transform $V_\lambda(f)(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t}\,dt$ is convex of order γ $(0 \le \gamma \le 1/2)$ when $f \in \mathcal{P}(\beta)$. Some interesting further consequences are also considered.

MSC: Primary 30C45; secondary 33C50

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1 Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{S}$, $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are univalent, starlike of order γ and convex of order γ in $\mathcal{U}$, respectively. In particular, the classes $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$ are the familiar ones of starlike and convex functions in $\mathcal{U}$, respectively.

We note that

$$f(z) \in \mathcal{K}(\gamma) \quad \Longleftrightarrow \quad zf'(z) \in \mathcal{S}^*(\gamma) \tag{1.2}$$

for $0 \le \gamma < 1$.

Let a, b, and c be complex numbers with $c \ne 0, -1, -2, \ldots$. Then the *Gaussian hypergeometric function* ${}_2F_1$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.3}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}). \end{cases}$$

For functions $f_j(z)$ $(j=1,2)$ of the forms

$$f_j(z) := \sum_{n=0}^{\infty} a_{j,n+1} z^{n+1} \quad (a_{j,1} := 1; j = 1,2),$$

let $(f_1 * f_2)(z)$ denote the *Hadamard product* or *convolution* of $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 * f_2)(z) := \sum_{n=0}^{\infty} a_{1,n+1} a_{2,n+1} z^{n+1} \quad (a_{j,1} := 1; j = 1,2).$$

By using (1.3), Hohlov [1] introduced the convolution operator $H_{a,b,c}$ by

$$H_{a,b,c}(f)(z) := z\,{}_2F_1(a,b;c;z) * f(z) \tag{1.4}$$

for $f \in \mathcal{A}$. The three-parameter family of operators given by (1.4) contains as special cases several of the known linear integral or differential operators studied by a number of authors. This operator has been studied extensively by Ponnusamy [2], Kim and Rønning [3] and many others [4, 5]. In particular, if $a = 1$ in (1.4), then $H_{1,b,c}$ is the operator $\mathcal{L}(b,c)$ due to Carlson and Shaffer [6] which was defined by

$$\mathcal{L}(b,c)f(z) = z\,{}_2F_1(1,b;c;z) * f(z).$$

Clearly, $\mathcal{L}(b,c)$ maps $\mathcal{A}$ onto itself, and $\mathcal{L}(c,b)$ is the inverse of $\mathcal{L}(b,c)$, provided that $b \neq 0,-1,-2,\ldots$. Furthermore, $\mathcal{L}(b,b)$ is the unit operator and

$$\mathcal{L}(b,c) = \mathcal{L}(b,e)\mathcal{L}(e,c) = \mathcal{L}(e,c)\mathcal{L}(b,e) \quad (c,e \neq 0,-1,-2,\ldots). \tag{1.5}$$

Also, we note that

$$\mathcal{K}(\gamma) = \mathcal{L}(1,2)\mathcal{S}^*(\gamma) \quad (0 \leq \gamma < 1)$$

and

$$\mathcal{S}^*(\gamma) = \mathcal{L}(2,1)\mathcal{K}(\gamma) \quad (0 \leq \gamma < 1). \tag{1.6}$$

Various definitions of fractional calculus operators are given by many authors. We use here the following definition due to Saigo [7] (see also [5, 8]).

Definition 1 For $\lambda > 0$, $\mu, \nu \in \mathbb{R}$, the fractional integral operator $\mathcal{I}_{0,z}^{\lambda,\mu,\nu}$ is defined by

$$\mathcal{I}_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-\zeta)^{\lambda-1} {}_2F_1\left(\lambda+\mu, -\nu; \lambda; 1-\frac{\zeta}{z}\right) f(\zeta)\, d\zeta,$$

where $f(z)$ is taken to be an analytic function in a simply connected region of the z-plane containing the origin with the order

$$f(z) = \mathcal{O}\bigl(|z|^{\epsilon}\bigr) \quad (z \to 0)$$

for $\epsilon > \max\{0, \mu - \nu\} - 1$, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$. With the aid of the above definition, Owa *et al.* [9] defined a modification of the fractional integral operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ by

$$\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(2-\mu+\nu)} z^{\mu} \mathcal{I}_{0,z}^{\lambda,\mu,\nu} f(z)$$

for $f(z) \in \mathcal{A}$ and $\min\{\lambda+\nu, -\mu+\nu, -\mu\} > -2$. Then it is observed that $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ also maps $\mathcal{A}$ onto itself and

$$\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) = \mathcal{L}(2, 2-\mu)\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu) f(z). \tag{1.7}$$

The function

$$s_{\alpha}(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \quad (0 \leq \alpha < 1)$$

is the well-known extremal function for the class $\mathcal{S}^*(\alpha)$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\alpha, \gamma)$ if

$$(f * s_{\alpha})(z) \in \mathcal{S}^*(\gamma) \quad (0 \leq \alpha < 1; 0 \leq \gamma < 1).$$

Note that

$$\mathcal{R}(\alpha, \gamma) = \mathcal{L}(1, 2-2\alpha)\mathcal{S}^*(\gamma) \tag{1.8}$$

and $\mathcal{R}(\alpha, \alpha) \equiv \mathcal{R}(\alpha)$ is the subclass of $\mathcal{A}$ consisting of *prestarlike functions of order* α which was introduced by Suffridge [10]. In [11], it is shown that $\mathcal{R}(\alpha) \subset \mathcal{S}$ if and only if $\alpha \leq 1/2$. For $\beta < 1$ we denote the class

$$\mathcal{P}(\beta) = \bigl\{ f \in \mathcal{A} : \exists \varphi \in \mathbb{R} \text{ such that } \operatorname{Re}\bigl[e^{i\varphi}\bigl(f'(z) - \beta\bigr)\bigr] > 0, z \in \mathcal{U} \bigr\}.$$

Throughout this paper we let $\lambda : [0,1] \to \mathbb{R}$ be a non-negative function with

$$\int_0^1 \lambda(t)\, dt = 1. \tag{1.9}$$

For certain specific subclasses of $f \in \mathcal{A}$, many authors considered the geometric properties of the integral transform of the form

$$V_{\lambda}(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t}\, dt. \tag{1.10}$$

More recently, starlikeness of this general operator $V_{\lambda}(f)$ was discussed by Fournier and Ruscheweyh [12] by assuming that $f \in \mathcal{P}(\beta)$. The method of proof is the duality principle developed mainly by Ruscheweyh [13]. This result was later extended by Ponnusamy and Rønning [14] by means of finding conditions such that $V_\lambda(f)$ carries $\mathcal{P}(\beta)$ into starlike functions of order γ, $0 \leq \gamma \leq 1/2$.

In this paper, we find conditions on β and the function $\lambda(t)$ such that $V_\lambda(f)$ carries $\mathcal{P}(\beta)$ into $\mathcal{K}(\gamma)$. As a consequence of this investigation, a number of new results are established.

2 Preliminaries

We begin by recalling the following results.

Lemma 1 ([15]; see also [5]) *If $f \in \mathcal{A}$ and $c - a + 1 > b > 0$, then*

$$H_{a,b,c}(f)(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{(1-t)^{c-a-b}}{\Gamma(c-a-b+1)} t^{b-2} {}_2F_1(c-a, 1-a; c-a-b+1; 1-t) f(tz)\, dt.$$

Remark 1 In view of Lemma 1, we see that the convolution operator (1.4) is an integral operator of the form (1.10) with

$$\lambda(t) = \frac{\Gamma(c)t^{b-1}(1-t)^{c-a-b}}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} {}_2F_1(c-a, 1-a; c-a-b+1; 1-t).$$

For $\Lambda : [0,1] \to \mathbb{R}$ being integrable and positive on $(0,1)$, we define

$$L_\Lambda(h_\gamma) = \inf_{z \in \mathcal{U}} \int_0^1 \Lambda(t) \left[\operatorname{Re} \frac{h_\gamma(zt)}{zt} - \frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2} \right] dt$$

and

$$M_\Lambda(h_\gamma) = \inf_{z \in \mathcal{U}} \int_0^1 \Lambda(t) \left[\operatorname{Re} h'_\gamma(zt) - \frac{1-t-\gamma(1+t)}{(1-\gamma)(1+t)^3} \right] dt,$$

where $0 \leq \gamma < 1$ and

$$h_\gamma(z) = \frac{z(1 + \frac{\epsilon+2\gamma-1}{2-2\gamma} z)}{(1-z)^2}, \quad |\epsilon| = 1. \tag{2.1}$$

In [16], Ponnusamy and Rønning proved the following lemmas.

Lemma 2 *Let $\Lambda(t)$ be integrable on $[0,1]$ and positive on $(0,1)$. If $\Lambda(t)/(1+t)(1-t)^{1+2\gamma}$ is decreasing on $(0,1)$, then for $0 \leq \gamma \leq 1/2$ we have $L_\Lambda(h_\gamma) \geq 0$.*

Lemma 3 *Let $0 \leq \gamma < 1$ and let $\lambda(t)$ be given by (1.9). Define $\beta < 1$ by*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) \left[\frac{1+\gamma-(1-\gamma)t}{(1-\gamma)(1+t)} - \frac{2\gamma}{1-\gamma} \frac{\log(1+t)}{t} \right] dt.$$

Assume that $\lim_{t\to 0+} t\Lambda(t) = 0$, where

$$\Lambda(t) = \int_t^1 \lambda(s)\, ds/s.$$

Then $V_\lambda(\mathcal{P}(\beta)) \subset \mathcal{S}^*(\gamma)$ if and only if $L_\Lambda(h_\gamma) \geq 0$.

We now find conditions on β and the non-negative weight function $\lambda(t)$ such that $V_\lambda(\mathcal{P}(\beta)) \subset \mathcal{K}(\gamma)$.

Lemma 4 (i) *Let* $\Lambda(t)$ *be monotone decreasing on* $[0,1]$ *satisfying* $\Lambda(1) = 0$ *and* $\lim_{t\to 0+} t\Lambda(t) = 0$. *For* $0 \le \gamma \le 1/2$ *if* $t\Lambda'(t)/(1+t)(1-t)^{1+2\gamma}$ *is increasing on* $(0,1)$, *then* $M_\Lambda(h_\gamma) \ge 0$.

(ii) *Let* $0 \le \gamma \le 1/2$ *and let* $\lambda(t)$ *and* $\Lambda(t)$ *be as in Lemma* 3. *Define* $\beta < 1$ *by*

$$\frac{\beta - \frac{1}{2}}{1-\beta} = -\int_0^1 \lambda(t) \frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\,dt.$$

Then $V_\lambda(\mathcal{P}(\beta)) \subset \mathcal{K}(\gamma)$ *if and only if* $M_\Lambda(h_\gamma) \ge 0$.

Proof (i) Let $M_\Lambda(h_\gamma) = \inf_{z\in\mathcal{U}} I_\gamma$. Then, by using the conditions $\Lambda(1) = 0$ and $\lim_{t\to 0+} t\Lambda(t) = 0$, an integration by parts yields

$$\begin{aligned} I_\gamma &= \int_0^1 \Lambda(t)\left[\operatorname{Re} h'_\gamma(zt) - \frac{1-t-\gamma(1+t)}{(1-\gamma)(1+t)^3}\right]dt \\ &= \int_0^1 \Lambda(t)\frac{d}{dt}\left[\operatorname{Re}\frac{h_\gamma(zt)}{z} - \frac{t(1-\gamma(1+t))}{(1-\gamma)(1+t)^2}\right]dt \\ &= -\int_0^1 t\Lambda'(t)\left[\operatorname{Re}\frac{h_\gamma(zt)}{zt} - \frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\right]dt. \end{aligned}$$

Since $t\Lambda'(t)/(1+t)(1-t)^{1+2\gamma}$ is increasing on $(0,1)$, by Lemma 2, $\inf_{z\in\mathcal{U}} I_\gamma \ge 0$, which evidently completes the proof of (i).

(ii) We state this proof only in outline here because the proof is similar to that of [3, Theorem 2.1]. Let $F(z) = V_\lambda(f)(z)$. Then, by convolution theory [13, p.94] and (1.2), we have

$$F(z) \in \mathcal{K}(\gamma) \quad\Longleftrightarrow\quad \frac{1}{z}\bigl(zF'(z) * h_\gamma(z)\bigr) \neq 0, \tag{2.2}$$

where $h_\gamma(z)$ is given by (2.1). Since $f \in \mathcal{P}(\beta)$, by the duality principle [13, p.23], it is enough to verify this with f given by

$$f'(z) = (1-\beta)\frac{1-xz}{1-yz} + \beta \quad \bigl(|x| = |y| = 1\bigr).$$

In the same way as in [3, Theorem 2.1], we conclude that (2.2) holds if and only if

$$\operatorname{Re}\int_0^1 \lambda(t)\left[\frac{h_\gamma(zt)}{zt} - \frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\right]dt > 0. \tag{2.3}$$

Integrating by parts, we find that the inequality (2.3) is equivalent to

$$\operatorname{Re}\int_0^1 \Lambda(t)\left[h'_\gamma(zt) - \frac{1-t-\gamma(1+t)}{(1-\gamma)(1+t)^3}\right]dt \ge 0,$$

which again is equivalent to $M_\Lambda(h_\gamma) \ge 0$. □

Remark 2 In particular, taking $\gamma = 0$ in Lemma 4, we obtain the result due to Ali and Singh [17, Theorem 1].

3 Main results

We define

$$\varphi(1-t) = 1 + \sum_{n=1}^{\infty} b_n(1-t)^n \quad (b_n \geq 0) \tag{3.1}$$

and

$$\lambda(t) = Ct^{b-1}(1-t)^{c-a-b}\varphi(1-t), \tag{3.2}$$

where C is a constant satisfying the condition (1.9). For $f \in \mathcal{A}$ Balasubramanian *et al.* [4] defined the operator $P_{a,b,c}$ by

$$P_{a,b,c}(f)(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t}\,dt,$$

where $\lambda(t)$ is given by (3.2). Special choices of $\varphi(1-t)$ and C led to various interesting geometric properties concerning certain linear operators. For example, if we take $\varphi(1-t) = {}_2F_1(c-a, 1-a; c-a-b+1; 1-t)$ and

$$C = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)},$$

by virtue of Remark 1,

$$P_{a,b,c}(f)(z) = H_{a,b,c}(f)(z). \tag{3.3}$$

First, by applying Lemma 4, we prove the following.

Theorem 1 *Let* $0 \leq \gamma \leq 1/2$, $a > 0$, $0 < b \leq 1$, *and* $c \geq a + b + 2\gamma + 1$, *and let* $\lambda(t)$ *be given by* (3.2). *Define* $\beta = \beta(a, b, c, \gamma)$ *by*

$$\frac{\beta - \frac{1}{2}}{1-\beta} = -\int_0^1 \lambda(t)\frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\,dt.$$

If $f(z) \in \mathcal{P}(\beta)$, *then* $P_{a,b,c}(f)(z) \in \mathcal{K}(\gamma)$. *The value of* β *is sharp.*

Proof Let $C > 0$ and

$$\Lambda(t) = \int_t^1 \frac{\lambda(s)}{s}\,ds,$$

where $\lambda(t)$ is given by (3.2). Then it is easily seen that $\Lambda(t)$ is monotone decreasing on $[0,1]$ and $\lim_{t\to 0+} t\Lambda(t) = 0$. In order to apply Lemma 4, we want to prove that the function

$$u(t) = \frac{\lambda(t)}{(1+t)(1-t)^{1+2\gamma}} \tag{3.4}$$

is decreasing on $(0,1)$, where $\lambda(t)$ is given by (3.2). Making use of the logarithmic differentiation of both sides in (3.4), we have

$$\frac{u'(t)}{u(t)} = \frac{\lambda'(t)}{\lambda(t)} + \frac{2(\gamma + (1+\gamma)t)}{1-t^2}. \tag{3.5}$$

Since

$$\lambda'(t) = Ct^{b-2}(1-t)^{c-a-b-1}\big[\varphi(1-t)\big\{(b-1)(1-t) - t(c-a-b)\big\} - t(1-t)\varphi'(1-t)\big],$$

from (3.4) and (3.5) we find that $u'(t) \le 0$ on $(0,1)$ is equivalent to

$$(c-a-3-2\gamma)t^2 + (c-a-b-2\gamma)t + 1 - b \ge -t\big(1-t^2\big)\frac{\varphi'(1-t)}{\varphi(1-t)} \quad (0 < t < 1). \tag{3.6}$$

In view of (3.1), $\varphi(1-t) > 0$ and $\varphi'(1-t) \ge 0$ on $(0,1)$, so that the right hand side of the inequality (3.6) is non-positive for all $t \in (0,1)$. If we assume that $0 \le \gamma \le 1/2$, $a > 0$, $0 < b \le 1$, and $c \ge a + b + 2\gamma + 1$, then $(c-a-3-2\gamma)t^2 + (c-a-b-2\gamma)t + 1 - b \ge 0$ for $t \in (0,1)$. Thus, the inequality (3.6) holds for all $t \in (0,1)$. Hence, from Lemma 4 we obtain $P_{a,b,c}(f)(z) \in \mathcal{K}(\gamma)$. □

The same techniques as in the proof of [5, Theorem 1] show that the value β is sharp. By using (3.3) and Theorem 1, we have the following.

Corollary 1 *Let* $0 \le \gamma \le 1/2$, $0 < a \le 1$, $0 < b \le 1$, *and* $c \ge a + b + 2\gamma + 1$. *Define* $\beta = \beta(a,b,c,\gamma)$ *by*

$$\begin{aligned}\frac{\beta - \frac{1}{2}}{1-\beta} = -\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{(1-t)^{c-a-b}t^{b-1}}{\Gamma(c-a-b+1)} \frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2} \\ \times {}_2F_1(c-a, 1-a; c-a-b+1; 1-t)\,dt.\end{aligned}$$

If $f(z) \in \mathcal{P}(\beta)$, *then* $H_{a,b,c}(f)(z) \in \mathcal{K}(\gamma)$. *The value of* β *is sharp.*

Proof If we put

$$\lambda(t) = \frac{\Gamma(c)t^{b-1}(1-t)^{c-a-b}}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}\,{}_2F_1(c-a, 1-a; c-a-b+1; 1-t),$$

then, by applying (3.3) and Theorem 1, we obtain the desired result. □

Setting $a = 1$ in Corollary 1, we obtain the following.

Corollary 2 *Let* $0 \le \gamma \le 1/2$, $0 < b \le 1$, *and* $c \ge b + 2\gamma + 2$. *Also let*

$$\beta(1,b,c,\gamma) = 1 - \frac{1-\gamma}{2[1 - {}_2F_1(2,b;c;-1) - \gamma(1 - {}_2F_1(1,b;c;-1))]}.$$

If $\beta(1,b,c,\gamma) < \beta < 1$ *and* $f(z) \in \mathcal{P}(\beta)$, *then* $\mathcal{L}(b,c)f(z) \in \mathcal{K}(\gamma)$.

Next we find a univalence criterion for the operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$.

Theorem 2 *Let* $0 \le \gamma \le 1/2$, $0 \le \mu < 2$, $\lambda \ge 2(1+\gamma) - \mu$, *and* $\mu - 2 < \nu \le \mu - 1$. *Define* $\beta = \beta(\lambda, \mu, \nu, \gamma)$ *by*

$$\beta = 1 - \frac{1-\gamma}{2[1 - {}_2F_1(2, 2-\mu+\nu; 2+\lambda+\nu; -1) - \gamma(1 - {}_2F_1(1, 2-\mu+\nu; 2+\lambda+\nu; -1))]}.$$

If $f(z) \in \mathcal{P}(\beta)$, *then* $\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) \in \mathcal{R}(\mu/2, \gamma)$.

Proof Making use of (1.5) and (1.7), we note that

$$\begin{aligned} \mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) &= \mathcal{L}(2, 2-\mu)\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu) f(z) \\ &= \mathcal{L}(1, 2-\mu)\mathcal{L}(2,1)\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu) f(z). \end{aligned} \tag{3.7}$$

By using Corollary 2, we obtain

$$\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu) f(z) \in \mathcal{K}(\gamma).$$

Since $0 \le \mu < 2$, from (1.6), (1.8) and (3.7) we have $\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) \in \mathcal{R}(\mu/2, \gamma)$, which completes the proof of Theorem 2. □

Taking $\mu = 2\gamma$ in Theorem 2, we get the following.

Corollary 3 *Let* $0 \le \gamma \le 1/2$, $\lambda \ge 2$, *and* $2(\gamma - 1) < \nu \le 2\gamma - 1$. *Define* $\beta = \beta(\lambda, \nu, \gamma)$ *by*

$$\beta = 1 - \frac{1-\gamma}{2[1 - {}_2F_1(2, 2(1-\gamma)+\nu; 2+\lambda+\nu; -1) - \gamma(1 - {}_2F_1(1, 2(1-\gamma)+\nu; 2+\lambda+\nu; -1))]}.$$

If $f(z) \in \mathcal{P}(\beta)$, *then* $\mathcal{J}_{0,z}^{\lambda,2\gamma,\nu} f(z) \in \mathcal{R}(\gamma) \subset \mathcal{S}$.

Proof If we put $\mu = 2\gamma$ in Theorem 2, then

$$\mathcal{J}_{0,z}^{\lambda,2\gamma,\nu} f(z) \in \mathcal{R}(\gamma, \gamma) = \mathcal{R}(\gamma).$$

Since $\gamma \le 1/2$, $\mathcal{R}(\gamma) \subset \mathcal{S}$, so that the proof is completed. □

Remark 3 In [4], Balasubramanian *et al.* found the conditions on the number β and the function $\lambda(t)$ such that $P_{a,b,c}(f)(z) \in \mathcal{S}^*(\gamma)$ $(0 \le \gamma \le 1/2)$. Since $\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) = P_{1-\nu,2,\lambda-\nu+2}(f)(z)$ with $\varphi(1-t) = {}_2F_1(\lambda+\mu, -\nu; \lambda; 1-t)$ and

$$C = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(\lambda)\Gamma(2-\mu+\nu)},$$

the condition on β and $\lambda(t)$ is easily found such that $\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) \in \mathcal{S}^*(\gamma)$.

Finally, by using Lemma 4 again, we investigate convexity of the operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$.

Theorem 3 *Let* $0 \le \gamma \le 1/2$, $0 < \lambda \le 1 + 2\gamma$, $2 < \mu < 3$, $0 < \nu \le 1$, *and* $\nu > \mu - 2$. *Define* $\beta = \beta(\lambda, \mu, \nu, \gamma)$ *by*

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = -\frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(\lambda)\Gamma(2-\mu+\nu)} \int_0^1 \frac{t(1-t)^{\lambda-1}(1-\gamma(1+t))}{(1-\gamma)(1+t)^2} {}_2F_1(\lambda+\mu, -\nu; \lambda; 1-t)\, dt.$$

If $f(z) \in \mathcal{P}(\beta)$, *then* $\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) \in \mathcal{K}(\gamma)$. *The value of* β *is sharp.*

Proof Let $0 \le \gamma \le 1/2$, $0 < \lambda \le 1 + 2\gamma$, $2 < \mu < 3$, and $\nu > \mu - 2$, and let

$$\lambda(t) = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(\lambda)\Gamma(2-\mu+\nu)} t(1-t)^{\lambda-1} {}_2F_1(\lambda+\mu, -\nu; \lambda; 1-t). \tag{3.8}$$

Then we can easily see that $\int_0^1 \lambda(t)\, dt = 1$, $\Lambda(t) = \int_t^1 \lambda(s)\, ds/s$ is monotone decreasing on $[0,1]$ and $\lim_{t\to 0+} t\Lambda(t) = 0$. Also we find that the function $u(t) = \lambda(t)/(1+t)(1-t)^{1+2\gamma}$ is decreasing on $(0,1)$, where $\lambda(t)$ is given by (3.8). Hence, $t\Lambda'(t)/(1+t)(1-t)^{1+2\gamma} = -u(t)$ is increasing on $(0,1)$. From Lemma 4, we obtain the desired result. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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