Journal of Inequalities and Applications
a SpringerOpen Journal

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Hybrid algorithms of nonexpansive semigroups for mixed equilibrium problems, variational inequalities, and fixed point problems

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Abstract
The purpose of this paper is to introduce two hybrid algorithms for the variational inequalities and mixed equilibrium problems over the common fixed points set of nonexpansive semigroups in Hilbert space. Under suitable conditions some strong convergence theorems for these two hybrid algorithms are proved. The results presented in the paper extend and improve some recent results.

Keywords: mixed equilibrium problem; variational inequalities; fixed point; nonexpansive semigroups

1 Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C is a nonempty closed convex subset of H and P_C is the metric projection of H onto C. In the sequel, we denote by $\rightarrow$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. Let $\phi: C \rightarrow \Re$ be a real-valued function and $\Theta: C \times C \rightarrow \Re$ be an equilibrium bifunctions, *i.e.*, $\Theta(u,u) = 0$ for each $u \in C$. We consider the mixed equilibrium problem (MEP) which is to find $x^* \in C$ such that

$$MEP: \Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C.$$

In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problem (EP), which is to find $x^* \in C$ such that

$$EP: \Theta(x^*, y) \geq 0.$$

Denote the set of solutions of MEP by Ω. The MEP includes fixed point problems, optimization problems, variational inequality problems, Nash EPS and the EP as special cases.

Recall that a mapping $T: C \rightarrow C$ is said to be nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$.

Let C be a closed convex subset of a Hilbert space H. A family of mappings $\mathcal{S} := \{S(s): 0 \leq s < \infty\}: C \rightarrow C$ is said to be a nonexpansive semigroup, if it satisfies the following conditions:

(i) $S(s+t) = S(s)S(t)$, $\forall s,t \in \Re^+$ and $S(0) = I$;
(ii) $\|S(s)x - S(s)y\| \leq \|x-y\|$, $\forall x,y \in C$, $s \geq 0$;
(iii) the mapping $t \mapsto S(t)x$ is continuous for each $x \in C$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, *i.e.*, $F(\mathcal{S}) = \bigcap_{s\geq 0} F(S(s))$. It is well known that $F(\mathcal{S})$ is closed and convex.

Now let $F : C \to C$ be a nonlinear operator. The variational inequality problem is formulated as finding a point $x^* \in C$ such that

$$VI(F,C) : \langle Fx^*, v - x^* \rangle \geq 0, \quad \forall v \in C.$$

It is well known that the $VI(F,C)$ is equivalent to the fixed point equation

$$x^* = P_C\bigl[x^* - \mu F\bigl(x^*\bigr)\bigr], \tag{1.1}$$

where $\mu > 0$ is an arbitrarily fixed constant. So, fixed point methods can be implemented to find a solution of the $VI(F,C)$ provided F satisfies some conditions and $\mu > 0$ is chosen appropriately. The fixed point formulation (1.1) involves the projection P_C, which may not be easy to compute, due to the complexity of the convex set C. In order to reduce the complexity probably caused by the projection P_C, Yamada [1] recently introduced a hybrid steepest-descent method for solving the $VI(F,C)$. Assume that F is an η-strongly monotone and κ-Lipschitzian mapping with $\kappa > 0$, $\eta > 0$ on C. An equally important problem is how to find an approximate solution of the $VI(F,C)$ if any. A great deal of effort has been done in this problem.

In 2007, Ceng and Yao [2] investigate the problem of finding a common element of the set of solutions of a mixed equilibrium problem (MEP) and the set of common fixed points of finitely many nonexpansive mappings in a real Hilbert space. Very recently, Yang *et al.* [3] introduce two hybrid algorithms for finding a common fixed point of a nonexpansive semigroup in Hilbert space.

Motivated and inspired by Ceng and Yao [2] and Yang *et al.* [3], the purpose of this paper is to introduce two hybrid algorithms for the variational inequalities and mixed equilibrium problems over the common fixed points set of nonexpansive semigroups in Hilbert space. Under suitable conditions some strong convergence theorem for these two hybrid algorithms are proved. The results presented in the paper extend and improve some recent results.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. For solving mixed equilibrium problems, let us assume that the function $\Theta : C \times C \to \Re$ satisfies the following conditions:

(H1) Θ is monotone, *i.e.*, $\Theta(x,y) + \Theta(y,x) \leq 0$, $\forall x,y \in C$;
(H2) for each fixed $y \in C$, the mapping $x \rightharpoonup \Theta(x,y)$ is concave and upper semicontinuous;
(H3) for each fixed $x \in C$, the mapping $y \rightharpoonup \Theta(x,y)$ is convex.

A mapping $F : C \to C$ is said to be:

(i) κ-Lipschitz continuous, if there exists a constant $\kappa > 0$ such that

$$\|Fx - Fy\| \leq \kappa\|x-y\|, \quad \forall x,y \in C;$$

(ii) η-strongly monotone, if there exists a constant $\eta > 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C.$$

A differentiable function $K : C \to \Re$ is said to be:

(i) ξ-convex [4], if

$$K(y) - K(x) \geq \langle K'(x), \xi(y, x)\rangle, \quad \forall x, y \in C,$$

where $K'(x)$ is the Fréchet derivative of K at x;

(ii) ξ-strongly convex [5], if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K'(x), \xi(y, x)\rangle \geq (\sigma/2)\|x - y\|^2, \quad \forall x, y \in C.$$

The following lemmas will be needed in proving our main results.

Lemma 2.1 [2] *Let C be a nonempty closed convex subset of a real Hilbert space H and $\varphi : C \to \Re$ be a lower semicontinuous and convex functional. Let $\theta : C \times C \to \Re$ be an equilibrium bifunction satisfying the conditions* (H1)-(H3). *Assume that*

(i) *$\xi : C \times C \to H$ is λ-Lipschitz continuous such that*
  - (a) $\xi(x, y) + \xi(y, x) = 0$, $\forall x, y \in C$;
  - (b) *$\xi(\cdot, \cdot)$ is affine in the first variable*;
  - (c) *for each fixed $x \in C$, the mapping $y \mapsto \xi(x, y)$ is sequentially continuous from the weak topology to the weak topology*;

(ii) *$K : C \to \Re$ is ξ-strongly convex with constant $\sigma > 0$, and its derivative K' is sequentially continuous from the weak topology to the strong topology*;

(iii) *for each $x \in C$ there exist a bounded subset $D_x \subset C$ and a point $z_x \in C$ such that, for any $y \in C \setminus D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \xi(z_x, y)\rangle < 0.$$

For given $r > 0$, let $J_r : C \to C$ be the mapping defined by

$$J_r(x) = \left\{y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \xi(z, y)\rangle \geq 0, \forall z \in C\right\}, \quad x \in C. \tag{2.1}$$

Then

(i) *J_r is single-valued*;

(ii) *J_r is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ and*

$$\langle K'(x_1) - K'(x_2), u_1 - u_2\rangle \geq \langle K'(u_1) - K'(u_2), u_1 - u_2\rangle, \quad \forall (x_1, x_2) \in C \times C,$$

where $u_i = J_r(x_i)$ for $i = 1, 2$;

(iii) $F(J_r) = \Omega(\Theta,\varphi)$, *where* $\Omega(\Theta,\varphi)$ *is the set of solutions of the following mixed equilibrium problem*:

$$\Theta(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.$$

(iv) $\Omega(\Theta,\varphi)$ *is closed and convex.*

Lemma 2.2 [6] *Let* x_n *and* y_n *be bounded sequences in a Banach space* E *and let* β_n *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 1$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.3 [7] *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n\delta_n + \gamma_n, \quad \forall n \geq n_0,$$

where n_0 *is some nonnegative integer,* $\{\lambda_n\}$, $\{\delta_n\}$, *and* $\{\gamma_n\}$ *are sequences satisfying*
(i) $\lambda_n \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n\delta_n < \infty$,
(iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 [8] *Let* C *be a bounded closed convex subset of* H *and* $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ *be a nonexpansive semigroup on* C, *then for any* $h > 0$

$$\lim_{s\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t S(s)x\,ds - S(h)\left(\frac{1}{t}\int_0^t S(s)x\,ds\right) \right\| = 0.$$

Lemma 2.5 [9] *Let* C *be a nonempty bounded closed convex subset of* H, x_n *be a sequence in* C *and* $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ *be a nonexpansive semigroup on* C. *If the following conditions are satisfied*:
(i) $x_n \rightharpoonup z$;
(ii) $\limsup_{n\to\infty} \limsup_{n\to\infty} \|S(s)x_n - x_n\| = 0$,
then $z \in F(\mathcal{S})$.

Lemma 2.6 [10] *Let* F *be an* η-*strongly monotone and* κ-*Lipschitzian operator on a Hilbert space* H *with* $0 < \eta < \kappa$ *and* $0 < t < \eta/\kappa^2$. *Then* $T = (I - tF) : H \to H$ *is a contraction with contraction coefficient* $\tau_t = \sqrt{1 - t(2\eta - t\kappa^2)}$.

Lemma 2.7 *In a real Hilbert space* H, *we have the inequality*

$$\|x-y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2,$$
$$\left\|\lambda x + (1-\lambda)y\right\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ *and* $\lambda \in [0,1]$.

Recall that a Banach space E is said to satisfy the Opial condition, if for any sequence x_n in E with $x_n \rightharpoonup x$, then for every $y \in E$ with $y \neq x$ we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

It is well known that each Hilbert space satisfies the Opial condition.

3 Main results

Now we will show our main results.

Theorem 3.1 *Let H be a real Hilbert space. Let $\varphi : H \to \Re$ be a lower semicontinuous and convex functional. Let $\Theta : H \times H \to \Re$ be an equilibrium functions satisfying conditions* (H1)-(H3). *Let $\mathcal{S} := \{S(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on H. Let F be an η-strongly monotone and κ-Lipschitzian operator on H. Let $\{\gamma_t\}_{0<t<1}$ be a continuous net of positive real numbers such that $\lim_{t\to 0+} \gamma_t = +\infty$. Putting $\tau_t = \sqrt{1 - t(2\eta - t\kappa^2)}$, for each $t \in (0, \eta/\kappa^2)$, let the net $\{x_t\}$ be defined by the following implicit scheme*:

$$x_t = \frac{1}{\gamma_t} \int_0^{\gamma_t} S(s)\big[(I - tF)J_r x_t\big]\, ds, \tag{3.1}$$

where $J_r : H \to H$ is the mapping defined by (2.1). *Suppose the following conditions are satisfied*:

- (i) *$\xi : H \times H \to H$ is λ-Lipschitz continuous such that*
  - (a) *$\xi(x,y) + \xi(y,x) = 0$, $\forall x, y \in H$;*
  - (b) *$x \mapsto \xi(x,y)$ is affine;*
  - (c) *$y \mapsto \xi(x,y)$ is sequentially continuous from the weak topology to the weak topology;*
- (ii) *$K : H \to \Re$ is ξ-strongly convex with constant $\sigma > 0$, and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with a Lipschitz constant $\nu > 0$ and $\sigma \ge \lambda\nu$;*
- (iii) *for each $x \in H$ there exist a bounded subset $D_x \subset H$ and a point $z_x \in H$ such that, for any $y \in H \setminus D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\big\langle K'(y) - K'(x), \xi(z_x, y)\big\rangle < 0,$$

and if $\Gamma := \Omega(\Theta, \varphi) \cap F(\mathcal{S}) \neq \emptyset$. Then, as $t \to 0^+$, the net $\{x_t\}$ converges strongly to an element x^* of Γ provided J_r is firmly nonexpansive which is the unique solution of the following variational inequality:

$$\big\langle Fx^*, x^* - u\big\rangle \le 0, \quad \forall u \in \Gamma. \tag{3.2}$$

Proof We divide the proof into several steps.

Step 1. First, we note that the net $\{x_t\}$ defined by (3.1) is well defined. In fact, we define a mapping

$$P_t x := \frac{1}{\gamma_t} \int_0^{\gamma_t} S(s)\big[(I - tF)J_r x\big]\, ds, \quad t \in \big(0, \eta/\kappa^2\big), x \in H. \tag{3.3}$$

Because J_r is nonexpansive. It follows from Lemma 2.6 that

$$\begin{aligned}\|P_t x - P_t y\| &\leq \frac{1}{\gamma_t}\int_0^{\gamma_t} \big\|S(s)\big[(I-tF)J_r x\big] - S(s)\big[(I-tF)J_r y\big]\big\|\,ds \\ &\leq \big\|(I-tF)x-(I-tF)y\big\| \leq \tau_t\|x-y\|.\end{aligned}$$

Hence, the P_t is a contraction, and so it has a unique fixed point. Therefore, the net $\{x_t\}$ defined by (3.1) is well defined.

Step 2. We prove that $\{x_t\}$ is bounded. Taking $u \in \Gamma$ and using Lemma 2.6, we have

$$\begin{aligned}\|x_t - u\| &= \left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)\big[(I-tF)J_r x_t\big]\,ds - u\right\| \\ &= \left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)\big[(I-tF)J_r x_t\big]\,ds - \frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)J_r u\,ds\right\| \\ &\leq \frac{1}{\gamma_t}\int_0^{\gamma_t}\big\|S(s)\big[(I-tF)J_r x_t\big] - S(s)J_r u\big\|\,ds \\ &\leq \big\|(I-tF)x_t - u\big\| \\ &\leq \big\|(I-tF)x_t - (I-tF)u - tFu\big\| \\ &\leq \tau_t\|x_t-u\| + t\|Fu\|.\end{aligned}$$

It follows that

$$\|x_t - u\| \leq \frac{t}{1-\tau_t}\|Fu\|. \tag{3.4}$$

Observe that

$$\lim_{t\to 0^+}\frac{t}{1-\tau_t} = \frac{1}{\eta}. \tag{3.5}$$

Thus, (3.4) and (3.5) imply that the net $\{x_t\}$ is bounded for small enough t. Without loss of generality, we may assume that the net $\{x_t\}$ is bounded for all $t \in (0, \eta/\kappa^2)$. Consequently, we deduce that $\{Fx_t\}$ and $\{J_r x_t\}$ are also bounded.

Step 3. On the other hand, from (3.1) and (3.4), we have

$$\begin{aligned}&\big\|x_t - S(s)x_t\big\| \\ &\quad\leq \left\|S(s)x_t - S(s)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right)\right\| \\ &\qquad + \left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds - x_t\right\| + \left\|S(s)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right) - \frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right\| \\ &\quad\leq 2\left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds - x_t\right\| + \left\|S(s)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right) - \frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right\| \\ &\quad= 2\left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds - \frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)\big[(I-tF)J_r x_t\big]\,ds\right\| \\ &\qquad + \left\|S(s)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right) - \frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right\|\end{aligned}$$

$$\begin{aligned}
&\leq 2\frac{1}{\gamma_t}\int_0^{\gamma_t}\big\|S(s)x_t - S(s)\big[(I-tF)J_r x_t\big]\big\|\,ds \\
&\quad + \left\|S(s)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right) - \frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right\| \\
&\leq 2\big\|x_t - (I-tF)J_r x_t\big\| + \left\|S(s)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right) - \frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right\| \\
&\leq 2\|x_t - J_r x_t\| + 2t\|FJ_r x_t\| + \left\|S(s)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right) - \frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)x_t\,ds\right\|. \qquad (3.6)
\end{aligned}$$

In fact, we have

$$\begin{aligned}
\|x_t - u\| &= \left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)\big[(I-tF)J_r x_t\big]\,ds - u\right\| \\
&= \left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)\big[(I-tF)J_r x_t\big]\,ds - \frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)u\,ds\right\| \\
&\leq \frac{1}{\gamma_t}\int_0^{\gamma_t}\big\|S(s)\big[(I-tF)J_r x_t\big] - S(s)u\big\|\,ds \\
&\leq \big\|(I-tF)J_r x_t - u\big\|,
\end{aligned}$$

so

$$\begin{aligned}
\|x_t - u\|^2 &\leq \big\|(I-tF)J_r x_t - u\big\|^2 \\
&\leq \|J_r x_t - u\|^2 + t^2\|FJ_r x_t\|^2 - 2t\langle J_r x_t - u, FJ_r x_t\rangle \\
&\leq \|x_t - u\|^2 - \|x_t - J_r x_t\|^2 + t^2\|FJ_r x_t\|^2 - 2t\langle J_r x_t - u, FJ_r x_t\rangle,
\end{aligned}$$

observe that

$$\|x_t - J_r x_t\|^2 \leq t^2\|FJ_r x_t\|^2 - 2t\langle J_r x_t - u, FJ_r x_t\rangle,$$

then

$$\lim_{t\to 0^+}\|x_t - J_r x_t\| = 0. \qquad (3.7)$$

This together with Lemma 2.4 and (3.6) implies that

$$\lim_{t\to 0^+}\big\|x_t - S(s)x_t\big\| = 0. \qquad (3.8)$$

Let $\{t_n\}\subset(0,1)$ be a sequence such that $t_n\to 0$ as $n\to\infty$. Put $x_n := x_{t_n}$ and $y_n = J_r x_n$. Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ which converges weakly to $w\in H$. Without loss of generality, we can assume that $y_{n_j}\rightharpoonup w$. Next we prove that

$$w\in\Gamma := \Omega(\Theta,\varphi)\cap F(\mathcal{S}).$$

(a) In fact, we have

$$\big\|y_n - S(s)y_n\big\| = \big\|y_n - x_n + x_n - S(s)x_n + S(s)x_n - S(s)y_n\big\| \leq 2\|y_n - x_n\| + \big\|x_n - S(s)x_n\big\|.$$

With (3.7) and (3.8), we have

$$\lim_{n\to\infty}\|y_n - S(s)y_n\| = 0. \tag{3.9}$$

Indeed, from Lemma 2.5 and (3.9) we know that $w \in F(\mathcal{S})$, *i.e.*, $w = S(s)w$, $\forall s \geq 0$.

(b) Now we prove that $w \in \Omega = \Omega(\Theta, \varphi)$. In fact, since $y_n = J_r x_n$, we have

$$\Theta(J_r x_n, x) + \varphi(x) - \varphi(J_r x_n) + \frac{1}{r}\langle K'(J_r x_n) - K'(x_n), \xi(x, J_r x_n)\rangle \geq 0, \quad \forall x \in H.$$

From the monotonicity of Θ, we have

$$\frac{1}{r}\langle K'(J_r x_n) - K'(x_n), \xi(x, J_r x_n)\rangle + \varphi(x) - \varphi(J_r x_n) \geq -\Theta(J_r x_n, x) \geq \Theta(x, J_r x_n)$$

and hence

$$\left\langle \frac{K'(J_r x_{n_j}) - K'(x_{n_j})}{r}, \xi(x, J_r x_{n_j})\right\rangle + \varphi(x) - \varphi(J_r x_{n_j}) \geq \Theta(x, J_r x_{n_j}). \tag{3.10}$$

Since (3.7), then $(K'(J_r x_{n_j}) - K'(x_{n_j}))/r \to 0$ and $y_{n_j} \rightharpoonup w$, from the weak lower semicontinuity of φ and $\Theta(x,y)$ in the second variable y, we have $\Theta(x,w) + \varphi(w) - \varphi(x) \leq 0$ for all $x \in H$. For $0 < t \leq 1$ and $x \in H$, let $x_t = tx + (1-t)w$. Since $x \in H$ and $w \in H$, we have $x_t \in H$ and hence $\Theta(x_t, w) + \varphi(w) - \varphi(x_t) \leq 0$. From the convexity of equilibrium bifunction $\Theta(x,y)$ in the second variable y, we have

$$\begin{aligned}
0 &= \Theta(x_t, x_t) + \varphi(x_t) - \varphi(x_t) \\
&\leq t\Theta(x_t, x) + (1-t)\Theta(x_t, w) + t\varphi(x) + (1-t)\varphi(w) - \varphi(x_t) \\
&\leq t\big[\Theta(x_t, x) + \varphi(x) - \varphi(x_t)\big],
\end{aligned}$$

and hence $\Theta(x_t, x) + \varphi(x) - \varphi(x_t) \geq 0$. Then we have $\Theta(w, x) + \varphi(x) - \varphi(w) \geq 0$ for all $x \in H$. So $w \in \Omega$.

We can obtain $w \in \Gamma := \Omega \cap F(\mathcal{S})$ and $x_n \rightharpoonup w$.

Step 4. Finally, from (3.1), we have

$$\begin{aligned}
\|x_t - u\|^2 &= \left\| \frac{1}{\gamma_t}\int_0^{\gamma_t} S(s)\big[(I - tF)J_r x_t\big]\,ds - u\right\|^2 \\
&= \left\| \frac{1}{\gamma_t}\int_0^{\gamma_t} \big[S(s)(I - tF)J_r x_t - S(s)J_r u\big]\,ds\right\|^2 \\
&\leq \big\|(I - tF)x_t - (I - tF)u - tFu\big\|^2 \\
&\leq \tau_t^2\|x_t - u\|^2 + t^2\|Fu\|^2 + 2t\big\langle (I - tF)u - (I - tF)x_t, Fu\big\rangle \\
&\leq \tau_t\|x_t - u\|^2 + t^2\|Fu\|^2 + 2t\langle u - x_t, Fu\rangle + 2t^2\langle Fx_t - Fu, Fu\rangle \\
&\leq \tau_t\|x_t - u\|^2 + t^2\|Fu\|^2 + 2t\langle u - x_t, Fu\rangle + 2\kappa t^2\|x_t - u\|\|Fu\|.
\end{aligned}$$

Therefore,

$$\|x_t - u\|^2 \leq \frac{t^2}{1-\tau_t}\|Fu\|^2 + \frac{2t}{1-\tau_t}\langle u - x_t, Fu\rangle + \frac{2\kappa t^2}{1-\tau_t}\|x_t - u\|\|Fu\|. \tag{3.11}$$

It follows that

$$\|x_n - w\|^2 \le \frac{t_n^2}{1-\tau_{t_n}}\|Fw\|^2 + \frac{2t_n}{1-\tau_{t_n}}\langle w - x_n, Fw\rangle + \frac{2\kappa t_n^2}{1-\tau_{t_n}}\|x_n - w\|\|Fw\|.$$

Thus, $x_n \rightharpoonup w$ implies that $x_n \to w$.

Again, from (3.11), we obtain

$$\|x_n - u\|^2 \le \frac{t_n^2}{1-\tau_{t_n}}\|Fu\|^2 + \frac{2t_n}{1-\tau_{t_n}}\langle u - x_n, Fu\rangle + \frac{2\kappa t_n^2}{1-\tau_{t_n}}\|x_n - u\|\|Fu\|. \tag{3.12}$$

It is clear that $\lim_{n\to\infty}(t_n^2/1-\tau_{t_n}) = 0$, $\lim_{n\to\infty}(2t_n/1-\tau_{t_n}) = 2/\eta$, and $\lim_{n\to\infty}(2\kappa t_n^2/1-\tau_{t_n}) = 0$. We deduce immediately from (3.12) that $\langle Fu, w-u\rangle \le 0$, which is equivalent to its dual variational inequality $\langle Fw, w-u\rangle \le 0$. That is, $w \in \Gamma$ is a solution of the variational inequality (3.2).

Suppose that $x^* \in \Gamma$ and $w \in \Gamma$ both are solutions to the variational inequality (3.2); then

$$\begin{aligned} &\langle Fx^*, x^* - w\rangle \le 0, \\ &\langle Fw, w - x^*\rangle \le 0. \end{aligned} \tag{3.13}$$

Adding up (3.13) and the last inequality yields $\langle Fx^* - Fw, x^* - w\rangle \le 0$. The strong monotonicity of F implies that $x^* = w$ and the uniqueness is proved. Later, we will use $x^* \in \Gamma$ to denote the unique solution of (3.2). This completes the proof. □

Next we introduce an explicit algorithm for finding an element of Γ.

Theorem 3.2 *Let H be a real Hilbert space. Let $\varphi : H \to \Re$ be a lower semicontinuous and convex functional. Let $\Theta : H \times H \to \Re$ be an equilibrium functions satisfying conditions* (H1)-(H3). *Let $\mathcal{S} := \{S(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on H such that $F(\mathcal{S}) \ne \emptyset$. Let F be an η-strongly monotone and κ-Lipschitzian operator on H with $0 < \eta < \kappa$. For given $x_0 \in H$ arbitrarily, define a sequence $\{x_n\}$ iteratively by*

$$\begin{aligned} &y_n = x_n - \lambda_n F(x_n), \\ &x_{n+1} = (1-\alpha_n)y_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} S(s)J_r y_n\,ds, \quad n \ge 0, \end{aligned} \tag{3.14}$$

where $\{\lambda_n\}$, $\{t_n\}$ are sequences in $(0,\infty)$, $\{\alpha_n\}$ is a sequence in $[0,1]$, and $J_r : H \to H$ is the mapping defined by (2.1). *Suppose the following conditions are satisfied*:

(i) *$\xi : H \times H \to H$ is λ-Lipschitz continuous such that*
  (a) *$\xi(x,y) + \xi(y,x) = 0$, $\forall x, y \in H$*;
  (b) *$x \mapsto \xi(x,y)$ is affine*;
  (c) *$y \mapsto \xi(x,y)$ is sequentially continuous from the weak topology to the weak topology*;

(ii) *$K : H \to \Re$ is ξ-strongly convex with constant $\sigma > 0$, and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with a Lipschitz constant $\nu > 0$ and $\sigma \ge \lambda\nu$*;

(iii) *For each* $x \in H$ *there exist a bounded subset* $D_x \subset H$ *and a point* $z_x \in H$ *such that, for any* $y \in H \setminus D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \xi(z_x, y)\rangle < 0;$$

(iv) $\limsup_{n\to\infty} \lambda_n < \eta/\kappa^2$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(vi) $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty}(t_{n+1}/t_n) = 1$;

(vii) $0 < \gamma \le \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, *for some* $\gamma \in (0, 1)$.

If $\Gamma := \Omega(\Theta, \varphi) \cap F(\mathcal{S}) \neq \emptyset$, *then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to an element* x^* *of* Γ *provided* J_r *is firmly nonexpansive if and only if* $\lambda_n F(x_n) \to 0$, *where* x^* *is the unique solution of the following variational inequality*:

$$\langle Fx^*, x^* - u\rangle \le 0, \quad \forall u \in \Gamma.$$

Proof The necessity is obvious. We only need to prove the sufficiency. Suppose that $\lambda_n F(x_n) \to 0$.

Step 1. First, we show that $\{x_n\}$, $\{y_n\}$, $\{S(s)y_n\}$, $\{Fx_n\}$, and $\{J_r x_n\}$ are bounded. In fact, letting $u \in \Gamma$, we have $u = S(s)J_r u$.

Then

$$\begin{aligned} \|x_{n+1} - u\| &= \left\|(1-\alpha_n)y_n + \alpha_n \frac{1}{t_n}\int_0 t_n S(s)J_r y_n\, ds - u\right\| \\ &= \left\|(1-\alpha_n)(y_n - u) + \alpha_n\left(\frac{1}{t_n}\int_0^{t_n} S(s)J_r y_n\, ds - u\right)\right\| \\ &\le (1-\alpha_n)\|y_n - u\| + \alpha_n \frac{1}{t_n}\int_0^{t_n} \|S(s)J_r y_n - S(s)J_r u\|\, ds \\ &\le (1-\alpha_n)\|y_n - u\| + \alpha_n\|y_n - u\| \\ &= \|y_n - u\|. \end{aligned} \tag{3.15}$$

From condition (iv), without loss of generality, we can assume that $\lambda_n \le a < \eta/\kappa^2$, $\forall n \ge 0$. By (3.14) and Lemma 2.6, we have

$$\begin{aligned} \|y_n - u\| &= \|x_n - \lambda_n F(x_n) - u\| = \|(I - \lambda_n F)x_n - (I - \lambda_n F)u - \lambda_n Fu\| \\ &\le \|(I - \lambda_n F)x_n - (I - \lambda_n F)u\| + \lambda_n\|Fu\| \le \tau_{\lambda_n}\|x_n - u\| + \lambda_n\|Fu\|, \end{aligned} \tag{3.16}$$

where $\tau_{\lambda_n} = \sqrt{1 - \lambda_n(2\eta - \lambda_n\kappa^2)} \in (0, 1)$.

Then, from (3.15) and (3.16), we obtain

$$\begin{aligned} \|x_{n+1} - u\| &\le \tau_{\lambda_n}\|x_n - u\| + \lambda_n\|Fu\| = [1 - (1 - \tau_{\lambda_n})]\|x_n - u\| + (1 - \tau_{\lambda_n})\frac{\lambda_n}{1 - \tau_{\lambda_n}}\|Fu\| \\ &\le \max\left\{\|x_n - u\|, \frac{\lambda_n}{1 - \tau_{\lambda_n}}\|Fu\|\right\}. \end{aligned}$$

Observe that $\lim_{n\to\infty}(\lambda_n/1 - \tau_{\lambda_n}) = 1/\eta$, we have by induction

$$\|x_{n+1} - u\| \le \max\{\|x_0 - u\|, M_1\|Fu\|\}, \tag{3.17}$$

where $M_1 = \sup_n\{\lambda_n/1-\tau_{\lambda_n}\} < \infty$. Hence $\{x_n\}$ is bounded. Consequently, we deduce that $\{y_n\}$, $\{S(s)y_n\}$, $\{Fx_n\}$ and $\{J_r x_n\}$ are also bounded.

Step 2. Define $x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n$, $\forall n \geq 0$, then $u_n = \frac{1}{\alpha_n}[x_{n+1} - (1-\alpha_n)x_n]$.

Observe that

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \left\| \frac{x_{n+2} - (1-\alpha_{n+1})x_{n+1}}{\alpha_{n+1}} - \frac{x_{n+1} - (1-\alpha_n)x_n}{\alpha_n} \right\| \\
&= \left\| \frac{(1-\alpha_{n+1})y_{n+1} + \alpha_{n+1}\frac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)J_r y_{n+1}\,ds - (1-\alpha_{n+1})x_{n+1}}{\alpha_{n+1}} \right. \\
&\quad \left. - \frac{(1-\alpha_n)y_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} S(s)J_r y_n\,ds - (1-\alpha_n)x_n}{\alpha_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}\frac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)J_r y_{n+1}\,ds - (1-\alpha_{n+1})\lambda_{n+1}F(x_{n+1})}{\alpha_{n+1}} \right. \\
&\quad \left. - \frac{\alpha_n \frac{1}{t_n}\int_0^{t_n} S(s)J_r y_n\,ds - (1-\alpha_n)\lambda_n F(x_n)}{\alpha_n} \right\| \\
&\leq \left\| \frac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)J_r y_{n+1}\,ds - \frac{1}{t_n}\int_0^{t_n} S(s)J_r y_n\,ds \right\| \\
&\quad + \frac{1-\alpha_{n+1}}{\alpha_{n+1}}\left\|\lambda_{n+1}F(x_{n+1})\right\| + \frac{1-\alpha_n}{\alpha_n}\left\|\lambda_n F(x_n)\right\|. \qquad (3.18)
\end{aligned}$$

Next, we estimate

$$\begin{aligned}
&\left\| \frac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)J_r y_{n+1}\,ds - \frac{1}{t_n}\int_0^{t_n} S(s)J_r y_n\,ds \right\| \\
&\quad \leq \left\| \frac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)J_r y_{n+1}\,ds - \frac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)J_r y_n\,ds \right\| \\
&\qquad + \left\| \frac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)J_r y_n\,ds - \frac{1}{t_n}\int_0^{t_n} S(s)J_r y_n\,ds \right\| \\
&\quad \leq \|y_{n+1} - y_n\| + \left\| \frac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)J_r y_n\,ds - \frac{1}{t_n}\int_0^{t_n} S(s)J_r y_n\,ds \right\| \\
&\quad \leq \|y_{n+1} - y_n\| + \left|\frac{1}{t_{n+1}} - \frac{1}{t_n}\right| \left\|\int_0^{t_n} S(s)J_r y_n\,ds\right\| + \frac{1}{t_{n+1}}\left\|\int_{t_n}^{t_{n+1}} S(s)J_r y_n\,ds\right\| \\
&\quad \leq \left\|x_{n+1} - \lambda_{n+1}F(x_{n+1}) - x_n + \lambda_n F(x_n)\right\| + \left|\frac{t_n}{t_{n+1}} - 1\right| M_2 \\
&\quad \leq \|x_{n+1} - x_n\| + \left\|\lambda_{n+1}F(x_{n+1})\right\| + \left\|\lambda_n F(x_n)\right\| + \left|\frac{t_n}{t_{n+1}} - 1\right| M_2, \qquad (3.19)
\end{aligned}$$

where $M_2 = \sup_n\{2\|S(s)J_r y_n\|\} < \infty$. From (3.18) and (3.19), we have

$$\begin{aligned}
\|u_{n+1} - u_n\| \leq{}& \frac{1-\alpha_{n+1}}{\alpha_{n+1}}\left\|\lambda_{n+1}F(x_{n+1})\right\| + \frac{1-\alpha_n}{\alpha_n}\left\|\lambda_n F(x_n)\right\| + \|x_{n+1} - x_n\| \\
&+ \left\|\lambda_{n+1}F(x_{n+1})\right\| + \left\|\lambda_n F(x_n)\right\| + \left|\frac{t_n}{t_{n+1}} - 1\right| M_2.
\end{aligned}$$

This together with condition (vii) implies that

$$\|u_{n+1}-u_n\| \le \frac{1}{\gamma}\bigl(\|\lambda_{n+1}F(x_{n+1})\| + \|\lambda_n F(x_n)\|\bigr) + \|x_{n+1}-x_n\| + \left|\frac{t_n}{t_{n+1}}-1\right|M_2. \tag{3.20}$$

Namely,

$$\|u_{n+1}-u_n\| - \|x_{n+1}-x_n\| \le \frac{1}{\gamma}\bigl(\|\lambda_{n+1}F(x_{n+1})\| + \|\lambda_n F(x_n)\|\bigr) + \left|\frac{t_n}{t_{n+1}}-1\right|M_2.$$

Since $\lambda_n F(x_n) \to 0$ and condition (vi), we get

$$\limsup_{n\to\infty}\bigl(\|u_{n+1}-u_n\| - \|x_{n+1}-x_n\|\bigr) \le 0.$$

Consequently, by Lemma 2.2, we deduce $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Therefore,

$$\|x_{n+1}-x_n\| = \alpha_n\|u_n - x_n\| \to 0 \quad (n\to\infty). \tag{3.21}$$

Step 3. Next, we claim that $\lim_{n\to\infty}\|x_n - S(s)x_n\| = 0$. Observe that

$$\begin{aligned}
\|x_n - S(s)x_n\| &\le \left\|S(s)x_n - S(s)\left(\frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds\right)\right\| + \left\|\frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds - x_n\right\| \\
&\quad + \left\|S(s)\left(\frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds\right) - \frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds\right\| \\
&\le 2\left\|\frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds - x_n\right\| \\
&\quad + \left\|S(s)\left(\frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds\right) - \frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds\right\|.
\end{aligned} \tag{3.22}$$

Note that

$$\begin{aligned}
&\left\|\frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds - x_n\right\| \\
&\quad \le \|x_n - x_{n+1}\| + \left\|x_{n+1} - \frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds\right\| \\
&\quad = \|x_n - x_{n+1}\| + \left\|(1-\alpha_n)y_n + \alpha_n\frac{1}{t_n}\int_0^{t_n} S(s)J_r y_n\,ds - \frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds\right\| \\
&\quad \le \|x_n - x_{n+1}\| + (1-\alpha_n)\left\|y_n - \frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds\right\| \\
&\qquad + \alpha_n\left\|\frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds - \frac{1}{t_n}\int_0^{t_n} S(s)J_r y_n\,ds\right\| \\
&\quad \le \|x_n - x_{n+1}\| + (1-\alpha_n)\|y_n - x_n\| + (1-\alpha_n)\left\|x_n - \frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds\right\| \\
&\qquad + \alpha_n\|x_n - J_r y_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}&\left\|\frac{1}{t_n}\int_0^{t_n} S(s)x_n\,ds - x_n\right\| \\ &\quad\le \frac{1}{\alpha_n}\bigl(\|x_n - x_{n+1}\| + (1-\alpha_n)\bigl\|\lambda_n F(x_n)\bigr\| + \alpha_n\|x_n - J_r y_n\|\bigr).\end{aligned}\tag{3.23}$$

From (3.15), we have

$$\begin{aligned}\|x_{n+1}-u\|^2 &= \left\|(1-\alpha_n)y_n + \alpha_n\frac{1}{t_n}\int_0 t_n S(s)J_r y_n\,ds - u\right\|^2 \\ &\le (1-\alpha_n)\|y_n - u\|^2 + \alpha_n\left\|\frac{1}{t_n}\int_0^{t_n} S(s)J_r y_n\,ds - u\right\|^2 \\ &\le (1-\alpha_n)\bigl\|x_n - \lambda_n F(x_n) - u\bigr\|^2 + \alpha_n\|J_r y_n - u\|^2 \\ &\le (1-\alpha_n)\|x_n - u\|^2 + (1-\alpha_n)\bigl\|\lambda_n F(x_n)\bigr\|^2 + \alpha_n\bigl(\|x_n-u\|^2 - \|x_n - J_r y_n\|^2\bigr) \\ &\le (1-\alpha_n)\bigl\|\lambda_n F(x_n)\bigr\|^2 + \|x_n-u\|^2 - \alpha_n\|x_n - J_r y_n\|^2,\end{aligned}$$

then

$$\begin{aligned}\alpha_n\|x_n - J_r y_n\|^2 &\le (1-\alpha_n)\bigl\|\lambda_n F(x_n)\bigr\|^2 + \|x_n-u\|^2 - \|x_{n+1}-u\|^2 \\ &\le (1-\alpha_n)\bigl\|\lambda_n F(x_n)\bigr\|^2 + \|x_n - x_{n+1}\|\bigl(\|x_n-u\| + \|x_{n+1}-u\|\bigr).\end{aligned}$$

This together with condition (vii), $\lambda_n F(x_n) \to 0$, and (3.21), we have

$$\lim_{n\to\infty}\|x_n - J_r y_n\| = 0.\tag{3.24}$$

By Lemma 2.4, (3.21)-(3.24), and $\lambda_n F(x_n) \to 0$, we derive

$$\lim_{n\to\infty}\bigl\|x_n - S(s)x_n\bigr\| = 0.\tag{3.25}$$

Step 4. Next, we show that $\limsup_{n\to\infty}\langle Fx^*, x^* - x_n\rangle \le 0$, where $x^* = \lim_{n\to\infty} x_{t_n}$ and x_{t_n} is defined by $x_{t_n} = (1/t_n)\int_0^{t_n} S(s)[(I - t_n F)J_r x_{t_n}]\,ds$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to w. Similarly be able to prove $w \in \Gamma$ like Theorem 3.1. Hence, by Theorem 3.1, we have

$$\limsup_{n\to\infty}\bigl\langle Fx^*, x^* - x_n\bigr\rangle = \lim_{k\to\infty}\bigl\langle Fx^*, x^* - x_{n_k}\bigr\rangle = \bigl\langle Fx^*, x^* - w\bigr\rangle \le 0.\tag{3.26}$$

Step 5. Finally, we prove that $\{x_n\}$ converges strongly to $x^* \in \Gamma$. From (3.14), we have

$$\begin{aligned}&\bigl\|x_{n+1} - x^*\bigr\|^2 \\ &\quad= \left\|(1-\alpha_n)y_n + \alpha_n\frac{1}{t_n}\int_0 t_n S(s)J_r y_n\,ds - x^*\right\|^2 \\ &\quad\le (1-\alpha_n)\bigl\|y_n - x^*\bigr\|^2 + \alpha_n\left\|\frac{1}{t_n}\int_0^{t_n} S(s)J_r y_n\,ds - x^*\right\|^2\end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n)\|y_n - x^*\|^2 + \alpha_n\|y_n - x^*\|^2 \\
&\le \|y_n - x^*\|^2 \\
&= \|x_n - \lambda_n F(x_n) - x^*\|^2 \\
&= \|(I-\lambda_n F)x_n - (I-\lambda_n F)x^* - \lambda_n F x^*\|^2 \\
&\le \tau_{\lambda_n}^2 \|x_n - x^*\|^2 + {\lambda_n}^2\|F(x^*)\|^2 + 2\lambda_n\langle (I-\lambda_n F)x^* - (I-\lambda_n F)x_n, F(x^*)\rangle \\
&\le \tau_{\lambda_n}\|x_n - x^*\|^2 + {\lambda_n}^2\|F(x^*)\|^2 + 2\lambda_n\langle x^* - x_n, Fx^*\rangle + 2\lambda_n\langle \lambda_n F x_n, Fx^*\rangle - 2\lambda_n^2\|Fx^*\|^2 \\
&\le [1-(1-\tau_{\lambda_n})]\|x_n - x^*\|^2 + 2\lambda_n\langle x^* - x_n, Fx^*\rangle + 2\lambda_n\|\lambda_n F(x_n)\|\|Fx^*\| - \lambda_n^2\|Fx^*\|^2 \\
&\le [1-(1-\tau_{\lambda_n})]\|x_n - x^*\|^2 + (1-\tau_{\lambda_n})\left[\frac{2\lambda_n}{1-\tau_{\lambda_n}}\langle x^* - x_n, Fx^*\rangle + \frac{2\lambda_n\|Fx^*\|}{1-\tau_{\lambda_n}}\|\lambda_n F(x_n)\|\right] \\
&= (1-\delta_n)\|x_n - x^*\|^2 + \delta_n\rho_n,
\end{aligned}
\tag{3.27}
$$

where $\delta_n = 1-\tau_{\lambda_n}$ and $\rho_n = \frac{2\lambda_n}{1-\tau_{\lambda_n}}\langle x^* - x_n, Fx^*\rangle + \frac{2\lambda_n\|Fx^*\|}{1-\tau_{\lambda_n}}\|\lambda_n F(x_n)\|$. Obviously, $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\rho_n \le 0$. Hence, all conditions of Lemma 2.3 are satisfied. Therefore, we immediately deduce that the sequence $\{x_n\}$ converges strongly to $x^* \in \Gamma$.

Observe that

$$
\|y_n - x^*\| \le \|y_n - x_n\| + \|x_n - x^*\| \le \|\lambda_n F(x_n)\| + \|x_n - x^*\| \to 0, \quad n\to\infty. \tag{3.28}
$$

It is clear that $\{y_n\}$ converges strongly to $x^* \in \Gamma$. From $x^* = \lim_{t\to 0} x_t$ and Theorem 3.1, we see that x^* is the unique solution of the variational inequality (3.2). This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Acknowledgements
The research was supported by Fujian Nature Science Foundation.

Received: 22 March 2014 Accepted: 14 April 2014 Published: 12 May 2014

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10.1186/1029-242X-2014-174
Cite this article as: Jiang and Wang: Hybrid algorithms of nonexpansive semigroups for mixed equilibrium problems, variational inequalities, and fixed point problems. *Journal of Inequalities and Applications* 2014, 2014:174