A note on norm inequalities for positive operators

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Abstract

In this short note, we present a generalization of a norm inequality due to Bhatia and Kittaneh (Lett. Math. Phys. 43:225-231, 1998), which is also a refinement and a generalization of a result obtained by Kittaneh (Commun. Math. Phys. 104:307-310, 1986).

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1 Introduction

Let M_n be the space of $n \times n$ complex matrices. Let $\| \cdot \|$ denote any unitarily invariant norm on M_n. We shall always denote the singular values of A by $s_1(A) \geq \cdots \geq s_n(A) \geq 0$, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. Let $A, B \in M_n$ be Hermitian; the order relation $A \geq B$ means, as usual, that $A - B$ is positive semidefinite.

Let $A, B \in M_n$ be positive semidefinite. Bhatia and Kittaneh [1, Theorem 2.2] proved that for any positive integer m,

$$\left\| A^m + B^m \right\| \leq \left\| (A + B)^m \right\|. \tag{1.1}$$

Let $A, B \in M_n$. Kittaneh [2, Theorem 2.2] proved that

$$\|A + B\|_F^2 \leq \frac{1}{2} \left(\left\| |A| + |B| \right\|_F^2 + \left\| |A^*| + |B^*| \right\|_F^2 \right), \tag{1.2}$$

where $\|X\|_F$ is the Frobenius norm of X. For more information on the Schatten p-norm and its applications, the reader is referred to [3].

In this note, we present a generalization of inequality (1.1), which is also a refinement and a generalization of (1.2).

2 Main results

Now, we show the generalization of inequality (1.1).

Theorem 2.1 *Let $A, B \in M_n$ and suppose that p, q be real numbers with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for any positive integer m,*

$$\left\| A|A|^{m-1} + B|B|^{m-1} \right\| \leq \left\| \left(|A|^m + |B|^m \right)^{p/2} \right\|^{1/p} \left\| \left(|A^*|^m + |B^*|^m \right)^{q/2} \right\|^{1/q}. \tag{2.1}$$

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Proof Let $A, B \in M_n$ with polar decompositions $A = U|A|$ and $B = V|B|$. It is known [4, p.15] that

$$
\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} |A| & |A| \\ |A| & |A| \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & U^* \end{bmatrix} \geq 0.
$$

It follows that

$$
\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix}^m + \begin{bmatrix} |B| & B^* \\ B & |B^*| \end{bmatrix}^m = 2^{m-1} \begin{bmatrix} |A|^m + |B|^m & |A|^m U^* + |B|^m V^* \\ U|A|^m + V|B|^m & |A^*|^m + |B^*|^m \end{bmatrix} \geq 0.
$$

So, by Proposition 1.3.2 of [4, p.13], we have

$$
U|A|^m + V|B|^m = \left(|A|^m + |B|^m\right)^{1/2} K \left(\left|A^*\right|^m + \left|B^*\right|^m\right)^{1/2}
$$

for some contraction K. For $k = 1, \dots, n$, by using Horn's inequality [5, p.72], we know that

$$
\prod_{j=1}^{k} s_j\left(U|A|^m + V|B|^m\right) \leq \prod_{j=1}^{k} s_j^{1/2}\left(|A|^m + |B|^m\right) s_j^{1/2}\left(\left|A^*\right|^m + \left|B^*\right|^m\right). \tag{2.2}
$$

Let

$$
X = \operatorname{diag}\left(s_1^{1/2}\left(|A|^m + |B|^m\right), \dots, s_n^{1/2}\left(|A|^m + |B|^m\right)\right)
$$

and

$$
Y = \operatorname{diag}\left(s_1^{1/2}\left(\left|A^*\right|^m + \left|B^*\right|^m\right), \dots, s_n^{1/2}\left(\left|A^*\right|^m + \left|B^*\right|^m\right)\right).
$$

Then inequality (2.2) is equivalent to

$$
\prod_{j=1}^{k} s_j\left(U|A|^m + V|B|^m\right) \leq \prod_{j=1}^{k} s_j(XY).
$$

Since weak log-majorization implies weak majorization, we get

$$
\sum_{j=1}^{k} s_j\left(U|A|^m + V|B|^m\right) \leq \sum_{j=1}^{k} s_j(XY). \tag{2.3}
$$

Thanks to the Fan dominance principle [5, p.93], we know that inequality (2.3) is equivalent to

$$
\left\| U|A|^m + V|B|^m \right\| \leq \|XY\|. \tag{2.4}
$$

By Hölder's inequality for unitarily invariant norms [5, p.95] (see also [6, Theorem 2.4]), we obtain

$$
\|XY\| \leq \left\| X^p \right\|^{1/p} \left\| Y^q \right\|^{1/q}. \tag{2.5}
$$

It follows from (2.4) and (2.5) that

$$\left\| A|A|^{m-1} + B|B|^{m-1} \right\| \le \left\| \left(|A|^m + |B|^m \right)^{p/2} \right\|^{1/p} \left\| \left(|A^*|^m + |B^*|^m \right)^{q/2} \right\|^{1/q}.$$

This completes the proof. □

Remark 2.2 Inequality (2.1) is a norm version of the following scalar triangle inequality:

$$\left| z_1|z_1|^{m-1} + z_2|z_2|^{m-1} \right| \le \left| \left(|z_1|^m + |z_2|^m \right)^{p/2} \right|^{1/p} \left| \left(|z_1^*|^m + |z_2^*|^m \right)^{q/2} \right|^{1/q}.$$

Corollary 2.3 *Let* $A, B \in M_n$. *Then for any positive integer* m,

$$\left\| A|A|^{m-1} + B|B|^{m-1} \right\| \le \left\| \left(|A| + |B| \right)^m \right\|^{1/2} \left\| \left(|A^*| + |B^*| \right)^m \right\|^{1/2}. \tag{2.6}$$

Proof Putting $p = q = 2$ in inequality (2.1), we have

$$\left\| A|A|^{m-1} + B|B|^{m-1} \right\| \le \left\| |A|^m + |B|^m \right\|^{1/2} \left\| |A^*|^m + |B^*|^m \right\|^{1/2}.$$

It follows from inequality (1.1) and this last inequality that

$$\left\| A|A|^{m-1} + B|B|^{m-1} \right\| \le \left\| \left(|A| + |B| \right)^m \right\|^{1/2} \left\| \left(|A^*| + |B^*| \right)^m \right\|^{1/2}.$$

This completes the proof. □

Remark 2.4 If A and B are positive semidefinite, then by inequality (2.6), we get (1.1).

Remark 2.5 For $m = 1$, by inequality (2.1), we get

$$\| A + B \| \le \left\| \left(|A| + |B| \right)^{p/2} \right\|^{1/p} \left\| \left(|A^*| + |B^*| \right)^{q/2} \right\|^{1/q}.$$

In particular,

$$\| A + B \| \le \left\| |A| + |B| \right\|^{1/2} \left\| |A^*| + |B^*| \right\|^{1/2}, \tag{2.7}$$

which is a generalization and a refinement of inequality (1.2). For the usual operator norm, it is known that

$$\| A + B \|_\infty \le \sqrt{2} \left\| |A| + |B| \right\|_\infty, \tag{2.8}$$

which is sharp. Since

$$\left\| |A^*| + |B^*| \right\| \le \left\| |A^*| \right\| + \left\| |B^*| \right\| \le 2 \left\| |A| + |B| \right\|,$$

we find that inequality (2.7) is a strengthening of inequality (2.8).

Remark 2.6 Recently, Zou [7, Theorem 2.2] and Zou and He [8, Theorem 2.2] gave some generalizations of inequality (1.1) for normal matrices. Our result is different from the ones obtained by these authors.

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