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A note on the equation $x^y + y^z = z^x$

Yulin Lu[1] and Xiaoxue Li[2*]

*Correspondence: lxx20072012@163.com
[2]School of Mathematics, Northwest University, Xi'an, Shaanxi, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, we shall use some simple inequalities and a deep result on the existence of primitive divisors of Lucas numbers to prove that the exponential Diophantine equation $x^y + y^z = z^x$ has no positive integer solution (x, y, z) with $2 \mid y$.

MSC: 11D61

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1 Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of all integers and positive integers, respectively. Recently, Zhang and Yuan [1] were interested in the equation

$$x^y + y^z = z^x, \quad x, y, z \in \mathbb{N}. \tag{1.1}$$

Using the Gel'fond-Baker method, they proved that all solutions (x, y, z) of (1.1) satisfy $\max\{x, y, z\} < \exp(\exp(\exp(5)))$. This upper bound is far beyond the computable scope at present. In this paper, we shall use some simple inequalities and a deep result on the existence of primitive divisors of Lucas numbers to prove the following result.

Theorem *Equation* (1.1) *has no solution* (x, y, z) *with* $2 \mid y$.

In addition, it is obvious that $(x, y, z) = (1, 1, 2)$ is a solution of (1.1). Because one have not found the other solutions, we propose a conjecture as follows:

Conjecture *Equation* (1.1) *has only the solution* $(x, y, z) = (1, 1, 2)$.

Our theorem supports the above mentioned conjecture.

2 Preliminaries

Lemma 2.1 *Let* $f(X) = X/\log X$, *where* X *is a real number. Then* $f(X)$ *is an increasing function for* $X > e$.

Proof Since $f'(X) = (\log X - 1)/(\log X)^2$, we have $f'(X) > 0$ for $X > e$. Thus, the lemma is proved. □

Lemma 2.2 *Let* $g(X) = \sqrt{X} - 2(2 + \log(4X))/\pi$, *where* X *is a real number. Then we have* $g(X) > 0$ *for* $X \geq 16$.

Proof Since $g'(X) = 1/2\sqrt{X} - 2/\pi X > 0$ for $X \geq 16$, g(X) is an increasing function satisfying $g(X) \geq g(16) > 0$ for $X \geq 16$. The lemma is proved. □

Lemma 2.3 ([2, 3]) *The equation*

$$X^2 + 2^m = Y^n, \quad X, Y, m, n \in \mathbb{N}, \gcd(X, Y) = 1, n > 2 \tag{2.1}$$

has only the solutions $(X, Y, m, n) = (5, 3, 1, 3)$ *and* $(7, 3, 5, 4)$.

Lemma 2.4 ([4, Theorem 8.4]) *The equation*

$$X^2 + Y^m = 2^n, \quad X, Y, m, n \in \mathbb{N}, 2 \nmid Y, Y > 1, m > 1 \tag{2.2}$$

has only the solution $(X, Y, m, n) = (13, 7, 3, 9)$.

Lemma 2.5 ([4, Theorem 8.4]) *The equation*

$$X^2 - Y^m = 2^n, \quad X, Y, m, n \in \mathbb{N}, 2 \nmid Y, Y > 1, m > 2, n > 1 \tag{2.3}$$

has only the solution $(X, Y, m, n) = (71, 17, 3, 7)$.

Let D be a positive integer, and let $h(-4D)$ denote the class number of positive binary quadratic primitive forms of discriminant $-4D$.

Lemma 2.6 $h(-4D) \leq D$.

Proof Notice that $h(-4) = h(-8) = h(-28) = 1$, $h(-12) = h(-16) = h(-20) = h(-24) = h(-32) = h(-36) = h(-40) = h(-52) = h(-60) = 2$, $h(-44) = 3$, $h(-48) = h(-56) = 4$. The lemma holds for $D \leq 15$. By Theorems 11.4.3, 12.10.1, and 12.14.3 of [5], if $D \geq 1$, then

$$h(-4D) < \frac{2\sqrt{D}}{\pi}\bigl(2 + \log(4D)\bigr). \tag{2.4}$$

Therefore, if $h(-4D) > D$, then from (2.4) we get

$$\sqrt{D} < \frac{2}{\pi}\bigl(2 + \log(4D)\bigr). \tag{2.5}$$

But, by Lemma 2.2, (2.5) is impossible for $D \geq 16$. Thus, the lemma is proved. □

Lemma 2.7 *Let* k *be a positive integer with* $\gcd(k, 2D) = 1$. *Every solution* (X, Y, Z) *of the equation*

$$X^2 + DY^2 = k^Z, \quad X, Y, Z \in \mathbb{Z}, \gcd(X, Y) = 1, Z > 0, \tag{2.6}$$

can be expressed as

$$Z = Z_1 t, \quad t \in \mathbb{N},$$

$$X + Y\sqrt{-D} = \lambda_1(X_1 + \lambda_2 Y_1\sqrt{-D})^t, \quad \lambda_1, \lambda_2 \in \{\pm 1\},$$

where X_1, Y_1, Z_1 *are positive integers satisfying*

$$X_1^2 + DY_1^2 = k^{Z_1}, \quad \gcd(X_1, Y_1) = 1, Z_1 \mid h(-4D).$$

Proof This lemma is the special case of [6, Theorems 1 and 2] for $D_1 = 1$ and $D_2 < 0$.

Let α, β be algebraic integers. If $\alpha + \beta$ and $\alpha\beta$ are nonzero coprime integers and α/β is not a root of unity, then (α, β) is called a Lucas pair. Further, let $a = \alpha + \beta$ and $c = \alpha\beta$. Then we have

$$\alpha = \frac{1}{2}(a + \lambda\sqrt{b}), \qquad \beta = \frac{1}{2}(a - \lambda\sqrt{b}), \quad \lambda \in \{\pm 1\},$$

where $b = a^2 - 4c$. We call (a, b) the parameters of the Lucas pair (α, β). Two Lucas pairs (α_1, β_1) and (α_2, β_2) are equivalent if $\alpha_1/\alpha_2 = \beta_1/\beta_2 = \pm 1$. Given a Lucas pair (α, β), one defines the corresponding sequence of Lucas numbers by

$$L_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n = 0, 1, 2, \ldots.$$

For equivalent Lucas pairs (α_1, β_1) and (α_2, β_2), we have $L_n(\alpha_1, \beta_1) = \pm L_n(\alpha_2, \beta_2)$ for any $n \geq 0$. A prime p is called a primitive divisor of $L_n(\alpha, \beta)$ $(n > 1)$ if $p \mid L_n(\alpha, \beta)$ and $p \nmid bL_1(\alpha, \beta) \cdots L_{n-1}(\alpha, \beta)$. A Lucas pair (α, β) such that $L_n(\alpha, \beta)$ has no primitive divisor will be called an n-defective Lucas pair. Further, a positive integer n is called totally non-defective if no Lucas pair is n-defective. □

Lemma 2.8 ([7]) *Let n satisfy* $4 < n \leq 30$ *and* $n \neq 6$. *Then, up to equivalence, all parameters of n-defective Lucas pairs are given as follows*:

(i) $n = 5$, $(a, b) = (1, 5), (1, -7), (2, -40), (1, -11), (1, -15), (12, -76), (12, -1{,}364)$.
(ii) $n = 7$, $(a, b) = (1, -7), (1, -19)$.
(iii) $n = 8$, $(a, b) = (2, -24), (1, -7)$.
(iv) $n = 10$, $(a, b) = (2, -8), (5, -3), (5, -47)$.
(v) $n = 12$, $(a, b) = (1, 5), (1, -7), (1, -11), (2, -56), (1, -15), (1, -19)$.
(vi) $n \in \{13, 18, 30\}$, $(a, b) = (1, -7)$.

Lemma 2.9 ([8]) *If* $n > 30$, *then n is totally non-defective*.

3 Further lemmas on the solutions of (1.1)

Throughout this section, we assume that (x, y, z) is a solution of (1.1) with $(x, y, z) \neq (1, 1, 2)$.

Lemma 3.1 ([1]) x, y *and* z *are coprime*.

Lemma 3.2 $\min\{x, y, z\} \geq 3$.

Proof Since $z^x = x^y + y^z > 1$, we have $z > 1$. If $x = 1$, since $(x, y, z) \neq (1, 1, 2)$, then $y > 1$ and $z = 1 + y^z \geq 1 + 2^z \geq z + 3$, a contradiction. Similarly, if $y = 1$, then $x > 1$ and $x + 1 = z^x \geq 2^x \geq x + 2$, a contradiction. Therefore, we have $\min\{x, y, z\} \geq 2$.

If $x = 2$, then

$$2^y + y^z = z^2. \tag{3.1}$$

Further, by Lemma 3.1, y and z are odd integers with $\min\{y,z\} \geq 3$. Hence, we see from (3.1) that (2.3) has the solution $(X,Y,m,n) = (z,y,z,y)$. But, by Lemma 2.5, it is impossible.

Similarly, if $y = 2$ or $z = 2$, then we have

$$x^2 + 2^z = z^x, \quad 2 \nmid xz, \min\{x,z\} \geq 3 \tag{3.2}$$

or

$$x^y + y^2 = 2^x, \quad 2 \nmid xy, \min\{x,y\} \geq 3. \tag{3.3}$$

But, by Lemmas 2.3 and 2.4, (3.2) and (3.3) are impossible. Thus, we get $\min\{x,y,z\} \geq 3$. The lemma is proved. □

Lemma 3.3 $y < x$.

Proof By (1.1), we have $z^x > x^y$ and $z^x > y^z$. Hence,

$$\frac{x}{\log x} > \frac{y}{\log z} \tag{3.4}$$

and

$$\frac{x}{\log y} > \frac{z}{\log z}. \tag{3.5}$$

In addition, by Lemmas 3.1 and 3.2, x, y and z are distinct.

If $x < y < z$, by Lemma 3.2, then $3 \leq x < y < z$. Hence, by Lemma 2.1, we get

$$\frac{z}{\log z} > \frac{x}{\log x} > \frac{x}{\log y}, \tag{3.6}$$

which contradicts (3.5). Similarly, we can remove the case that $x < z < y$.

If $z < x < y$, then $3 \leq z < x < y$ and

$$\frac{y}{\log z} > \frac{y}{\log y} > \frac{x}{\log x}, \tag{3.7}$$

which contradicts (3.4). Thus, we get $y < x$. The lemma is proved. □

4 Proof of theorem

We now assume that (x,y,z) is a solution of (1.1) with $2 \mid y$. Since $(x,y,z) \neq (1,1,2)$, by Lemmas 3.1, 3.2 and 3.3, we have $2 \nmid xz$, $\gcd(y,z) = 1$, $\min\{x,y,z\} \geq 3$ and $x > y$.

We see from (1.1) that the equation

$$X^2 + yY^2 = z^Z, \quad X,Y,Z \in \mathbb{Z}, \gcd(X,Y) = 1, Z > 0 \tag{4.1}$$

has the solution

$$(X,Y,Z) = \left(x^{y/2}, y^{(z-1)/2}, x\right). \tag{4.2}$$

Applying Lemma 2.7 to (4.1) and (4.2), we have

$$x = Z_1 t, \quad t \in \mathbb{N}, \tag{4.3}$$

$$x^{y/2} + y^{(z-1)/2}\sqrt{-y} = \lambda_1\left(X_1 + \lambda_2 Y_1\sqrt{-y}\right)^t, \quad \lambda_1, \lambda_2 \in \{\pm 1\}, \tag{4.4}$$

where X_1, Y_1, Z_1 are positive integers satisfying

$$X_1^2 + yY_1^2 = z^{Z_1}, \quad \gcd(X_1, Y_1) = 1 \tag{4.5}$$

and

$$Z_1 \mid h(-4y). \tag{4.6}$$

Let

$$\alpha = X_1 + Y_1\sqrt{-y}, \qquad \beta = X_1 - Y_1\sqrt{-y}. \tag{4.7}$$

We see from (4.5) and (4.7) that $\alpha + \beta = 2X_1$ and $\alpha\beta = z^{Z_1}$ are coprime nonzero integers, $\alpha/\beta = ((X_1^2 - yY_1^2) + 2X_1Y_1\sqrt{-y})/z^{Z_1}$ is not a root of unity. Hence, (α, β) is a Lucas pair with parameters $(2X_1, -4yY_1^2)$. Further, Let $L_n(\alpha, \beta)$ $(n = 0, 1, 2, \ldots)$ denote the corresponding Lucas numbers. By (4.4) and (4.7), we have

$$y^{(z-1)/2} = \left|L_t(\alpha, \beta)\right|. \tag{4.8}$$

We find from (4.7) and (4.8) that the Lucas number $L_t(\alpha, \beta)$ has no primitive divisor. Therefore, by Lemma 2.9, we have $t \leq 30$. Further, since $2 \nmid x$ and $2 \nmid t$ by (4.3), it is easy to remove all cases in Lemma 2.8 and conclude that $t \in \{1, 3\}$.

If $t = 3$, then from (4.4) we get

$$y^{(z-1)/2} = \lambda_1\lambda_2 Y_1\left(3X_1^2 - yY_1^2\right). \tag{4.9}$$

Let $d = \gcd(Y_1, 3X_1^2 - yY_1^2)$. Since $\gcd(X_1, Y_1) = 1$, we have $d \mid 3$ and $d \in \{1, 3\}$. Further, since $t \mid x$, we get $3 \mid x$, $3 \nmid y$ and $d \neq 3$ by (4.9). Therefore, we have $d = 1$ and, by (4.9), $\gcd(y, 3X_1^2 - yY_1^2) = 1$ and

$$Y_1 = y^{(z-1)/2}, \qquad 3X_1^2 - yY_1^2 = \pm 1. \tag{4.10}$$

It implies that

$$3X_1^2 \mp 1 = y^z. \tag{4.11}$$

But, since $2 \mid y$ and $z \geq 3$, we get from (4.11) that $2 \nmid X_1$ and $0 \equiv y^z \equiv 3X_1^2 \mp 1 \equiv 3 \mp 1 \not\equiv 0 \pmod 8$, a contradiction.

If $t = 1$, then from (4.3) and (4.6) that $x = Z_1$, $x \mid h(-4y)$ and

$$x \leq h(-4y). \tag{4.12}$$

But recall that $x > y$, by Lemma 2.6, (4.12) is impossible. Thus, (1.1) has no solution (x, y, z) with $2 \mid y$. The theorem is proved.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YL obtained the theorems and completed the proof. XL corrected and improved the final version. Both authors read and approved the final manuscript.

Author details
[1]College of Mathematics and Information Science, Weinan Normal University, Weinan, Shaanxi, P.R. China. [2]School of Mathematics, Northwest University, Xi'an, Shaanxi, P.R. China.

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References
1. Zhang, Z-F, Yuan, P-Z: On the diophantine equation $ax^y + by^z + cz^x = 0$. Int. J. Number Theory **8**(3), 813-821 2012
2. Cohn, JHE: The diophantine equation $x^2 + 2^k = y^n$. Arch. Math. Basel **59**(3), 341-343 (1992)
3. Le, M-H: On Cohn's conjecture concerning the diophantine equation $x^2 + 2^m = y^n$. Arch. Math. Basel **78**(1), 26-35 (2002)
4. Bennett, MA, Skinner, CM: Ternary diophantine equations via Galois representations and modular forms. Can. J. Math. **56**(1), 23-54 (2004)
5. Hua, L-K: Introduction to Number Theory. Springer, Berlin (1982)
6. Le, M-H: Some exponential diophantine equations I: the equation $D_1x^2 - D_2y^2 = \lambda k^z$. J. Number Theory **55**(2), 209-221 (1995)
7. Voutier, PM: Primitive divisors of Lucas and Lehmer sequences. Math. Comput. **64**, 869-888 (1995)
8. Bilu, Y, Hanrot, G, Voutier, PM: Existence of primitive divisors of Lucas and Lehmer numbers (with an appendix by M Mignotte). J. Reine Angew. Math. **539**, 75-122 (2001)

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