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Bernstein properties for α-complete hypersurfaces

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Abstract
In the first part of this paper we focus on the Bernstein property of relative surfaces with complete α-metric. As a corollary, we give a new Bernstein type theorem for affine maximal surface and relative extremal surface. In the second part, we offer a relative simple proof of the Bernstein theorem for the affine Kähler-Ricci flat graph with complete α-metric, which was proved in (Li and Xu in Results Math. 54:329-340, 2009), based on a new observation on α-Ricci curvature.

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1 Introduction

In affine differential geometry, there are two famous conjectures about complete affine maximal surfaces, stated by Calabi and Chern, respectively. Chern assumes that the maximal surface is a convex graph on the whole R^2, while Calabi supposes it is complete with respect to the Blaschke metric. Both versions of affine Bernstein problems attracted many mathematicians, and they were solved during the last decade. For Chern's conjecture, please see related work [1–4] and [5]. For Calabi's conjecture see [4, 6] and [7]. The two conjectures differ in the assumptions on the completeness of the affine maximal surface considered. On the other hand, Li and Jia in [8] considered the Bernstein problem of an affine maximal hypersurface with complete Calabi metric and proved the following.

Theorem *Let $x_{n+1} = f(x_1, \dots, x_n)$ be a locally strongly convex function defined in a domain $\Omega \subset A^n$. If $M = \{(x, f(x)) \mid (x_1, \dots, x_n) \in \Omega\}$ is an affine maximal hypersurface, and if M is complete with respect to the Calabi metric $G = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j$, then, in the case where $n = 2$ or $n = 3$, M must be an elliptic paraboloid.*

Here we consider the Bernstein property of affine maximal surfaces with a more general complete relative metric, which is called an α-metric in [9] (or a Li metric in [10] and [11])

$$G^{(\alpha)} := \rho^\alpha \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j, \qquad \rho := \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^{-\frac{1}{n+2}},$$

where α is a constant. Obviously, it is a conformal metric to the Blaschke metric ($\alpha = 1$) and the Calabi metric ($\alpha = 0$). In fact, we study more general surfaces satisfying a fourth order partial differential equation (PDE), which include affine maximal hypersurface equations (see [8]), an α-relative extremal hypersurface equation (see [12]) and the Abreu equation (see [13]),

$$\Delta\rho = -\beta\frac{\|\nabla\rho\|^2}{\rho}, \tag{1.1}$$

where β is a constant, and the Laplacian Δ and the norm are defined with respect to the Calabi metric. The first result we obtain is as follows.

Theorem 1.1 *Let $f(x_1, x_2)$ be a strictly convex function defined on a convex domain $\Omega \in R^2$, which satisfies the PDE* (1.1). *Assume that the graph surface $(x,f(x))$ is complete with respect to the α-metric. Then, if $\alpha + \alpha\beta + 2(\beta+1)^2 - 1 > 0$, f must be a quadratic polynomial.*

From Theorem 1.1 we can obtain the following three corollaries.

For affine maximal surfaces (the case $\beta = 0$ in Theorem 1.1), we have the following.

Corollary 1.2 *If an affine maximal surface given by a strictly convex function is complete with respect to α-metric with $\alpha > -1$, then it must be an elliptic paraboloid.*

This is a new result about affine maximal surface, which generalizes the above theorem in [8] in dimension 2.

If α-relative extremal hypersurfaces are given by a strictly convex function f, then f should satisfy the PDE (1.1) with $\beta = \frac{n\alpha-2}{2}$; see [12]. For α-relative extremal surfaces, we have the following.

Corollary 1.3 *Let $y : M \to R^3$ be a locally strongly convex α-relative extremal surface, complete with respect to the α-metric, which is given by a locally strongly convex function. Then, if $\alpha^2 > \frac{1}{3}$, M is an elliptic paraboloid.*

In [12] Xu-Xiong-Sheng used an affine blow up analysis to prove the Bernstein theorems for α-relative extremal hypersurfaces with complete α-metrics. Corollary 1.3 gives a new analytic and relative simple proof of the Bernstein property for α-relative extremal surfaces with complete α-metric.

In [14], Jia and Li obtained a Bernstein property for the Abreu equation in dimension $n \le 5$ under the assumption of completeness with respect to the Calabi metric. Later Xiong and Sheng used a blow up analysis to prove a Bernstein property for the Abreu equation under the assumption of completeness with respect to the $\frac{n+2}{2}$-metric in [13]. By Theorem 1.1, we get the following.

Corollary 1.4 *Let $f(x_1, x_2)$ be a strictly convex function defined on a convex domain $\Omega \in R^2$, which satisfies the Abreu equation ($\beta = -3$ in PDE* (1.1)). *Assume that the graph surface $(x,f(x))$ is complete with respect to the α-metric. Then, if $\alpha < \frac{7}{2}$, f must be a quadratic polynomial.*

Remark The restriction on α and β is necessary in Theorem 1.1. Namely, there exist some α and β such that the solutions of the PDE (1.1) are not unique. For example, the function

$$f(x) = e^{x_1} + \frac{1}{2}x_2^2,$$

which satisfies the PDE (1.1) with $\beta = -3$. Its graph is complete with respect to the $\alpha = 4$ metric.

In the second part of this paper we study the following PDE:

$$\det\left(\frac{\partial^2 f}{\partial x_i\,\partial x_j}\right) = \exp\left\{\sum_{i=1}^{n} d_i x_i + d_0\right\}, \tag{1.2}$$

where $d_0, d_1, \ldots, d_n$ are real constants. In [15] and [16], Li and the first author have shown that the affine Kähler-Ricci flat graph hypersurface has a rigidity property under different completeness conditions. By deriving a differential inequality for the Laplacian ΔJ of the relative Pick invariant J and combining with estimating the norm of the relative Tchebychev vector Φ and J on a geodesic ball, we proved in [15] the following.

Theorem 1.5 *Let f be a strictly convex C^∞-function defined on a convex domain $\Omega \subset \mathbb{R}^n$ satisfying the PDE* (1.2). *Define*

$$M := \left\{\left(x, f(x)\right) \mid x_{n+1} = f(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in \Omega\right\}.$$

If $\alpha \neq n+2$ and M is complete with respect to the α-metric $G^{(\alpha)}$ then M must be an elliptic paraboloid.

Here we shall give it a relative simple proof, and do not need to estimate the relative Pick invariant J based on a new observation on α-Ricci curvature (see (4.8)).

2 α-Relative geometry

Let f be a strictly convex C^∞-function defined on a domain $\Omega \subset \mathbb{R}^n$. Again we write

$$M := \left\{\left(x, f(x)\right) \mid x_{n+1} = f(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in \Omega\right\}.$$

We consider the Riemannian metric g on M, defined by

$$g = \sum f_{ij}\, dx_i\, dx_j,$$

which is called the *Calabi metric*. It is the relative metric with respect to Calabi's normalization $Y = (0, 0, \ldots, 1)$. For the position vector $y = (x_1, \ldots, x_n, f(x_1, \ldots, x_n))$ we have

$$y_{,ij} = \sum A_{ij}^k y_k + f_{ij} Y,$$

and for the conormal field

$$U = (-f_1, \ldots, -f_n, 1). \tag{2.1}$$

Let q be a given function defined on M such that $q > 0$ everywhere. Li introduced a relative normalization of graph hypersurface M, given by its conormal vector field $U^{(q)} = qU$, where $q = \rho^\alpha$. Then there is a unique transversal vector field $Y^{(q)}$ such that

$$\left\langle U_i^{(q)}, Y^{(q)}\right\rangle = 0, \qquad \left\langle U^{(q)}, Y^{(q)}\right\rangle = 1. \tag{2.2}$$

Under this normalization, the corresponding components of the relative metric $G^{(q)}_{ij}$, the relative Pick tensor $A^{(q)}_{ijk}$, and the relative Weingarten tensor $B^{(q)}_{ij}$ satisfy (for details see [9])

$$G^{(q)}_{ij} = qf_{ij} = \rho^{\alpha} f_{ij}, \tag{2.3}$$

$$\begin{aligned} A^{(q)}_{ijk} &= -\frac{1}{2}\left(f_{ij}\frac{\partial q}{\partial x_k} + f_{ik}\frac{\partial q}{\partial x_j} + f_{jk}\frac{\partial q}{\partial x_i} + qf_{ijk}\right) \\ &= -\frac{1}{2}\left(\alpha\left(G_{ij}\frac{\rho_k}{\rho} + G_{ik}\frac{\rho_j}{\rho} + G_{jk}\frac{\rho_i}{\rho}\right) + \rho^{\alpha} f_{ijk}\right), \end{aligned} \tag{2.4}$$

$$\begin{aligned} B^{(q)}_{ij} &= -\frac{1}{q}\frac{\partial^2 q}{\partial x_i \partial x_j} + \frac{2}{q^2}\frac{\partial q}{\partial x_i}\frac{\partial q}{\partial x_j} + \sum \frac{f^{kl}}{q}\frac{\partial q}{\partial x_l}\frac{\partial f_{ij}}{\partial x_k} \\ &= -\alpha\frac{\rho_{ij}}{\rho} + \alpha(\alpha+1)\frac{\rho_i\rho_j}{\rho^2} + \alpha f^{kl} f_{ijk}\frac{\rho_l}{\rho}. \end{aligned} \tag{2.5}$$

To simplify, we omit the upper index (q) in (2.4) and later. From the integrability conditions, the components of the Ricci tensor read

$$R_{ik} = \sum_{m,l}(A_{iml}A_{mlk} - A_{imk}A_{mll}) + \frac{n-2}{2}B_{ik} + \frac{n}{2}L_1 G_{ik}. \tag{2.6}$$

Under the above α-relative normalization,

$$\sum_l A_{mll} = \sum_{i,j} G^{ij} A_{ijm} = \frac{(n+2)(1-\alpha)}{2}\frac{\rho_m}{\rho}. \tag{2.7}$$

The α-relative mean curvature is

$$L_1 = \frac{1}{n}\sum G^{ij}B_{ij} = -\frac{\alpha}{n}\rho^{-\alpha-1} f^{ij}\left(\rho_{ij} - (\alpha - n - 1)\frac{\rho_i\rho_j}{\rho}\right). \tag{2.8}$$

By (2.8) and the PDE (1.1), we have

$$L_1 = \frac{\alpha}{n}\left(\beta + \alpha - \frac{n}{2}\right)\Phi, \tag{2.9}$$

where

$$\Phi := \rho^{-\alpha}\sum f^{ij}\frac{\rho_i}{\rho}\frac{\rho_j}{\rho}. \tag{2.10}$$

3 Proof of Theorem 1.1

In the following we will use the α-metric to do the calculations. That is to say, the norms and the Laplacian operator are defined with respect to the α-metric. Firstly we will estimate $\Delta\Phi$. Using the relationship between conformal metrics and Proposition 3.1 in [2] (see also Proposition 4.5.2 in [4]), we can prove the following.

Proposition 3.1 *Let f be a strictly convex function satisfying the PDE* (1.1). *Then the Laplacian of Φ satisfies the following inequality*:

$$\begin{aligned} \Delta\Phi \geq{}& \frac{n}{2(n-1)}\frac{\|\nabla\Phi\|^2}{\Phi} + \left(\frac{n^2-5n+6}{2(n-1)}\alpha - \frac{(n-2)(2\beta+2)}{n-1}\right)\langle\nabla\Phi, \nabla\ln\rho\rangle \\ &+ \left(\frac{2-n}{2(n-1)}\alpha^2 + \frac{3-n}{n-1}(\alpha+\alpha\beta) + \frac{2(\beta+1)^2}{n-1} - \frac{(n+2)^2(n-1)}{8n}\right)\Phi^2. \end{aligned}$$

In particular, for $n = 2$, we have

$$\Delta\Phi \geq \frac{\|\nabla\Phi\|^2}{\Phi} + \left(\alpha + \alpha\beta + 2(\beta+1)^2 - 1\right)\Phi^2. \tag{3.1}$$

Secondly we shall prove $\Phi \equiv 0$. Denote by $s(p_0, p)$ the geodesic distance function from $p_0 \in M$ with respect to the α-metric. For any positive number a, let $B_a(p_0) := \{p \in M \mid s(p_0, p) \leq a\}$. In the following we derive an estimate of Φ in a geodesic ball $B_a(p_0)$. Denote

$$\mathcal{A} := \max_{B_a(p_0)} \left\{\left(a^2 - s^2\right)^2 \Phi\right\}.$$

Lemma 3.2 *Let $f(x_1, x_2)$ be a strictly convex C^∞-function satisfying the PDE* (1.1). *If $\alpha + \alpha\beta + 2(\beta+1)^2 - 1 > 0$ and $B_a(p_0)$ is compact, then there exists a constant $C > 0$ depending only on β and α such that*

$$\mathcal{A} \leq Ca^2.$$

Proof Consider the function

$$F := \left(a^2 - s^2\right)^2 \Phi \tag{3.2}$$

defined on $B_a(p_0)$. Obviously, F attains its supremum at some interior point p^*. We may assume that s^2 is a C^2-function in a neighborhood of p^*, and $\Phi(p^*) > 0$. Choose an orthonormal frame field on M around p^* with respect to the α-metric. Then, at p^*,

$$\frac{\Phi_{,i}}{\Phi} - \frac{4ss_{,i}}{a^2 - s^2} = 0, \tag{3.3}$$

$$\frac{\Delta\Phi}{\Phi} - \frac{\sum(\Phi_{,i})^2}{\Phi^2} - \frac{8s^2}{(a^2 - s^2)^2} - \frac{4s\Delta s}{a^2 - s^2} - \frac{4}{a^2 - s^2} \leq 0, \tag{3.4}$$

where we use the fact

$$\|\nabla s\|^2 = 1,$$

',' denotes the covariant derivative with respect to the α metric. Inserting (3.1) into (3.4) we get

$$\left(\alpha + \alpha\beta + 2(\beta+1)^2 - 1\right)\Phi - \frac{12a^2}{(a^2 - s^2)^2} - \frac{4s\Delta s}{a^2 - s^2} \leq 0. \tag{3.5}$$

Now we calculate the term $\frac{4s\Delta s}{a^2 - s^2}$. Denote $a^* = s(p_0, p^*)$. Assume that

$$\max_{B_{a^*}(p_0)} \Phi = \Phi(\tilde{p}).$$

In the case

$$\alpha + \alpha\beta + 2(\beta+1)^2 - 1 > 0,$$

by (3.1), we know that

$$\max_{p\in B_{a^*}(p_0)} \Phi(p) = \max_{p\in \partial B_{a^*}(p_0)} \Phi(p).$$

On the other hand, we have $a^2 - r^2 = a^2 - (a^*)^2$ on $\partial B_{a^*}(p_0)$, and it follows that

$$\max_{p\in B_{a^*}(p_0)} \Phi(p) = \Phi\big(p^*\big).$$

By (2.6), (2.7), and (2.9) we obtain

$$R_{ik} = A_{iml}A_{mlk} - 2(1-\alpha)A_{imk}\frac{\rho_m}{\rho} + \frac{\alpha}{2}(\beta+\alpha-1)\Phi G_{ik}. \tag{3.6}$$

For any $p \in B_{a^*}(p_0, G^{(\alpha)})$, by a coordinate transformation we may assume that $f_{ij}(p) = \delta_{ij}$ and $R_{ij}(p) = 0$ for $i \neq j$. Then from (3.6), using the Schwartz inequality we know that the Ricci curvature $\mathrm{Ric}(M, G)$ with respect to the α-metric on $B_{a^*}(p_0)$ is bounded from below by

$$\mathrm{Ric}(M,G) \geq -C_1\Phi\big(p^*\big)G, \tag{3.7}$$

for some positive constant C_1, depending only on β and α. By the Laplacian comparison theorem (see [17]), we get

$$s\Delta s \leq \big(1 + C_1\sqrt{\Phi}\big(p^*\big)s\big). \tag{3.8}$$

Substituting (3.8) into (3.5) yields

$$\big(\alpha + \alpha\beta + 2(\beta+1)^2 - 1\big)\Phi \leq \frac{16a^2}{(a^2-s^2)^2} + \frac{4aC_1}{a^2-s^2}\sqrt{\Phi}. \tag{3.9}$$

Multiplying both sides of (3.9) by $(a^2 - s^2)^2(p^*)$ we have

$$\big(\alpha + \alpha\beta + 2(\beta+1)^2 - 1\big)\mathcal{A} \leq C_1\mathcal{A}^{\frac{1}{2}}a + 16a^2. \tag{3.10}$$

In the case

$$\alpha + \alpha\beta + 2(\beta+1)^2 - 1 > 0$$

we have

$$\mathcal{A} \leq C_2a^2, \tag{3.11}$$

where C_2 is a positive constant depending only on α and β. □

Proof of Theorem 1.1 Using Lemma 3.2, at any interior point of $B_a(p_0)$, we obtain

$$\Phi \leq C\frac{a^2}{(a^2-s^2)^2}.$$

For $a \to \infty$ we get

$$\Phi \equiv 0.$$

This means that M is an affine complete parabolic hypersphere. We apply a result of Calabi (see [17], p.128) and conclude that M must be an elliptic paraboloid. This completes the proof of Theorem 1.1. □

4 Proof of Theorem 1.5

As before, using the relationship between conformal metrics and Proposition 3.1 in [16], we have the following.

Proposition 4.1 *Let f be a strictly convex C^∞-function satisfying the PDE* (1.2). *Then the Laplacian of Φ satisfies the following inequality*:

$$\Delta\Phi \geq \frac{n}{n-1}\frac{\|\nabla\Phi\|^2}{\Phi} + \left(\frac{n^2-3n+6}{2(n-1)}\alpha + \frac{n^2-3n-10}{2(n-1)}\right)\langle\nabla\Phi, \nabla\ln\rho\rangle + \frac{(n+2-\alpha)^2}{n-1}\Phi^2.$$

Lemma 4.2 *Let f be a strictly convex C^∞-function satisfying the PDE* (1.2). *If $\alpha \neq n+2$ and $B_a(p_0)$ is compact, then there exists a constant $C > 0$ depending only on n such that*

$$\Phi \leq C\frac{a^2}{(a^2-s^2)^2}.$$

Proof Consider the function

$$F := (a^2-s^2)^2\Phi$$

defined on $B_a(p_0)$. Obviously, F attains its supremum at some interior point p^*. We may assume that s^2 is a C^2-function in a neighborhood of p^*, and $\Phi(p^*) > 0$. Choose an orthonormal frame field on M around p^* with respect to the α-metric. Then, at p^*,

$$\frac{\Phi_{,i}}{\Phi} - \frac{4ss_{,i}}{a^2-s^2} = 0, \tag{4.1}$$

$$\frac{\Delta\Phi}{\Phi} - \frac{\sum(\Phi_{,i})^2}{\Phi^2} - \frac{8s^2}{(a^2-s^2)^2} - \frac{4s\Delta s}{a^2-s^2} - \frac{4}{a^2-s^2} \leq 0. \tag{4.2}$$

Inserting Proposition 4.1 into (4.2) we get

$$\begin{aligned}&\frac{1}{n-1}\frac{\|\nabla\Phi\|^2}{\Phi^2} + \left(\frac{n^2-3n+6}{2(n-1)}\alpha + \frac{n^2-3n-10}{2(n-1)}\right)\langle\ln\nabla\Phi, \nabla\ln\rho\rangle + \frac{(n+2-\alpha)^2}{n-1}\Phi \\ &\quad - \frac{12a^2}{(a^2-s^2)^2} - \frac{4s\Delta s}{a^2-s^2} \leq 0.\end{aligned} \tag{4.3}$$

Using (4.1) and the Schwartz inequality we obtain

$$\left(\frac{(n+2-\alpha)^2}{n-1} - \epsilon\right)\Phi - \frac{C_1a^2}{(a^2-s^2)^2} - \frac{4s\Delta s}{a^2-s^2} \leq 0, \tag{4.4}$$

where ϵ is a positive constant to be determined later, and C_1 is a positive constant depending only n, α, and ϵ.

Next we calculate the term $\frac{4s\Delta s}{a^2-s^2}$. Denote $a^* = s(p_0, p^*)$. Reasoning as in Section 3, in the case $\alpha \neq n+2$, by Proposition 4.1, we see that

$$\max_{p\in B_{a^*}(p_0)} \Phi(p) = \Phi(p^*).$$

By (2.4), (2.5), and (1.2) we get

$$B_{ik} = \alpha^2 \frac{\rho_i\rho_k}{\rho^2} + \alpha f^{lj} f_{ikj}\frac{\rho_l}{\rho} = -2\alpha G^{lj} A_{ikj}\frac{\rho_l}{\rho} - \alpha^2\left(\frac{\rho_i\rho_k}{\rho^2} + \Phi G_{ik}\right). \tag{4.5}$$

By (2.6), (2.7), (2.9), (1.2), and (4.5) we obtain

$$\begin{aligned} R_{ik} &= A_{il}^m A_{mk}^l - \frac{(n+2)+(n-6)\alpha}{2} A_{ik}^m \frac{\rho_m}{\rho} \\ &\quad - \frac{n-2}{2}\alpha^2\frac{\rho_i\rho_k}{\rho^2} + \left(\frac{3-n}{2}\alpha^2 - \frac{n+2}{2}\alpha\right)\Phi G_{ik}. \end{aligned} \tag{4.6}$$

For any $p \in B_{a^*}(p_0)$, by a coordinate transformation we may assume that $f_{ij}(p) = \delta_{ij}$ and $R_{ij}(p) = 0$ for $i \neq j$. Then from (4.6), using the Schwartz inequality we know

$$\begin{aligned} R_{ii}(p) &\geq \sum_{l,m} A_{ilm}^2 - \frac{(n+2)+(n-6)\alpha}{2}\sum_m A_{iim}\frac{\rho_m}{\rho} + \left(\frac{5-2n}{2}\alpha^2 - \frac{n+2}{2}\alpha\right)\Phi \\ &\geq \sum_m \left(A_{iim}^2 - \frac{(n+2)+(n-6)\alpha}{2} A_{iim}\frac{\rho_m}{\rho}\right) + \left(\frac{5-2n}{2}\alpha^2 - \frac{n+2}{2}\alpha\right)\Phi \\ &\geq \left(\frac{5-2n}{2}\alpha^2 - \frac{n+6}{2}\alpha - \frac{[(n+2)+(n-6)\alpha]^2}{16}\right)\Phi \\ &:= -C_2\Phi(p), \end{aligned} \tag{4.7}$$

where C_2 is a positive constant depending only on n and α. Then the Ricci curvature $\mathrm{Ric}(M, G)$ with respect to the α-metric on $B_{a^*}(p_0)$ is bounded from below by

$$\mathrm{Ric}(M, G) \geq -C_2\Phi(p^*)G. \tag{4.8}$$

By the Laplacian comparison theorem (see [17]), we get

$$s\Delta s \leq (n-1)\left(1 + C_2\sqrt{\Phi}(p^*)s\right). \tag{4.9}$$

Substituting (4.9) into (4.4) yields

$$\left(\frac{(n+2-\alpha)^2}{n-1} - \epsilon\right)\Phi \leq \frac{C_3a^2}{(a^2-s^2)^2} + \frac{4aC_2}{a^2-s^2}\sqrt{\Phi}. \tag{4.10}$$

Multiplying both sides of (4.10) by $(a^2-s^2)^2(p^*)$ we have

$$\left(\frac{(n+2-\alpha)^2}{n-1} - \epsilon\right)\mathcal{A} \leq 4C_2\mathcal{A}^{\frac{1}{2}}a + C_3a^2, \tag{4.11}$$

where $\mathcal{A}$ is defined as before. In the case $\alpha \neq n+2$, we may choose ϵ small enough such that $\frac{(n+2-\alpha)^2}{n-1} - \epsilon > 0$. Then

$$\mathcal{A} \leq C_4 a^2, \tag{4.12}$$

where C_4 is a positive constant depending only on α and n. □

Using the same method as in the proof of Theorem 1.1, we complete the proof of Theorem 1.5.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RX participated in the affine geometry and conformal metric studies in the paper, LC carried out the Laplacian operator and the applications of inequalities studies. Both authors read and approved the final manuscript.

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