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Strong convergence algorithm for approximating the common solutions of a variational inequality, a mixed equilibrium problem and a hierarchical fixed-point problem

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This paper is dedicated to Yassmine Bnouhachem

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Abstract

This paper investigates the common set of solutions of a variational inequality, a mixed equilibrium problem, and a hierarchical fixed-point problem in a Hilbert space. A numerical method is proposed to find the approximate element of this common set. The strong convergence of this method is proved under some conditions. The proposed method is shown to be an improvement and extension of some known results.

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1 Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Let C be a nonempty closed convex subset of H and A be a mapping from C into H. A classical variational inequality problem, denoted by $\mathrm{VI}(A,C)$, is to find a vector $u \in C$ such that

$$\langle v-u, Au\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

The solution of $\mathrm{VI}(A,C)$ is denoted by Ω^*. It is easy to observe that

$$u^* \in \Omega^* \quad \Longleftrightarrow \quad u^* = P_C\bigl[u^* - \rho Au^*\bigr], \quad \text{where } \rho > 0.$$

We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and the related optimization problems; see [1–25]. The fixed-point theory has played an important role in the development of various algorithms for solving variational inequalities. Using the projection operator technique, one usually establishes an equivalence between the variational inequalities and the fixed-point problem. This alternative equivalent formulation was used by Lions and Stampacchia [1] to study the existence of a solution of the variational inequalities.

We introduce the following definitions, which are useful in the following analysis.

Definition 1.1 The mapping $T : C \to H$ is said to be

(a) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(b) strongly monotone if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C;$$

(c) α-inverse strongly monotone if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \quad \forall x, y \in C;$$

(d) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(e) k-Lipschitz continuous if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C;$$

(f) contraction on C if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

It is easy to observe that every α-inverse strongly monotone T is monotone and Lipschitz continuous. A mapping $T : C \to H$ is called k-strict pseudo-contraction if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C. \tag{1.2}$$

The fixed-point problem for the mapping T is to find $x \in C$ such that

$$Tx = x. \tag{1.3}$$

We denote by $F(T)$ the set of solutions of (1.3). It is well known that the class of strict pseudo-contractions includes the class of Lipschitzian mappings, then $F(T)$ is closed and convex and $P_{F(T)}$ is well defined (see [2]).

The mixed equilibrium problem, denoted by *MEP*, is to find $x \in C$ such that

$$F_1(x, y) + \langle Dx, y - x\rangle \geq 0, \quad \forall y \in C, \tag{1.4}$$

where $F_1 : C \times C \to \mathbb{R}$ is a bifunction, and $D : C \to H$ is a nonlinear mapping. This problem was introduced and studied by Moudafi and Théra [3] and Moudafi [4]. The set of solutions of (1.4) is denoted by

$$MEP(F_1) := \{x \in C : F_1(x,y) + \langle Dx, y - x\rangle \geq 0, \forall y \in C\}. \tag{1.5}$$

If $D = 0$, then it is reduced to the equilibrium problem, which is to find $x \in C$ such that

$$F_1(x,y) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The solution set of (1.6) is denoted by $EP(F_1)$. Numerous problems in physics, optimization, and economics reduce to finding a solution of (1.6); see [5–9]. In 1997, Combettes and Hirstoaga [10] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty. Recently Plubtieng and Punpaeng [7] introduced an iterative method for finding the common element of the set $F(T) \cap \Omega^* \cap EP(F_1)$.

Let $S : C \to H$ be a nonexpansive mapping. The following problem is called a hierarchical fixed point problem: Find $x \in F(T)$ such that

$$\langle x - Sx, y - x\rangle \geq 0, \quad \forall y \in F(T). \tag{1.7}$$

It is known that the hierarchical fixed-point problem (1.7) links with some monotone variational inequalities and convex programming problems; see [11, 12, 26]. Various methods have been proposed to solve the hierarchical fixed-point problem; see Moudafi [13], Mainge and Moudafi in [14], Marino and Xu in [15] and Cianciaruso *et al.* [16]. Very recently, Yao *et al.* [12] introduced the following strong convergence iterative algorithm to solve problem (1.7):

$$\begin{aligned} y_n &= \beta_n S x_n + (1 - \beta_n)x_n, \\ x_{n+1} &= P_C[\alpha_n f(x_n) + (1 - \alpha_n)Ty_n], \quad \forall n \geq 0, \end{aligned} \tag{1.8}$$

where $f : C \to H$ is a contraction mapping, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. Under some certain restrictions on the parameters, Yao *et al.* proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (I - f)z, y - z\rangle \geq 0, \quad \forall y \in F(T). \tag{1.9}$$

In 2011, Ceng *et al.* [17] investigated the following iterative method:

$$x_{n+1} = P_C[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n))], \quad \forall n \geq 0, \tag{1.10}$$

where U is a Lipschitzian mapping, and F is a Lipschitzian and strongly monotone mapping. They proved that under some approximate assumptions on the operators and parameters, the sequence $\{x_n\}$ generated by (1.10) converges strongly to the unique solution of the variational inequality

$$\langle \rho U(z) - \mu F(z), x - z\rangle \geq 0, \quad \forall x \in \operatorname{Fix}(T).$$

In this paper, motivated by the work of Yao *et al.* [12], Ceng *et al.* [17], Bnouhachem [18, 19] and by the recent work going in this direction, we give an iterative method for finding the approximate element of the common set of solutions of (1.1), (1.4), and (1.7) in a real Hilbert space. We establish a strong convergence theorem based on this method. We would like to mention that our proposed method is quite general and flexible and includes many known results for solving equilibrium problems, variational inequality problems, and hierarchical fixed-point problems; see, *e.g.*, [11, 12, 14–17, 24, 25] and relevant references cited therein.

2 Preliminaries

In this section, we list some fundamental lemmas that are useful in the consequent analysis. The first lemma provides some basic properties of projection onto C.

Lemma 2.1 *Let P_C denote the projection of H onto C. Then we have the following inequalities*:

$$\langle z - P_C[z], P_C[z] - v\rangle \geq 0, \quad \forall z \in H, v \in C; \tag{2.1}$$

$$\langle u - v, P_C[u] - P_C[v]\rangle \geq \|P_C[u] - P_C[v]\|^2, \quad \forall u, v \in H; \tag{2.2}$$

$$\|P_C[u] - P_C[v]\| \leq \|u - v\|, \quad \forall u, v \in H; \tag{2.3}$$

$$\|u - P_C[z]\|^2 \leq \|z - u\|^2 - \|z - P_C[z]\|^2, \quad \forall z \in H, u \in C. \tag{2.4}$$

Lemma 2.2 [20] *Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions*:

(i) $F_1(x,x) = 0$, $\forall x \in C$;
(ii) *F_1 is monotone, i.e.*, $F_1(x,y) + F_1(y,x) \leq 0$, $\forall x, y \in C$;
(iii) *for each $x, y, z \in C$*, $\lim_{t\to 0} F_1(tz + (1-t)x, y) \leq F_1(x,y)$;
(iv) *for each $x \in C$, $y \to F_1(x,y)$ is convex and lower semicontinuous.*

Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that

$$F_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.3 [10] *Assume that $F_1 : C \times C \to \mathbb{R}$ satisfies assumptions* (i)-(iv) *of Lemma* 2.2, *and for $r > 0$ and $\forall x \in H$, define a mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \left\{ z \in C : F_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}.$$

Then the following hold:

(i) *T_r is single-valued*;
(ii) *T_r is firmly nonexpansive, i.e.*,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H;$$

(iii) $F(T_r) = EP(F_1)$;
(iv) *$EP(F_1)$ is closed and convex.*

Lemma 2.4 [21] *Let C be a nonempty closed convex subset of a real Hilbert space H. If $T : C \to C$ is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, then the mapping $I - T$ is demiclosed at 0, i.e., if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)x = 0$.*

Lemma 2.5 [17] *Let $U : C \to H$ be a τ-Lipschitzian mapping, and let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone mapping, then for $0 \leq \rho\tau < \mu\eta$, $\mu F - \rho U$ is $\mu\eta - \rho\tau$-strongly monotone, i.e.,*

$$\langle (\mu F - \rho U)x - (\mu F - \rho U)y, x - y \rangle \geq (\mu\eta - \rho\tau)\|x - y\|^2, \quad \forall x, y \in C.$$

Lemma 2.6 [22] *Suppose that $\lambda \in (0,1)$ and $\mu > 0$. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator. In association with a nonexpansive mapping $T : C \to C$, define the mapping $T^\lambda : C \to H$ by*

$$T^\lambda x = Tx - \lambda\mu FT(x), \quad \forall x \in C.$$

Then T^λ is a contraction provided $\mu < \frac{2\eta}{k^2}$, that is,

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\nu)\|x - y\|, \quad \forall x, y \in C,$$

where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$.

Lemma 2.7 [23] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, and δ_n is a sequence such that

(1) $\sum_{n=1}^\infty \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^\infty |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 [27] *Let C be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that*

(i) *the weak w-limit set $w_w(x_n) \subset C$, where $w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\}$;*
(ii) *for each $z \in C$, $\lim_{n\to\infty} \|x_n - z\|$ exists.*

Then $\{x_n\}$ is weakly convergent to a point in C.

3 The proposed method and some properties

In this section, we suggest and analyze our method for finding the common solutions of the variational inequality (1.1), the mixed equilibrium problem (1.4), and the hierarchical fixed-point problem (1.7).

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D, A : C \to H$ be θ, α-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (i)-(iv) of Lemma 2.2 and $S, T : C \to C$ be a nonexpansive mappings such that $F(T) \cap \Omega^* \cap MEP(F_1) \neq \emptyset$. Let $F : C \to C$ be a k-Lipschitzian mapping and be η-strongly monotone, and let $U : C \to C$ be a τ-Lipschitzian mapping.

Algorithm 3.1 For an arbitrary given $x_0 \in C$, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated by

$$\begin{aligned} &F_1(u_n, y) + \langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C; \\ &z_n = P_C[u_n - \lambda_n A u_n]; \\ &y_n = \beta_n S x_n + (1-\beta_n) z_n; \\ &x_{n+1} = P_C\bigl[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)\bigl(T(y_n)\bigr)\bigr], \quad \forall n \geq 0, \end{aligned} \tag{3.1}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{r_n\} \subset (0, 2\theta)$. Suppose that the parameters satisfy $0 < \mu < \frac{2\eta}{k^2}$, $0 \leq \rho\tau < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Also, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $\lim_{n\to\infty} (\beta_n/\alpha_n) = 0$,
(c) $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,
(d) $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$,
(e) $\liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 2\alpha$ and $\sum_{n=1}^{\infty} |\lambda_{n-1} - \lambda_n| < \infty$.

Remark 3.1 Our method can be viewed as an extension and improvement for some well-known results, for example, the following.

- If $A = 0$, we obtain an extension and improvement of the method of Wang and Xu [24] for finding the approximate element of the common set of solutions of a mixed equilibrium problem and a hierarchical fixed-point problem in a real Hilbert space.
- If we have the Lipschitzian mapping $U = f$, $F = I$, $\rho = \mu = 1$, and $A = 0$, we obtain an extension and improvement of the method of Yao *et al.* [12] for finding the approximate element of the common set of solutions of a mixed equilibrium problem and a hierarchical fixed-point problem in a real Hilbert space.
- The contractive mapping f with a coefficient $\alpha \in [0,1)$ in other papers [12, 15, 22, 25] is extended to the cases of the Lipschitzian mapping U with a coefficient constant $\gamma \in [0, \infty)$.

This shows that Algorithm 3.1 is quite general and unifying.

Lemma 3.1 *Let $x^* \in F(T) \cap \Omega^* \cap MEP(F_1)$. Then $\{x_n\}$, $\{u_n\}$, $\{z_n\}$, and $\{y_n\}$ are bounded.*

Proof First, we show that the mapping $(I - r_n D)$ is nonexpansive. For any $x, y \in C$,

$$\begin{aligned} \bigl\|(I - r_n D)x - (I - r_n D)y\bigr\|^2 &= \bigl\|(x-y) - r_n(Dx - Dy)\bigr\|^2 \\ &= \|x-y\|^2 - 2r_n\langle x - y, Dx - Dy\rangle + r_n^2\|Dx - Dy\|^2 \\ &\leq \|x-y\|^2 - r_n(2\theta - r_n)\|Dx - Dy\|^2 \\ &\leq \|x-y\|^2. \end{aligned}$$

Similarly, we can show that the mapping $(I - \lambda_n A)$ is nonexpansive. It follows from Lemma 2.3 that $u_n = T_{r_n}(x_n - r_n D x_n)$. Let $x^* \in F(T) \cap \Omega^* \cap MEP(F_1)$; we have $x^* =$

$T_{r_n}(x^* - r_n Dx^*)$.

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_n Dx_n) - T_{r_n}(x^* - r_n Dx^*)\|^2 \\
&\le \|(x_n - r_n Dx_n) - (x^* - r_n Dx^*)\|^2 \\
&\le \|x_n - x^*\|^2 - r_n(2\theta - r_n)\|Dx_n - Dx^*\|^2 \\
&\le \|x_n - x^*\|^2. \qquad (3.2)
\end{aligned}$$

Since the mapping A is α-inverse strongly monotone, we have

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C[u_n - \lambda_n Au_n) - P_C[x^* - \lambda_n Ax^*]\|^2 \\
&\le \|u_n - x^* - \lambda_n(Au_n - Ax^*)\|^2 \\
&\le \|u_n - x^*\|^2 - \lambda_n(2\alpha - \lambda_n)\|Au_n - Ax^*\|^2 \\
&\le \|u_n - x^*\|^2 \\
&\le \|x_n - x^*\|^2. \qquad (3.3)
\end{aligned}$$

We define $V_n = \alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n))$. Next, we prove that the sequence $\{x_n\}$ is bounded, and without loss of generality we can assume that $\beta_n \le \alpha_n$ for all $n \ge 1$. From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|P_C[V_n] - P_C[x^*]\| \\
&\le \|\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n)) - x^*\| \\
&\le \alpha_n\|\rho U(x_n) - \mu F(x^*)\| + \|(I - \alpha_n \mu F)(T(y_n)) - (I - \alpha_n \mu F)T(x^*)\| \\
&= \alpha_n\|\rho U(x_n) - \rho U(x^*) + (\rho U - \mu F)x^*\| \\
&\quad + \|(I - \alpha_n \mu F)(T(y_n)) - (I - \alpha_n \mu F)T(x^*)\| \\
&\le \alpha_n \rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + (1 - \alpha_n \nu)\|y_n - x^*\| \\
&\le \alpha_n \rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| \\
&\quad + (1 - \alpha_n \nu)\|\beta_n Sx_n + (1 - \beta_n)z_n - x^*\| \\
&\le \alpha_n \rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| \\
&\quad + (1 - \alpha_n \nu)(\beta_n\|Sx_n - Sx^*\| + \beta_n\|Sx^* - x^*\| + (1 - \beta_n)\|z_n - x^*\|) \\
&\le \alpha_n \rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| \\
&\quad + (1 - \alpha_n \nu)(\beta_n\|Sx_n - Sx^*\| + \beta_n\|Sx^* - x^*\| + (1 - \beta_n)\|x_n - x^*\|) \\
&\le \alpha_n \rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| \\
&\quad + (1 - \alpha_n \nu)(\beta_n\|x_n - x^*\| + \beta_n\|Sx^* - x^*\| + (1 - \beta_n)\|x_n - x^*\|) \\
&= (1 - \alpha_n(\nu - \rho\tau))\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + (1 - \alpha_n \nu)\beta_n\|Sx^* - x^*\| \\
&\le (1 - \alpha_n(\nu - \rho\tau))\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + \beta_n\|Sx^* - x^*\| \\
&\le (1 - \alpha_n(\nu - \rho\tau))\|x_n - x^*\| + \alpha_n(\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|)
\end{aligned}$$

$$
\begin{aligned}
&= \bigl(1-\alpha_n(\nu-\rho\tau)\bigr)\|x_n-x^*\| + \frac{\alpha_n(\nu-\rho\tau)}{\nu-\rho\tau}\bigl(\|(\rho U-\mu F)x^*\| + \|Sx^*-x^*\|\bigr) \\
&\le \max\left\{\|x_n-x^*\|, \frac{1}{\nu-\rho\tau}\bigl(\|(\rho U-\mu F)x^*\| + \|Sx^*-x^*\|\bigr)\right\},
\end{aligned}
$$

where the third inequality follows from Lemma 2.6.

By induction on n, we obtain $\|x_n - x^*\| \le \max\{\|x_0 - x^*\|, \frac{1}{\nu-\rho\tau}(\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|)\}$ for $n \ge 0$ and $x_0 \in C$. Hence, $\{x_n\}$ is bounded and, consequently, we deduce that $\{u_n\}$, $\{z_n\}$, $\{v_n\}$, $\{y_n\}$, $\{S(x_n)\}$, $\{T(x_n)\}$, $\{F(T(y_n))\}$, and $\{U(x_n)\}$ are bounded. □

Lemma 3.2 *Let $x^* \in F(T) \cap \Omega^* \cap MEP(F_1)$ and $\{x_n\}$ be the sequence generated by Algorithm* 3.1. *Then we have*:

(a) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

(b) *The weak w-limit set $w_w(x_n) \subset F(T)$, $(w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\})$.*

Proof From the nonexpansivity of the mapping $(I - \lambda_n A)$ and P_C, we have

$$
\begin{aligned}
\|z_n - z_{n-1}\| &\le \bigl\|(u_n - \lambda_n Au_n) - (u_{n-1} - \lambda_{n-1}Au_{n-1})\bigr\| \\
&= \bigl\|(u_n - u_{n-1}) - \lambda_n(Au_n - Au_{n-1}) - (\lambda_n - \lambda_{n-1})Au_{n-1}\bigr\| \\
&\le \bigl\|(u_n - u_{n-1}) - \lambda_n(Au_n - Au_{n-1})\bigr\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\| \\
&\le \|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|.
\end{aligned} \tag{3.4}
$$

Next, we estimate that

$$
\begin{aligned}
\|y_n - y_{n-1}\| &= \bigl\|\beta_n Sx_n + (1-\beta_n)z_n - \bigl(\beta_{n-1}Sx_{n-1} + (1-\beta_{n-1})z_{n-1}\bigr)\bigr\| \\
&= \bigl\|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})Sx_{n-1} \\
&\quad + (1-\beta_n)(z_n - z_{n-1}) + (\beta_{n-1} - \beta_n)z_{n-1}\bigr\| \\
&\le \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|z_n - z_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\bigl(\|Sx_{n-1}\| + \|z_{n-1}\|\bigr).
\end{aligned} \tag{3.5}
$$

It follows from (3.4) and (3.5) that

$$
\begin{aligned}
\|y_n - y_{n-1}\| &\le \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\bigl\{\|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|\bigr\} \\
&\quad + |\beta_n - \beta_{n-1}|\bigl(\|Sx_{n-1}\| + \|z_{n-1}\|\bigr).
\end{aligned} \tag{3.6}
$$

On the other hand, $u_n = T_{r_n}(x_n - r_nDx_n)$ and $u_{n-1} = T_{r_{n-1}}(x_{n-1} - r_{n-1}Dx_{n-1})$, we have

$$
F_1(u_n, y) + \langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C \tag{3.7}
$$

and

$$
F_1(u_{n-1}, y) + \langle Dx_{n-1}, y - u_{n-1}\rangle + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C. \tag{3.8}
$$

Take $y = u_{n-1}$ in (3.7) and $y = u_n$ in (3.8), we get

$$F_1(u_n, u_{n-1}) + \langle Dx_n, u_{n-1} - u_n\rangle + \frac{1}{r_n}\langle u_{n-1} - u_n, u_n - x_n\rangle \geq 0 \tag{3.9}$$

and

$$F_1(u_{n-1}, u_n) + \langle Dx_{n-1}, u_n - u_{n-1}\rangle + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0. \tag{3.10}$$

Adding (3.9) and (3.10) and using the monotonicity of F_1, we have

$$\langle Dx_{n-1} - Dx_n, u_n - u_{n-1}\rangle + \left\langle u_n - u_{n-1}, \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n}\right\rangle \geq 0,$$

which implies that

$$\begin{aligned}
0 &\leq \left\langle u_n - u_{n-1}, r_n(Dx_{n-1} - Dx_n) + \frac{r_n}{r_{n-1}}(u_{n-1} - x_{n-1}) - (u_n - x_n)\right\rangle \\
&= \left\langle u_{n-1} - u_n, u_n - u_{n-1} + \left(1 - \frac{r_n}{r_{n-1}}\right)u_{n-1}\right. \\
&\quad \left. + (x_{n-1} - r_nDx_{n-1}) - (x_n - r_nDx_n) - x_{n-1} + \frac{r_n}{r_{n-1}}x_{n-1}\right\rangle \\
&= \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)u_{n-1} + (x_{n-1} - r_nDx_{n-1}) - (x_n - r_nDx_n) - x_{n-1} + \frac{r_n}{r_{n-1}}x_{n-1}\right\rangle \\
&\quad - \|u_n - u_{n-1}\|^2 \\
&= \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)(u_{n-1} - x_{n-1}) + (x_{n-1} - r_nDx_{n-1}) - (x_n - r_nDx_n)\right\rangle \\
&\quad - \|u_n - u_{n-1}\|^2 \\
&\leq \|u_{n-1} - u_n\|\left\{\left|1 - \frac{r_n}{r_{n-1}}\right|\|u_{n-1} - x_{n-1}\| + \big\|(x_{n-1} - r_nDx_{n-1}) - (x_n - r_nDx_n\big\|\right\} \\
&\quad - \|u_n - u_{n-1}\|^2 \\
&\leq \|u_{n-1} - u_n\|\left\{\left|1 - \frac{r_n}{r_{n-1}}\right|\|u_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|\right\} - \|u_n - u_{n-1}\|^2,
\end{aligned}$$

and then

$$\|u_{n-1} - u_n\| \leq \left|1 - \frac{r_n}{r_{n-1}}\right|\|u_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|.$$

Without loss of generality, let us assume that there exists a real number μ such that $r_n > \mu > 0$ for all positive integers n. Then we get

$$\|u_{n-1} - u_n\| \leq \|x_{n-1} - x_n\| + \frac{1}{\mu}|r_{n-1} - r_n|\|u_{n-1} - x_{n-1}\|. \tag{3.11}$$

It follows from (3.6) and (3.11) that

$$\|y_n - y_{n-1}\| \leq \beta_n\|x_n - x_{n-1}\| + (1 - \beta_n)\left\{\|x_n - x_{n-1}\| + \frac{1}{\mu}|r_n - r_{n-1}|\|u_{n-1} - x_{n-1}\|\right.$$

$$
\begin{aligned}
&\quad + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|\Bigg\} + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big) \\
&= \|x_n - x_{n-1}\| + (1-\beta_n)\left\{\frac{1}{\mu}|r_n - r_{n-1}|\|u_{n-1} - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|\right\} \\
&\quad + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big). \qquad (3.12)
\end{aligned}
$$

Next, we estimate that

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \big\|P_C[V_n] - P_C[V_{n-1}]\big\| \\
&\leq \big\|\alpha_n\rho\big(U(x_n) - U(x_{n-1})\big) + (\alpha_n - \alpha_{n-1})\rho U(x_{n-1}) + (I - \alpha_n\mu F)\big(T(y_n)\big) \\
&\quad - (I - \alpha_n\mu F)T(y_{n-1}) + (I - \alpha_n\mu F)\big(T(y_{n-1})\big) - (I - \alpha_{n-1}\mu F)\big(T(y_{n-1})\big)\big\| \\
&\leq \alpha_n\rho\tau\|x_n - x_{n-1}\| + (1 - \alpha_n\nu)\|y_n - y_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\big(\big\|\rho U(x_{n-1})\big\| + \big\|\mu F\big(T(y_{n-1})\big)\big\|\big), \qquad (3.13)
\end{aligned}
$$

where the second inequality follows from Lemma 2.6. From (3.12) and (3.13), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq \alpha_n\rho\tau\|x_n - x_{n-1}\| + (1 - \alpha_n\nu) \\
&\quad \times \left(\|x_n - x_{n-1}\| + \frac{1}{\mu}|r_n - r_{n-1}|\|u_{n-1} - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|\right) \\
&\quad + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big) \\
&\quad + |\alpha_n - \alpha_{n-1}|\big(\big\|\rho U(x_{n-1})\big\| + \big\|\mu F\big(T(y_{n-1})\big)\big\|\big) \\
&\leq \big(1 - (\nu - \rho\tau)\alpha_n\big)\|x_n - x_{n-1}\| \\
&\quad + \frac{1}{\mu}|r_n - r_{n-1}|\|u_{n-1} - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big) \\
&\quad + |\alpha_n - \alpha_{n-1}|\big(\big\|\rho U(x_{n-1})\big\| + \big\|\mu F\big(T(y_{n-1})\big)\big\|\big) \\
&\leq \big(1 - (\nu - \rho\tau)\alpha_n\big)\|x_n - x_{n-1}\| \\
&\quad + M\left(\frac{1}{\mu}|r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|\right). \qquad (3.14)
\end{aligned}
$$

Here

$$
\begin{aligned}
M = \max\Big\{&\sup_{n\geq 1}\|u_{n-1} - x_{n-1}\|, \sup_{n\geq 1}\|Au_{n-1}\|, \sup_{n\geq 1}\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big), \\
&\sup_{n\geq 1}\big(\big\|\rho U(x_{n-1})\big\| + \big\|\mu F\big(T(y_{n-1})\big)\big\|\big)\Big\}.
\end{aligned}
$$

It follows by conditions (a)-(e) of Algorithm 3.1 and Lemma 2.7 that

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.
$$

Next, we show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Since $x^* \in F(T)\cap\Omega^*\cap MEP(F_1)$, by using (3.2) and (3.3), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \langle P_C(V_n)-x^*, x_{n+1}-x^*\rangle \\
&= \langle P_C(V_n)-V_n, P_C(V_n)-x^*\rangle + \langle V_n - x^*, x_{n+1}-x^*\rangle \\
&\le \langle \alpha_n(\rho U(x_n)-\mu F(x^*)) + (I-\alpha_n\mu F)(T(y_n)) \\
&\quad - (I-\alpha_n\mu F)(T(x^*)), x_{n+1}-x^*\rangle \\
&= \langle \alpha_n\rho(U(x_n)-U(x^*)), x_{n+1}-x^*\rangle + \alpha_n\langle \rho U(x^*)-\mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \langle (I-\alpha_n\mu F)(T(y_n)) - (I-\alpha_n\mu F)(T(x^*)), x_{n+1}-x^*\rangle \\
&\le \alpha_n\rho\tau\|x_n-x^*\|\|x_{n+1}-x^*\| + \alpha_n\langle \rho U(x^*)-\mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + (1-\alpha_n\nu)\|y_n-x^*\|\|x_{n+1}-x^*\| \\
&\le \frac{\alpha_n\rho\tau}{2}(\|x_n-x^*\|^2 + \|x_{n+1}-x^*\|^2) + \alpha_n\langle \rho U(x^*)-\mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \frac{(1-\alpha_n\nu)}{2}(\|y_n-x^*\|^2 + \|x_{n+1}-x^*\|^2) \\
&\le \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1}-x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n-x^*\|^2 \\
&\quad + \alpha_n\langle \rho U(x^*)-\mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \frac{(1-\alpha_n\nu)}{2}(\beta_n\|Sx_n-x^*\|^2 + (1-\beta_n)\|z_n-x^*\|^2) \\
&\le \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1}-x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n-x^*\|^2 \\
&\quad + \alpha_n\langle \rho U(x^*)-\mu F(x^*), x_{n+1}-x^*\rangle + \frac{(1-\alpha_n\nu)\beta_n}{2}\|Sx_n-x^*\|^2 \\
&\quad + \frac{(1-\alpha_n\nu)(1-\beta_n)}{2}\{\|x_n-x^*\|^2 - r_n(2\theta-r_n)\|Dx_n-Dx^*\|^2 \\
&\quad - \lambda_n(2\alpha-\lambda_n)\|Au_n-Ax^*\|^2\},
\end{aligned} \tag{3.15}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(x^*)-\mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2 \\
&\quad + \frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\{\|x_n-x^*\|^2 - r_n(2\theta-r_n)\|Dx_n-Dx^*\|^2 \\
&\quad - \lambda_n(2\alpha-\lambda_n)\|Au_n-Ax^*\|^2\} \\
&\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(x^*)-\mu F(x^*), x_{n+1}-x^*\rangle
\end{aligned}$$

$$\begin{aligned}
&+ \|x_n - x^*\|^2 + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n - x^*\|^2 \\
&- \frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\big\{r_n(2\theta - r_n)\|Dx_n - Dx^*\|^2 \\
&+ \lambda_n(2\alpha - \lambda_n)\|Au_n - Ax^*\|^2\big\}.
\end{aligned}$$

Then, from the inequality above, we get

$$\begin{aligned}
&\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\big\{r_n(2\theta - r_n)\|Dx_n - Dx^*\|^2 + \lambda_n(2\alpha - \lambda_n)\|Au_n - Ax^*\|^2\big\} \\
&\quad \le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\big\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\big\rangle \\
&\qquad + \beta_n\|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad \le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\big\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\big\rangle \\
&\qquad + \beta_n\|Sx_n - x^*\|^2 + \big(\|x_n - x^*\| + \|x_{n+1} - x^*\|\big)\|x_{n+1} - x_n\|.
\end{aligned}$$

Since $\liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2\alpha$, $\{r_n\} \subset (0, 2\theta)$, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\alpha_n \to 0$, and $\beta_n \to 0$, we obtain $\lim_{n\to\infty}\|Dx_n - Dx^*\| = 0$ and $\lim_{n\to\infty}\|Au_n - Ax^*\| = 0$.

Since T_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \big\|T_{r_n}(x_n - r_nDx_n) - T_{r_n}\big(x^* - r_nDx^*\big)\big\|^2 \\
&\le \big\langle u_n - x^*, (x_n - r_nDx_n) - \big(x^* - r_nDx^*\big)\big\rangle \\
&= \frac{1}{2}\big\{\|u_n - x^*\|^2 + \big\|(x_n - r_nDx_n) - \big(x^* - r_nDx^*\big)\big\|^2 \\
&\quad - \big\|u_n - x^* - \big[(x_n - r_nDx_n) - \big(x^* - r_nDx^*\big)\big]\big\|^2\big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|u_n - x^*\|^2 &\le \big\|(x_n - r_nDx_n) - \big(x^* - r_nDx^*\big)\big\|^2 - \big\|u_n - x_n + r_n\big(Dx_n - Dx^*\big)\big\|^2 \\
&\le \|x_n - x^*\|^2 - \big\|u_n - x_n + r_n\big(Dx_n - Dx^*\big)\big\|^2 \\
&\le \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2r_n\|u_n - x_n\|\|Dx_n - Dx^*\|.
\end{aligned}$$

From (3.15), (3.3), and the inequality above, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\big\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\big\rangle \\
&\quad + \frac{(1-\alpha_n\nu)}{2}\big(\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\|z_n - x^*\|^2\big) \\
&\le \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\big\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\big\rangle \\
&\quad + \frac{(1-\alpha_n\nu)}{2}\big(\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\|u_n - x^*\|^2\big)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1}-x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n-x^*\|^2 \\
&\quad + \alpha_n\langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \frac{(1-\alpha_n\nu)}{2}\big\{\beta_n\|Sx_n-x^*\|^2 + (1-\beta_n)\big(\|x_n-x^*\|^2 - \|u_n-x_n\|^2 \\
&\quad + 2r_n\|u_n-x_n\|\|Dx_n-Dx^*\|\big)\big\},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2 \\
&\quad + \frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\big\{\|x_n-x^*\|^2 - \|u_n-x_n\|^2 \\
&\quad + 2r_n\|u_n-x_n\|\|Dx_n-Dx^*\|\big\} \\
&\leq \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2 \\
&\quad + \|x_n-x^*\|^2 + \frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\big\{-\|u_n-x_n\|^2 \\
&\quad + 2r_n\|u_n-x_n\|\|Dx_n-Dx^*\|\big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\|u_n-x_n\|^2 &\leq \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2 \\
&\quad + \frac{2(1-\alpha_n\nu)(1-\beta_n)r_n}{1+\alpha_n(\nu-\rho\tau)}\|u_n-x_n\|\|Dx_n-Dx^*\| \\
&\quad + \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 \\
&= \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^*\rangle \\
&\quad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2(1-\alpha_n\nu)(1-\beta_n)r_n}{1+\alpha_n(\nu-\rho\tau)}\|u_n - x_n\|\|Dx_n - Dx^*\| \\
&+ (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\alpha_n \to 0$, $\beta_n \to 0$, and $\lim_{n\to\infty}\|Dx_n - Dx^*\| = 0$, we obtain

$$
\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.16}
$$

From (2.2), we get

$$
\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C[u_n - \lambda_n Au_n] - P_C[x^* - \lambda_n Ax^*]\|^2 \\
&\le \langle z_n - x^*, (u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*)\rangle \\
&= \frac{1}{2}\{\|z_n - x^*\|^2 + \|u_n - x^* - \lambda_n(Au_n - Ax^*)\|^2 \\
&\quad - \|u_n - x^* - \lambda_n(Au_n - Ax^*) - (z_n - x^*)\|^2\} \\
&\le \frac{1}{2}\{\|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - z_n - \lambda_n(Au_n - Ax^*)\|^2\} \\
&\le \frac{1}{2}\{\|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\langle u_n - z_n, Au_n - Ax^*\rangle\} \\
&\le \frac{1}{2}\{\|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\|\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|z_n - x^*\|^2 &\le \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\| \\
&\le \|x_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\|.
\end{aligned}
$$

From (3.15) and the inequality above, we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\
&\quad + \frac{(1-\alpha_n\nu)}{2}(\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\|z_n - x^*\|^2) \\
&\le \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\
&\quad + \frac{(1-\alpha_n\nu)}{2}\{\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)(\|x_n - x^*\|^2 \\
&\quad - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\|)\},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\left\|Sx_n-x^*\right\|^2\\
&+\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\left\{\left\|x_n-x^*\right\|^2-\|u_n-z_n\|^2\right.\\
&\left.+2\lambda_n\|u_n-z_n\|\left\|Au_n-Ax^*\right\|\right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\|u_n-z_n\|^2\\
&\quad\le\frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\left\|x_n-x^*\right\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\left\langle\rho U\left(x^*\right)-\mu F\left(x^*\right),x_{n+1}-x^*\right\rangle\\
&\qquad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\left\|Sx_n-x^*\right\|^2+\left\|x_n-x^*\right\|^2-\left\|x_{n+1}-x^*\right\|^2\\
&\qquad+2\lambda_n\|u_n-z_n\|\left\|Au_n-Ax^*\right\|\\
&\quad=\frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\left\|x_n-x^*\right\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\left\langle\rho U\left(x^*\right)-\mu F\left(x^*\right),x_{n+1}-x^*\right\rangle\\
&\qquad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\left\|Sx_n-x^*\right\|^2+\left(\left\|x_n-x^*\right\|+\left\|x_{n+1}-x^*\right\|\right)\|x_{n+1}-x_n\|\\
&\qquad+2\lambda_n\|u_n-z_n\|\left\|Au_n-Ax^*\right\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\alpha_n\to 0$, $\beta_n\to 0$, and $\lim_{n\to\infty}\|Au_n-Ax^*\|=0$, we obtain

$$
\lim_{n\to\infty}\|u_n-z_n\|=0. \tag{3.17}
$$

It follows from (3.16) and (3.17) that

$$
\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.18}
$$

Since $T(x_n)\in C$, we have

$$
\begin{aligned}
\left\|x_n-T(x_n)\right\|&\le\|x_n-x_{n+1}\|+\left\|x_{n+1}-T(x_n)\right\|\\
&=\|x_n-x_{n+1}\|+\left\|P_C[V_n]-P_C\left[T(x_n)\right]\right\|\\
&\le\|x_n-x_{n+1}\|+\left\|\alpha_n\left(\rho U(x_n)-\mu F\left(T(y_n)\right)\right)+T(y_n)-T(x_n)\right\|\\
&\le\|x_n-x_{n+1}\|+\alpha_n\left\|\rho U(x_n)-\mu F\left(T(y_n)\right)\right\|+\|y_n-x_n\|\\
&\le\|x_n-x_{n+1}\|+\alpha_n\left\|\rho U(x_n)-\mu F\left(T(y_n)\right)\right\|+\left\|\beta_nSx_n+(1-\beta_n)z_n-x_n\right\|\\
&\le\|x_n-x_{n+1}\|+\alpha_n\left\|\rho U(x_n)-\mu F\left(T(y_n)\right)\right\|\\
&\quad+\beta_n\|Sx_n-x_n\|+(1-\beta_n)\|z_n-x_n\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\alpha_n\to 0$, $\beta_n\to 0$, and $\|\rho U(x_n)-\mu F(T(y_n))\|$ and $\|Sx_n-x_n\|$ are bounded, and $\lim_{n\to\infty}\|x_n-z_n\|=0$, we obtain

$$
\lim_{n\to\infty}\left\|x_n-T(x_n)\right\|=0.
$$

Since $\{x_n\}$ is bounded, without loss of generality we can assume that $x_n \rightharpoonup x^* \in C$. It follows from Lemma 2.4 that $x^* \in F(T)$. Therefore, $w_w(x_n) \subset F(T)$. □

Theorem 3.1 *The sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to* z, *which is the unique solution of the variational inequality*

$$\langle \rho U(z) - \mu F(z), x - z \rangle \leq 0, \quad \forall x \in \Omega^* \cap MEP(F_1) \cap F(T). \tag{3.19}$$

Proof Since $\{x_n\}$ is bounded $x_n \rightharpoonup w$ and from Lemma 3.2, we have $w \in F(T)$. Next, we show that $w \in MEP(F_1)$. Since $u_n = T_{r_n}(x_n - r_n Dx_n)$, we have

$$F_1(u_n, y) + \langle Dx_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

It follows from the monotonicity of F_1 that

$$\langle Dx_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq F_1(y, u_n), \quad \forall y \in C$$

and

$$\langle Dx_{n_k}, y - u_{n_k} \rangle + \left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle \geq F_1(y, u_{n_k}), \quad \forall y \in C. \tag{3.20}$$

Since $\lim_{n\to\infty} \|u_n - x_n\| = 0$, and $x_n \rightharpoonup w$, it is easy to observe that $u_{n_k} \to w$. For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$, and we have $y_t \in C$. Then from (3.20), we obtain

$$\begin{aligned}
\langle Dy_t, y_t - u_{n_k} \rangle &\geq \langle Dy_t, y_t - u_{n_k} \rangle - \langle Dx_{n_k}, y_t - u_{n_k} \rangle - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle + F_1(y_t, u_{n_k}) \\
&= \langle Dy_t - Du_{n_k}, y_t - u_{n_k} \rangle + \langle Du_{n_k} - Dx_{n_k}, y_t - u_{n_k} \rangle \\
&\quad - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle + F_1(y_t, u_{n_k}).
\end{aligned} \tag{3.21}$$

Since D is Lipschitz continuous and $\lim_{n\to\infty} \|u_n - x_n\| = 0$, we obtain $\lim_{k\to\infty} \|Du_{n_k} - Dx_{n_k}\| = 0$. From the monotonicity of D and $u_{n_k} \to w$, it follows from (3.21) that

$$\langle Dy_t, y_t - w \rangle \geq F_1(y_t, w). \tag{3.22}$$

Hence, from assumptions (i)-(iv) of Lemma 2.2 and (3.22), we have

$$\begin{aligned}
0 &= F_1(y_t, y_t) \leq tF_1(y_t, y) + (1-t)F_1(y_t, w) \\
&\leq tF_1(y_t, y) + (1-t)\langle Dy_t, y_t - w \rangle \\
&\leq tF_1(y_t, y) + (1-t)t\langle Dy_t, y - w \rangle,
\end{aligned} \tag{3.23}$$

which implies that $F_1(y_t, y) + (1-t)\langle Dy_t, y - w \rangle \geq 0$. Letting $t \to 0_+$, we have

$$F_1(w, y) + \langle Dw, y - w \rangle \geq 0, \quad \forall y \in C,$$

which implies that $w \in MEP(F_1)$.

Furthermore, we show that $w \in \Omega^*$. Let

$$Tv = \begin{cases} Av + N_C v, & \forall v \in C, \\ \emptyset, & \text{otherwise}, \end{cases}$$

where $N_C v := \{w \in H : \langle w, v - u\rangle \geq 0, \forall u \in C\}$ is the normal cone to C at $v \in C$. Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega^*$ (see [28]). Let $G(T)$ denote the graph of T, and let $(v,u) \in G(T)$; since $u - Av \in N_C v$ and $z_n \in C$, we have

$$\langle v - z_n, u - Av\rangle \geq 0. \tag{3.24}$$

On the other hand, it follows from $z_n = P_C[u_n - \lambda_n A u_n]$ and $v \in C$ that

$$\bigl\langle v - z_n, z_n - (u_n - \lambda_n A u_n)\bigr\rangle \geq 0$$

and

$$\left\langle v - z_n, \frac{z_n - u_n}{\lambda_n} + Au_n\right\rangle \geq 0.$$

Therefore, from (3.24) and the inverse strong monotonicity of A, we have

$$\begin{aligned}
\langle v - z_{n_k}, u\rangle &\geq \langle v - z_{n_k}, Av\rangle \\
&\geq \langle v - z_{n_k}, Av\rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}} + Au_{n_k}\right\rangle \\
&\geq \langle v - z_{n_k}, Av - Az_{n_k}\rangle + \langle v - z_{n_k}, Az_{n_k} - Au_{n_k}\rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}}\right\rangle \\
&\geq \langle v - z_{n_k}, Az_{n_k} - Au_{n_k}\rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}}\right\rangle.
\end{aligned}$$

Since $\lim_{n\to\infty} \|u_n - z_n\| = 0$ and $u_{n_k} \to w$, it is easy to observe that $z_{n_k} \to w$. Hence, we obtain $\langle v - w, u\rangle \geq 0$. Since T is maximal monotone, we have $w \in T^{-1}0$, and hence $w \in \Omega^*$. Thus we have

$$w \in \Omega^* \cap MEP(F_1) \cap F(T).$$

Observe that the constants satisfy $0 \leq \rho\tau < \nu$ and

$$\begin{aligned}
k \geq \eta \quad &\Longleftrightarrow \quad k^2 \geq \eta^2 \\
&\Longleftrightarrow \quad 1 - 2\mu\eta + \mu^2 k^2 \geq 1 - 2\mu\eta + \mu^2\eta^2 \\
&\Longleftrightarrow \quad \sqrt{1 - \mu\bigl(2\eta - \mu k^2\bigr)} \geq 1 - \mu\eta \\
&\Longleftrightarrow \quad \mu\eta \geq 1 - \sqrt{1 - \mu\bigl(2\eta - \mu k^2\bigr)} \\
&\Longleftrightarrow \quad \mu\eta \geq \nu,
\end{aligned}$$

therefore, from Lemma 2.5, the operator $\mu F - \rho U$ is $\mu\eta - \rho\tau$ strongly monotone, and we get the uniqueness of the solution of the variational inequality (3.19) and denote it by $z \in \Omega^* \cap MEP(F_1) \cap F(T)$.

Next, we claim that $\limsup_{n\to\infty}\langle \rho U(z) - \mu F(z), x_n - z\rangle \le 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\begin{aligned}\limsup_{n\to\infty}\big\langle \rho U(z) - \mu F(z), x_n - z\big\rangle &= \limsup_{k\to\infty}\big\langle \rho U(z) - \mu F(z), x_{n_k} - z\big\rangle \\ &= \big\langle \rho U(z) - \mu F(z), w - z\big\rangle \le 0.\end{aligned}$$

Next, we show that $x_n \to z$. We have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \big\langle P_C(V_n) - z, x_{n+1} - z\big\rangle \\ &= \big\langle P_C(V_n) - V_n, P_C(V_n) - z\big\rangle + \langle V_n - z, x_{n+1} - z\rangle \\ &\le \big\langle \alpha_n\big(\rho U(x_n) - \mu F(z)\big) + (I - \alpha_n\mu F)\big(T(y_n)\big) - (I - \alpha_n\mu F)\big(T(z)\big), x_{n+1} - z\big\rangle \\ &= \big\langle \alpha_n\rho\big(U(x_n) - U(z)\big), x_{n+1} - z\big\rangle + \alpha_n\big\langle \rho U(z) - \mu F(z), x_{n+1} - z\big\rangle \\ &\quad + \big\langle (I - \alpha_n\mu F)\big(T(y_n)\big) - (I - \alpha_n\mu F)\big(T(z)\big), x_{n+1} - z\big\rangle \\ &\le \alpha_n\rho\tau\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\big\langle \rho U(z) - \mu F(z), x_{n+1} - z\big\rangle \\ &\quad + (1 - \alpha_n\nu)\|y_n - z\|\|x_{n+1} - z\| \\ &\le \alpha_n\rho\tau\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\big\langle \rho U(z) - \mu F(z), x_{n+1} - z\big\rangle \\ &\quad + (1 - \alpha_n\nu)\big\{\beta_n\|Sx_n - Sz\| + \beta_n\|Sz - z\| + (1 - \beta_n)\|z_n - z\|\big\}\|x_{n+1} - z\| \\ &\le \alpha_n\rho\tau\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\big\langle \rho U(z) - \mu F(z), x_{n+1} - z\big\rangle \\ &\quad + (1 - \alpha_n\nu)\big\{\beta_n\|x_n - z\| + \beta_n\|Sz - z\| + (1 - \beta_n)\|x_n - z\|\big\}\|x_{n+1} - z\| \\ &= \big(1 - \alpha_n(\nu - \rho\tau)\big)\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\big\langle \rho U(z) - \mu F(z), x_{n+1} - z\big\rangle \\ &\quad + (1 - \alpha_n\nu)\beta_n\|Sz - z\|\|x_{n+1} - z\| \\ &\le \frac{1 - \alpha_n(\nu - \rho\tau)}{2}\big(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\big) + \alpha_n\big\langle \rho U(z) - \mu F(z), x_{n+1} - z\big\rangle \\ &\quad + (1 - \alpha_n\nu)\beta_n\|Sz - z\|\|x_{n+1} - z\|,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \frac{1 - \alpha_n(\nu - \rho\tau)}{1 + \alpha_n(\nu - \rho\tau)}\|x_n - z\|^2 + \frac{2\alpha_n}{1 + \alpha_n(\nu - \rho\tau)}\big\langle \rho U(z) - \mu F(z), x_{n+1} - z\big\rangle \\ &\quad + \frac{2(1 - \alpha_n\nu)\beta_n}{1 + \alpha_n(\nu - \rho\tau)}\|Sz - z\|\|x_{n+1} - z\| \\ &\le \big(1 - \alpha_n(\nu - \rho\tau)\big)\|x_n - z\|^2 + \frac{2\alpha_n(\nu - \rho\tau)}{1 + \alpha_n(\nu - \rho\tau)} \\ &\quad \times \left\{\frac{1}{\nu - \rho\tau}\big\langle \rho U(z) - \mu F(z), x_{n+1} - z\big\rangle + \frac{(1 - \alpha_n\nu)\beta_n}{\alpha_n(\nu - \rho\tau)}\|Sz - z\|\|x_{n+1} - z\|\right\}.\end{aligned}$$

Let $\gamma_n = \alpha_n(\nu - \rho\tau)$ and $\delta_n = \frac{2\alpha_n(\nu-\rho\tau)}{1+\alpha_n(\nu-\rho\tau)}\{\frac{1}{\nu-\rho\tau}\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle + \frac{(1-\alpha_n\nu)\beta_n}{\alpha_n(\nu-\rho\tau)}\|Sz - z\|\|x_{n+1} - z\|\}$.

We have

$$\sum_{n=1}^{\infty} \alpha_n = \infty$$

and

$$\limsup_{n\to\infty} \left\{ \frac{1}{\nu - \rho\tau} \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle + \frac{(1 - \alpha_n \nu)\beta_n}{\alpha_n(\nu - \rho\tau)} \|Sz - z\| \|x_{n+1} - z\| \right\} \leq 0.$$

It follows that

$$\sum_{n=1}^{\infty} \gamma_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0.$$

Thus all the conditions of Lemma 2.7 are satisfied. Hence we deduce that $x_n \to z$. This completes the proof. □

4 Applications

In this section, we obtain the following results by using a special case of the proposed method for example.

Putting $A = 0$ in Algorithm 3.1, we obtain the following result which can be viewed as an extension and improvement of the method of Wang and Xu [24] for finding the approximate element of the common set of solutions of a mixed equilibrium problem and a hierarchical fixed-point problem in a real Hilbert space.

Corollary 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D : C \to H$ be θ-inverse strongly monotone mappings. Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions* (i)-(iv) *of Lemma* 2.2 *and $S, T : C \to C$ be a nonexpansive mappings such that $F(T) \cap MEP(F_1) \neq \emptyset$. Let $F : C \to C$ be a k-Lipschitzian mapping and be η-strongly monotone, and let $U : C \to C$ be a τ-Lipschitzian mapping. For an arbitrary given $x_0 \in C$, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated by*

$$\begin{aligned} &F_1(u_n, y) + \langle Dx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C; \\ &y_n = \beta_n S x_n + (1 - \beta_n) u_n; \\ &x_{n+1} = P_C \big[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)\big(T(y_n)\big) \big], \quad \forall n \geq 0, \end{aligned}$$

where $\{r_n\} \subset (0, 2\theta)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$. Suppose that the parameters satisfy $0 < \mu < \frac{2\eta}{k^2}$, $0 \leq \rho\tau < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Also, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ are sequences satisfying conditions (a)-(d) *of Algorithm* 3.1. *The sequence $\{x_n\}$ converges strongly to z, which is the unique solution of the variational inequality*

$$\langle \rho U(z) - \mu F(z), x - z \rangle \leq 0, \quad \forall x \in MEP(F_1) \cap F(T).$$

Putting $U = f$, $F = I$, $\rho = \mu = 1$, and $A = 0$, we obtain an extension and improvement of the method of Yao et al. [12] *for finding the approximate element of the common set of solutions* *of a mixed equilibrium problem and a hierarchical fixed-point problem in a real Hilbert space.*

Corollary 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D : C \to H$ be θ-inverse strongly monotone mappings. Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions* (i)-(iv) *of Lemma* 2.2 *and $S, T : C \to C$ be a nonexpansive mappings such that $F(T) \cap MEP(F_1) \neq \emptyset$. Let $f : C \to C$ be a τ-Lipschitzian mapping. For an arbitrary given $x_0 \in C$, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated by*

$$F_1(u_n, y) + \langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C;$$

$$y_n = \beta_n Sx_n + (1 - \beta_n)u_n;$$

$$x_{n+1} = P_C\bigl[\alpha_n f(x_n) + (1 - \alpha_n)T(y_n)\bigr], \quad \forall n \geq 0,$$

where $\{r_n\} \subset (0, 2\theta)$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0, 1)$ *satisfying conditions* (a)-(d) *of Algorithm* 3.1. *The sequence $\{x_n\}$ converges strongly to z, which is the unique solution of the variational inequality*

$$\bigl\langle f(z) - z, x - z\bigr\rangle \leq 0, \quad \forall x \in MEP(F_1) \cap F(T).$$

Remark 4.1 Some existing methods (*e.g.*, [12, 14, 16, 17, 25]) can be viewed as special cases of Algorithm 3.1. Therefore, the new algorithm is expected to be widely applicable.

To verify the theoretical assertions, we consider the following example.

Example 4.1 Let $\alpha_n = \frac{1}{3n}$, $\beta_n = \frac{1}{n^3}$, $\lambda_n = \frac{1}{2(n+1)}$, and $r_n = \frac{n}{n+1}$.
We have

$$\lim_{n\to\infty} \alpha_n = \frac{1}{3} \lim_{n\to\infty} \frac{1}{n} = 0$$

and

$$\sum_{n=1}^{\infty} \alpha_n = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

The sequence $\{\alpha_n\}$ satisfies condition (a).

$$\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \lim_{n\to\infty} \frac{3}{n^2} = 0.$$

Condition (b) is satisfied. We compute

$$\alpha_{n-1} - \alpha_n = \frac{1}{3}\left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{1}{3n(n-1)}.$$

It is easy to show $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$. Similarly, we can show $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$. The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy condition (c). We have

$$\liminf_{n\to\infty} r_n = \liminf_{n\to\infty} \frac{n}{n+1} = 1$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}|r_{n-1}-r_n| &= \sum_{n=1}^{\infty}\left|\frac{n-1}{n}-\frac{n}{n+1}\right| \\ &= \sum_{n=1}^{\infty}\frac{1}{n(n+1)} \\ &\le \sum_{n=1}^{\infty}\frac{1}{n^2} \\ &< \infty.\end{aligned}$$

Then, the sequence $\{r_n\}$ satisfies condition (d). We compute

$$\begin{aligned}\sum_{n=1}^{\infty}|\lambda_{n-1}-\lambda_n| &= \sum_{n=1}^{\infty}\left|\frac{1}{2n}-\frac{1}{2(n+1)}\right| \\ &= \frac{1}{2} \\ &< \infty.\end{aligned}$$

Then, the sequence $\{\lambda_n\}$ satisfies condition (e).

Let $\mathbb{R}$ be the set of real numbers, $D = 0$, and let the mapping $A : \mathbb{R} \to \mathbb{R}$ be defined by

$$Ax = \frac{x}{2}, \quad \forall x \in \mathbb{R},$$

let the mapping $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$T(x) = \frac{x}{2}, \quad \forall x \in \mathbb{R},$$

let the mapping $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \frac{2x+5}{7}, \quad \forall x \in \mathbb{R},$$

let the mapping $S : \mathbb{R} \to \mathbb{R}$ be defined by

$$S(x) = \frac{x}{2}, \quad \forall x \in \mathbb{R},$$

let the mapping $U : \mathbb{R} \to \mathbb{R}$ be defined by

$$U(x) = \frac{x}{14}, \quad \forall x \in \mathbb{R},$$

and let the mapping $F_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F_1(x,y) = -3x^2 + xy + 2y^2, \quad \forall (x,y) \in \mathbb{R} \times \mathbb{R}.$$

It is easy to show that A is a 1-inverse strongly monotone mapping, T and S are non-expansive mappings, F is a 1-Lipschitzian mapping and $\frac{1}{7}$-strongly monotone and U is

Table 1 The values of $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, and $\{x_n\}$ with initial values $x_1 = 30$ and $x_1 = -30$

	$x_1 = 30$				$x_1 = -30$			
	u_n	z_n	y_n	x_n	u_n	z_n	y_n	x_n
$n = 1$	8.571429	4.285714	15.000000	30.000000	–8.571429	–4.285714	–15.000000	–30.0000000
$n = 2$	1.710361	0.427590	0.837364	7.411565	–1.726060	–0.431515	–0.845050	–7.479592
$n = 3$	0.085202	0.000000	0.007495	0.404709	–0.093178	0.000000	–0.008196	–0.442594
$n = 4$	–0.001479	0.000370	0.000306	–0.007393	–0.003130	0.000783	0.000648	–0.015652
$n = 5$	–0.001617	0.000808	0.000769	–0.008354	–0.001585	0.000792	0.000753	–0.008187
$n = 6$	–0.001215	0.000911	0.000892	–0.006422	–0.001216	0.000912	0.000893	–0.006430
$n = 7$	–0.000972	0.000972	0.000962	–0.005226	–0.000972	0.000972	0.000962	–0.005225
$n = 8$	–0.000805	0.001006	0.000999	–0.004380	–0.000805	0.001006	0.000999	–0.004380
$n = 9$	–0.000682	0.001024	0.001020	–0.003754	–0.000682	0.001024	0.001020	–0.003754
$n = 10$	–0.000590	0.001032	0.001030	–0.003271	–0.000590	0.001032	0.001030	–0.003271

$\frac{1}{7}$-Lipschitzian. It is clear that

$$\Omega^* \cap MEP(F_1) \cap F(T) = \{0\}.$$

By the definition of F_1, we have

$$\begin{aligned} 0 &\leq F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \\ &= -3u_n^2 + u_n y + 2y^2 + \frac{1}{r_n}(y - u_n)(u_n - x_n). \end{aligned}$$

Then

$$\begin{aligned} 0 &\leq r_n\bigl(-3u_n^2 + u_n y + 2y^2\bigr) + \bigl(yu_n - yx_n - u_n^2 + u_n x_n\bigr) \\ &= 2r_n y^2 + (r_n u_n + u_n - x_n)y - 3r_n u_n^2 - u_n^2 + u_n x_n. \end{aligned}$$

Let $B(y) = 2r_n y^2 + (r_n u_n + u_n - x_n)y - 3r_n u_n^2 - u_n^2 + u_n x_n$. $B(y)$ is a quadratic function of y with coefficient $a = 2r_n$, $b = r_n u_n + u_n - x_n$, $c = -3r_n u_n^2 - u_n^2 + u_n x_n$. We determine the discriminant Δ of B as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (r_n u_n + u_n - x_n)^2 - 8r_n\bigl(-3r_n u_n^2 - u_n^2 + u_n x_n\bigr) \\ &= u_n^2 + 10r_n u_n^2 + 25u_n^2 r_n^2 - 2x_n u_n - 10x_n u_n r_n + x_n^2 \\ &= (u_n + 5u_n r_n)^2 - 2x_n(u_n + 5u_n r_n) + x_n^2 \\ &= (u_n + 5u_n r_n - x_n)^2. \end{aligned}$$

We have $B(y) \geq 0$, $\forall y \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta = 0$, we obtain

$$u_n = \frac{x_n}{1 + 5r_n}. \tag{4.1}$$

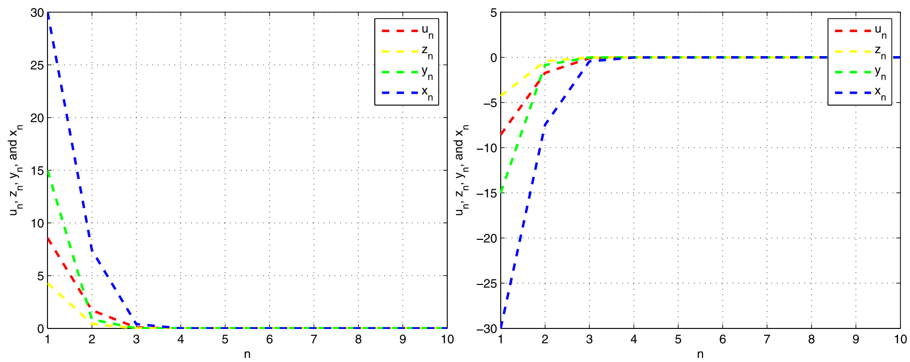

Figure 1 The convergence of $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, and $\{x_n\}$ with initial values $x_1 = 30$ and $x_1 = -30$.

For every $n \geq 1$, from (4.1), we rewrite (3.1) as follows:

$$\begin{cases} z_n = \frac{x_n}{1+5r_n} - \frac{x_n}{4(n+1)(1+5r_n)}; \\ y_n = \frac{x_n}{2n^3} + (1 - \frac{1}{n^3})z_n; \\ x_{n+1} = \rho\frac{x_n}{42n} + \frac{y_n}{2} - \mu\frac{y_n+5}{21n}. \end{cases}$$

In all the tests we take $\rho = \frac{1}{15}$ and $\mu = \frac{1}{7}$. In our example, $\eta = \frac{1}{7}$, $k = 1$, $\tau = \frac{1}{7}$. It is easy to show that the parameters satisfy $0 < \mu < \frac{2\eta}{k^2}$, $0 \leq \rho\tau < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. All codes were written in Matlab, the values of $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, and $\{x_n\}$ with different n are reported in Table 1.

Remark 4.2 Table 1 and Figure 1 show that the sequences $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, and $\{x_n\}$ converge to 0, where $\{0\} = \Omega^* \cap MEP(F_1) \cap F(T)$.

Competing interests
The author declares that he has no competing interests.

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