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On parametric implicit vector variational inequality problems

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Abstract

Recently Huang *et al.* (Math. Comput. Model. 43:1267-1274, 2006) introduced a class of parametric implicit vector equilibrium problems (for short PIVEP) and they presented some existence results for a solution of PIVEP. Also, they provided two theorems about upper and lower semi-continuity of the solution set of PIVEP in a locally convex Hausdorff topological vector space. The paper extends the corresponding results obtained in the setting of topological vector spaces with mild assumptions and removing the notion of locally non-positiveness at a point and lower semi-continuity of the parametric mapping.

Keywords: topological vector space; equilibrium problem; KKM mapping

1 Introduction and preliminaries

Equilibrium problems have been extensively studied in recent years, the origin of which can be traced back to Takahashi [1, Lemma 1], Blum and Oettli [2], and Noor and Oettli [3]. It is well known that vector equilibrium problems provide a unified model for several classes of problems, for example, vector variational inequality problems, vector complementarity problems, vector optimization problems, and vector saddle point problems; see [2–4] and the references therein. In 2003, Huang *et al.* [5] considered the implicit vector equilibrium problem (for short IVEP) which consists of finding $x \in E$ such that

$$f\big(g(x), y\big) \notin -\operatorname{int} C(x), \quad \forall y \in E,$$

where $f : E \times E \to Y$ and $g : E \to E$, are mappings, X and Y are two Hausdorff topological vector spaces, E is a nonempty closed convex subset of X and $C : E \to 2^Y$ be a set-valued mapping such that for any $x \in E$, $C(x)$ is a closed and convex cone with $C(x) \cap -C(x) = \{0\}$, that is pointed, with nonempty interior. They continued their research and introduced the parametric implicit vector equilibrium problem, which consists of finding $x^* \in \mathbf{K}(\lambda)$, for each given $(\lambda, \epsilon) \in \Lambda_1 \times \Lambda_2$ such that

$$f\big(\epsilon, g\big(x^*\big), y\big) \notin -\operatorname{int} C\big(x^*\big), \quad \forall y \in \mathbf{K}(\lambda),$$

where Λ_i $(i = 1, 2)$ are Hausdorff topological vector spaces (the parametric spaces), $\mathbf{K} : \Lambda_1 \to 2^X$ a set-valued mapping such that for any $\lambda \in \Lambda_1$, $\mathbf{K}(\lambda)$ is a nonempty, closed and convex subset of X with $\mathbf{K}(\Lambda_1) = \bigcup_{\lambda\in\Lambda_1} \mathbf{K}(\lambda) \subseteq E$ and $f : \Lambda_1 \times \Lambda_2 \times E \to Y$. They obtained some existence results for a solution of PIVEP and further they studied upper and lower semi-continuity of the solution of PIVEP in locally convex Hausdorff topological vector spaces. This paper is motivated and inspired by the recent paper [5] and its aim is to extend the results to the setting of Hausdorff topological vector spaces with mild assumptions and removing the condition of being locally non-positive at a point has been applied in Proposition 3.3 of [6] and lower semi-continuity of the parametric mapping used in Theorem 3.2 of [6]. More precisely, we first establish an existence result for a solution of IVEP and then by using it we will deal with the behavior of the solution set of PIVEP when the parameters (λ, ϵ) start to change. In fact we will show that the solution set as a mapping $S : \Lambda_1 \times \Lambda_2 \to 2^X$ is upper semi-continuous and lower semi-continuous under special conditions. In the rest of this section we recall some definitions and results that we need in the next section.

A subset P of Y is called a pointed and convex cone if and only if $P + P \subseteq P$, $tP \subseteq P$, for all $t \geq 0$, and $P \cap -P = \{0\}$. The domain of a set-valued mapping $W : X \to 2^Y$ is defined as $D(W) = \{x \in X : W(x) \neq \emptyset\}$ and its graph is defined as

$$\text{Graph}(W) = \{(x,z) \in X \times Y : z \in W(x)\}.$$

Also W is said to be closed if its graph, that is, Graph(W), is a closed subset of $X \times Y$. A set-valued mapping $T : X \to 2^Y$ is called upper semi-continuous (u.s.c.) at $x \in X$ if for every open set V containing $T(x)$ there exists an open set U containing x such that $T(u) \subseteq V$, for all $u \in U$. The mapping T is said to be lower semi-continuous (l.s.c.) if for every open set V with $T(x) \cap V \neq \emptyset$ there exists an open set U containing x such that $T(u) \cap V \neq \emptyset$. The mapping T is continuous at x if it is both u.s.c. and l.s.c. at x. Moreover, T is u.s.c. (l.s.c.) on X if T is u.s.c. (l.s.c.) at each point of X.

We need the following lemma in the sequel.

Lemma 1.1 ([7]) *Let X and Y be topological spaces and $T : X \to 2^Y$ be a mapping. The following statements are true*:

(i) *If for any $x \in X$, $T(x)$ is compact, then T is u.s.c. at $x \in X$ if and only if for any net $\{x_i\} \subseteq X$ such that $x_i \to x$ and for every $y_i \in T(x_i)$, there exist $y \in T(x)$ and a subnet $\{y_j\}$ of $\{y_i\}$ such that $y_j \to y$.*

(ii) *T is l.s.c. at $x \in X$ if and only if for any net $\{x_i\} \subseteq X$ with $x_i \to x$ and for any $y \in T(x)$, there exists a net $\{y_i\}$ such that $y_i \in T(x_i)$ and $y_i \to y$.*

Definition 1.2 ([8, 9]) Let X be a topological vector space. A mapping $F : K \subseteq X \to 2^X$ is said to be a KKM mapping, if, for any finite set $A \subseteq K$,

$$\operatorname{co} A \subseteq F(A) = \bigcup_{x \in A} F(x),$$

where $\operatorname{co} A$ denotes the convex hull of A.

The following lemma plays a crucial rule in this paper.

Lemma 1.3 ([8]) *Let K be a nonempty subset of a topological vector space X and $F : K \to 2^X$ be a KKM mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x \in B} F(x)$ is compact. Then $\bigcap_{x \in K} F(x) \neq \emptyset$.*

2 Main results

The next result provides an existence result for a solution of IVEP.

Theorem 2.1 *Let K be closed convex subset of a t.v.s. X and $f: K \times K \to Y$ and $g: K \to K$ be two mappings. If the following assumptions are satisfied*:

(a) $f(g(x), x) \notin -\operatorname{int} C(x)$, $\forall x \in K$,

(b) *the mapping* $x \to f(g(x), y)$ *is continuous, for all* $y \in K$,

(c) *for each* $x \in K$, *the set* $\{y \in K : f(g(x), y) \in -\operatorname{int} C(x)\}$ *is convex*,

(d) *the mapping* $W: K \to 2^Y$ *defined by* $W(x) = Y \setminus (-\operatorname{int} C(x))$ *is closed*,

(e) *there exist subsets* M *and* N *of* K, *compact convex and compact, respectively, such that for all* $x \in K \setminus N$ *there is* $y \in M$ *such that* $f(g(x), y) \in -\operatorname{int} C(x)$,

then the solution set of IVEP is nonempty and compact.

Proof Define the set-valued mapping $F: K \to 2^K$ by

$$F(y) = \{x \in K : f(g(x), y) \notin -\operatorname{int} C(x)\}.$$

We show that F satisfies all the assumptions of Lemma 1.3. By (b) and (d), $F(y)$ is a closed subset of K for all $y \in K$. It follows from (c) and (a) that F is a KKM mapping. Indeed, on the contrary of the assertion if there exist $y_1, y_2, \ldots, y_n$ in K and $z = \sum_{i=1}^{n} \lambda_i y_i \in \operatorname{co}\{y_1, y_2, \ldots, y_n\} \setminus \bigcup_{i=1}^{n} F(y_i)$, then $f(g(z), y_i) \in -\operatorname{int} C(z)$ and so by (c) we deduce that

$$f(g(z), z) \in -\operatorname{int} C(z),$$

which is a contradiction (by (a)). Then F is a KKM mapping. Also, it is obvious from (e) that $\bigcap_{y \in M} F(y) \subseteq N$ and so $\bigcap_{y \in M} F(y)$ is compact (note that $F(y)$ is closed for each $y \in Y$ and M is compact). Hence by Lemma 1.3 there exists $\overline{x} \in K$ such that

$$\overline{x} \in \bigcap_{x \in K} F(x)$$

and it is easy to see that the solution set of IVEP is equal to the set $\bigcap_{x \in K} F(x)$ and hence $\bar{x}$ is a solution of IVEP and further it is compact (note $\bigcap_{x \in K} F(x) \subseteq \bigcap_{x \in M} F(x) \subseteq N$) and hence the proof is complete. □

We note that if g is continuous and f is continuous with respect to the first variable then the mapping $x \to f(g(x), y)$ is continuous and so condition (b) holds while the simple example $g(x) = 1$ if x is rational, and $g(x) = 0$ if x is irrational, and $f(x, y) = 1$, for x rational, and $f(x, y) = 0$ if x is irrational, shows that it is easy to check that the mapping $x \to f(g(x), y)$ is continuous; nevertheless, neither g nor f is continuous, which shows that the converse does not hold in general. Moreover, in the example, if we take $K = [0, 1]$ then f and g satisfy all the assumptions of Theorem 2.1 and so the solution set of IVEP is nonempty and compact but the example cannot fulfill all the conditions of Proposition 3.1 in [6]. Hence Theorem 2.1 extends Proposition 3.1 in [6]. Also one can easily see the C-convexity of f at the second variable, that is, for each $x \in K$,

$$tf(x, y_1) + (1 - t)f(x, y_2) - f(x, ty_1 + (1 - t)y_2) \in C(x), \quad \forall t \in [0, 1],$$

which implies condition (c) of Theorem 2.1, while if we take $X = \Re$ and let K be any nonempty convex and compact subset of X and define $f(x,y) = -y^2$, for all $x, y \in K$ and we let g be an arbitrary mapping, we take the example $f(x,y) = -y^2$, for $x, y \in K$ and g an arbitrary mapping, then this example fulfills condition (c) (note that it satisfies all the assumptions of Theorem 2.1) but f is not convex at the second variable and hence condition (c) improves condition (3) in Proposition 3.1 of [6].

Definition 2.2 ([6]) A mapping $f : E \times E \to Y$ is said to be locally non-positive at $x_0 \in E$ with respect to a mapping $g : E \to E$ if there exist a neighborhood $V(x_0)$ of x_0 and a point $z_0 \in E \cap \operatorname{int} V(x_0)$ such that

$$f\bigl(g(x), z_0\bigr) \in -C(x), \quad \forall x \in E \cap \partial V(x_0),$$

where $\partial V(x_0)$ is the boundary of $V(x_0)$. In the case that g is the identity mapping, the mapping f is called locally non-positive at $x_0 \in E$.

The following corollary is an extension of Proposition 3.3 in [6] for topological vector spaces. Furthermore, the condition that f is locally non-positive at $x_0 \in K$ has been omitted.

Corollary 2.3 *Let K be a nonempty closed and convex subset of a Hausdorff topological vector space X and let $f : K \times K \to Y$ and $C : K \to Y$ be two mappings such that*:

(a) $f(g(x), x) = 0$, $\forall x \in K$,
(b) *the mapping $x \to f(g(x), y)$ is continuous, for all $y \in K$,*
(c) *for each $x \in K$ the mapping $y \to f(g(x), y)$ is $C(x)$-convex,*
(d) *the mapping $W : K \to 2^Y$ defined by $W(x) = Y \setminus (-\operatorname{int} C(x))$ is closed,*
(e) *there exist a nonempty compact and convex subset D of $K \cap V(x_0)$ and $y_0 \in D$ such that for all $x \in (K \cap V(x_0)) \setminus D$*

$$f\bigl(g(x), y\bigr) \in -\operatorname{int} C(x).$$

Then IVEP has a solution in the neighborhood $V(x_0)$ of x_0, that is, there exists $x^* \in (K \cap V(x_0))$ such that

$$f\bigl(g\bigl(x^*\bigr), y\bigr) \notin \operatorname{int} C\bigl(x^*\bigr), \quad \forall y \in K.$$

Moreover, the solution set is a compact subset of $K \cap V(x_0)$.

Proof There is neighborhood U of x_0 such that $\operatorname{co} U \subseteq V(x_0)$ (see, for example, [10]). Hence by Theorem 2.1, IVEP has a solution on $B = K \cap \overline{\operatorname{co}(\operatorname{co} U \cup \{y_0\})}$. Then there exists $x^* \in B$ such that

$$f\bigl(g\bigl(x^*\bigr), y\bigr) \notin -\operatorname{int} C\bigl(x^*\bigr), \quad \forall y \in B.$$

We claim that

$$f\bigl(g\bigl(x^*\bigr), y\bigr) \notin -\operatorname{int} C\bigl(x^*\bigr), \quad \forall y \in K.$$

Indeed, if the sentence is not true then there is $y \in K$ so that

$$f(g(x^*), y) \in -\operatorname{int} C(x^*).$$

Put $y_t = x^* + t(y - x^*)$, for $t > 0$. It is clear that $y_t \in B$, for t that is small enough. Then by condition (c) we have

$$f(g(x^*), y_t) \in (1-t)f(g(x^*), y) + tf(g(x^*), x^*) \in -\operatorname{int} C(x^*) + 0 = -\operatorname{int} C(x^*),$$

which is a contradiction. Hence x^* is a solution of IVEP. The second part follows from condition (e). This completes the proof. □

The next theorem is an extension of Theorems 3.1, 3.3 and Corollary 3.3 in [6] with mild assumptions for mappings which do not need to satisfy the locally non-positive condition. In fact this condition has been removed.

Theorem 2.4 *Let $F : \Lambda_2 \times E \times E \to Y$ and $g : E \to E$ be two mappings. If the following assumptions hold*:

(i) $\mathbf{K} : \Lambda_1 \to 2^E$ *is a continuous mapping with nonempty convex compact values*;
(ii) $(\varepsilon, x, y) \to F(\varepsilon, g(x), y)$ *is continuous*;
(iii) *the set* $\{y : F(\varepsilon, g(x), y) \in -\operatorname{int} C(x)\}$ *is convex and* $F(\varepsilon, g(x), x) = 0$, *for each* $(\epsilon, x) \in \Lambda_1 \times E$;
(iv) *the mapping* $W : E \to 2^Y$ *defined by* $W(x) = Y \backslash -\operatorname{int} C(x)$ *is closed*;

then

(i) *for each* $(\lambda, \varepsilon) \in \Lambda_1 \times \Lambda_2$, *the solution set*

$$S(\lambda, \varepsilon) = \{x \in \mathbf{K}(\lambda) : F(\varepsilon, g(x), y) \notin -\operatorname{int} C(x), \forall y \in \mathbf{K}(\lambda)\},$$

is nonempty and compact,

(ii) *the solution set mapping* $S : \Lambda_1 \times \Lambda_2 \to 2^X$ *defined by*

$$(\lambda, \varepsilon) \to S(\lambda, \varepsilon)$$

is continuous.

Proof The first part, that is, (i), follows from Theorem 2.1 by taking, for each $(\lambda, \varepsilon) \in \Lambda_1 \times \Lambda_2$, $M = N = \mathbf{K}(\lambda)$ and defining $f(x, y) = F(\varepsilon, x, y)$, for all $(x, y) \in \mathbf{K}(\lambda) \times \mathbf{K}(\lambda)$. To prove (ii), let $\{(\lambda_i, \epsilon_i)\}_{i \in I} \subseteq \Lambda_1 \times \Lambda_2$ be a net with $(\lambda_i, \epsilon_i) \to (\lambda, \epsilon)$ and $z_i \in S(\lambda_i, \epsilon_i) \subset K(\lambda_i)$. Since $\mathbf{K}$ is u.s.c., $\lambda_i \to \lambda$ and $z_i \in \mathbf{K}(\lambda_i)$, using Lemma 1.1(i), there exist $z \in \mathbf{K}(\lambda)$ and a subnet $\{z_{i_j}\}$ of $\{z_i\}$ which converges to z. So

$$F(\epsilon_{i_j}, g(z_{i_j}), y) \in Y \backslash (-\operatorname{int} C(z_{i_j})) = W(z_{i_j}), \quad \forall y \in \mathbf{K}(\lambda_{i_j}). \tag{1}$$

We claim that $z \in S(\lambda, \epsilon)$ (note that if we show the claim then according to Lemma 1.1 the mapping S will be an u.s.c.). If the claim is not true then there is $y \in \mathbf{K}(\lambda)$ such that

$$F(\epsilon, g(z), y) \in -\operatorname{int} C(z). \tag{2}$$

So, since $\mathbf{K}$ is l.s.c., there exists net $w_j \in \mathbf{K}(\lambda_j)$ such that $w_j \to y$. Then it follows from (1) that

$$F(\epsilon_{i_j}, g(z_{i_j}), w_j) \in W(z_{i_j}),$$

and so by (ii) and (iv) we get

$$F(\epsilon, g(z), y) \in Y \backslash -\operatorname{int} C(z) = W(z),$$

which is contradicted by (2) and so $z \in S(\lambda, \epsilon)$. It follows from $x_{i_j} \in S(\lambda_{i_j}, \epsilon_{i_j})$ that $f(\epsilon_{i_j}, g(x_{i_j}), \epsilon_{i_j}) \notin -\operatorname{int} C(x_{i_j})$ and so $f(\epsilon_{i_j}, g(x_{i_j}), y_{i_j}) \in W(x_{i_j})$ and by the closedness of W we get $f(\epsilon, g(x), y) \in W(x)$, which is a contradiction, and then the solution set mapping S is u.s.c. Now we show that S is l.s.c. Let $\{(\lambda_i, \epsilon_i)\}_{i\in I} \subseteq \Lambda_1 \times \Lambda_2$ be a net with $(\lambda_i, \epsilon_i) \to (\lambda, \epsilon)$ and z an arbitrary element of $S(\lambda, \epsilon) \subseteq \mathbf{K}(\lambda)$. Put $\Xi = \{V : V \text{ is a neighborhood of } z\}$ (note by the relation $V \preceq W$ if and only if $V \supseteq W$, the set Ξ is a directed set). Then for each $(V, i) \in \Xi \times I$, there is a closed and convex neighborhood $H_{V,i}$ of z such that $H_{V,i} \subset V$ (see [10]) and so it follows from Theorem 2.1 that there is $z_i \in H_{V,i} \cap \mathbf{K}(\lambda_i)$ such that

$$F(\epsilon_i, g(z_i), y) \notin -\operatorname{int} C(z_i), \quad \forall y \in H_{V,i} \cap \mathbf{K}(\lambda_i). \tag{3}$$

Now if there exists $y \in \mathbf{K}(\lambda_i)$ such that

$$F(\epsilon_i, g(z_i), y) \in -\operatorname{int} C(z_i),$$

it follows from $F(\epsilon_i, g(z_i), z_i) = 0$ (see condition (i)) and (iii) that

$$F(\epsilon_i, g(z_i), z_i + t(y - z_i)) \in -\operatorname{int} C(z_i), \quad \forall t \in [0, 1],$$

which is a contradiction, for $t \in [0, 1]$ small enough, by (3) (note $H_{V,i}$ is an open set and $z_i \in H_{V,i}$). So

$$F(\epsilon_i, g(z_i), y) \notin -\operatorname{int} C(z_i), \quad \forall y \in \mathbf{K}(\lambda_i),$$

and hence $z_i \in S(\epsilon_i, \lambda_i)$. Consequently, for each point $(V, i) \in \Xi \times I$ there is $z_i \in S(\epsilon_i, \lambda_i)$, and so $z_i \to z$. Hence it follows from Lemma 1.1(ii) that S is l.s.c. and the proof is completed. □

Inspired by the proof of the second part of the previous theorem we can deduce the lower semi-continuity of the solution set mapping. Indeed the next theorem is an improvement of Theorem 3.2 in [6] without using the lower semi-continuity of the mapping $\mathbf{K} : \Lambda_1 \to 2^E$; its proof is similar to the proof presented for the second part of Theorem 2.4 and so we omit the proof.

Theorem 2.5 *Let $F : \Lambda_2 \times E \times E \to Y$ and $g : E \to E$ be two mappings. For a given $(\lambda_0, \epsilon_0) \in (\Lambda_1, \Lambda_2)$ there exist neighborhoods $U(\lambda_0)$ of λ_0 and $M(\epsilon_0)$ of ϵ_0 such that the following assumptions are satisfied*:

(i) $\mathbf{K}: U(\lambda_0) \to 2^X$ *is a mapping with nonempty convex and compact values*;
(ii) *mapping* $(\varepsilon, x, y) \to F(\varepsilon, g(x), y)$ *is continuous, for each* $(\varepsilon, x, y) \in M(\epsilon_0) \times E \times E$;
(iii) *for each* $x \in E$ *the mapping* $y \to F(\varepsilon, g(x), y)$ *is* $C(x)$*-convex and* $F(\varepsilon, g(x), x) = 0$, *for each* $(\epsilon, x) \in M(\epsilon_0) \times E$;
(iv) *the mapping* $W: E \to 2^Y$ *defined by* $W(x) = Y \setminus -\operatorname{int} C(x)$ *is closed*.

Then

(i) $S(\lambda, \varepsilon)$, *for each* $(\lambda, \varepsilon) \in U(\lambda_0) \times M(\varepsilon_0)$ *is nonempty and compact*;
(ii) *the mapping* S *is l.s.c. at* $(\lambda_0, \varepsilon_0)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work presented have was carried out in collaboration between all authors. The main idea of this paper was proposed by AF and SP prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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