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An extension of the Golden-Thompson theorem

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Abstract

In this paper, we shall prove $|\operatorname{tr} e^{A+B}| \le \operatorname{tr}(|e^A||e^B|)$ for normal matrices A, B. In particular, $\operatorname{tr} e^{A+B} \le \operatorname{tr}(e^A e^B)$ if A, B are Hermitian matrices, yielding the Golden-Thompson inequality.

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1 Introduction and preliminaries

The famous Golden-Thompson inequality [1–4] for Hermitian matrices A, B states that $\operatorname{tr} e^{A+B} \le \operatorname{tr}(e^A e^B)$. This inequality is a basic tool in quantum statistical mechanics and extensions to infinite dimension have an extensive literature [5, 6]. In this paper, we extend the classical Golden-Thompson theorem to normal matrices.

Throughout this paper, we adopt the following notation. Let M_n be the set of all $n \times n$ complex matrices. For a matrix $A \in M_n$, as usual, its conjugate transpose is denoted by A^*. A matrix A is called Hermitian if $A = A^*$, normal if $A^*A = AA^*$, and unitary if $A^*A = AA^* = I_n$ (I_n is the identity matrix of order n). Given a matrix $A \in M_n$, the eigenvalues and singular values of A are denoted by $\lambda_1(A), \dots, \lambda_n(A)$, and $s_1(A), \dots, s_n(A)$, respectively, where $|\lambda_1(A)| \ge \cdots \ge |\lambda_n(A)|$ and $s_1(A) \ge \cdots \ge s_n(A)$. In particular, when A is positive semidefinite ($A \ge 0$), then $\lambda_1(A) \ge \cdots \ge \lambda_n(A) \ge 0$. For simplicity, we denote $\lambda(A) \equiv (\lambda_1(A), \dots, \lambda_n(A))$ and $s(A) \equiv (s_1(A), \dots, s_n(A))$. Recall that the singular values of a matrix $A \in M_n$ are defined to be the eigenvalues of $|A| \equiv (A^*A)^{1/2}$, *i.e.*, $s(A) = \lambda(|A|)$. Here $s_1(A) = \|A\|$ is the spectral norm of A. It is known that the spectral norm $\|\cdot\|$ over M_n is unitarily invariant, *i.e.*, $\|UAV\| = \|A\|$ for all unitary matrices U, V.

We now recall the concept of majorization (details can be found in [7–9]). We have the following basic majorant relations. For real vectors $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in coordinates in decreasing order, we say that x is weakly majorized by y, denoted by $x \prec_w y$, if

$$\sum_{j=1}^{k} x_j \le \sum_{j=1}^{k} y_j, \quad k = 1, \dots, n,$$

and the weak log-majorant relation $x \prec_{\mathrm{wlog}} y$ means

$$\prod_{j=1}^{k} x_j \le \prod_{j=1}^{k} y_j, \quad k = 1, \dots, n.$$

If in addition to $x \prec_{\mathrm{wlog}} y$, $\prod_1^n x_j = \prod_1^n y_j$ holds, we say that x is log-majorized by y, denoted briefly by symbols $x \prec_{\log} y$. The following statement (see [8, 10]) is well known: $x \prec_{\mathrm{wlog}} y$ yields $x \prec_w y$ for vectors $x, y \in R_+^n$.

Remark 1.1 For $x = (x_1, \ldots, x_n)$, we denote $|x| = (|x_1|, \ldots, |x_n|)$. Weyl's majorant theorem [10] says that $|\lambda(A)| \prec_{\log} s(A)$ for $A \in M_n$, that is,

$$\left(\left|\lambda_1(A)\right|, \ldots, \left|\lambda_n(A)\right|\right) \prec_{\log} \left(\left|s_1(A)\right|, \ldots, \left|s_n(A)\right|\right).$$

The formula above implies that $|\lambda(A)| \prec_w s(A)$.

2 Lemmas

In this section, we shall propose some lemmas, laying the foundations of our main results in the next section.

Lemma 2.1 [11] *If A, B are positive semidefinite matrices, then*

$$\|AB\|^t \le \left\|A^t B^t\right\|, \quad \textit{and} \quad \lambda_1^t(AB) \le \lambda_1\left(A^t B^t\right), \quad \textit{for } t \ge 1.$$

Here note that $\|X\| = s_1(X)$ is the spectral norm of X.

Lemma 2.2 *If $A, B \in M_n$ are normal matrices, then for any integer $m \ge 1$*

$$\|AB\|^m \le \left\|\left|A^m\right| \cdot |B|^m\right\| = \left\|A^m B^m\right\|.$$

Proof Take the polar decompositions $A = U|A|$ and $B = V|B|$. Here U, V are unitary matrices. Since A, B are normal, we can derive that $U|A| = |A|U$ and $V|B| = |B|V$ (see [12, 13]). Thus

$$AB = U|A||B|V, \qquad A^m B^m = U^m |A|^m |B|^m V^m.$$

Since the norm $\|\cdot\|$ is unitary invariant, we obtain the following:

$$\|AB\|^m = \left\|U\left(|A| \cdot |B|\right)V\right\|^m = \left\|\left(|A| \cdot |B|\right)\right\|^m,$$

and

$$\left\|A^m B^m\right\| = \left\|U^m \cdot |A|^m \cdot |B|^m \cdot V^m\right\| = \left\||A|^m \cdot |B|^m\right\|.$$

From Lemma 2.1, $\||A| \cdot |B|\|^m \le \|(|A|^m \cdot |B|^m)\|$, we therefore conclude that

$$\|AB\|^m \le \left\|A^m B^m\right\|. \qquad \square$$

Lemma 2.3 *If $A, B \in M_n$ are normal matrices, then*

$$\|AB\|^2 = s_1^2\left(|A| \cdot |B|\right) = \lambda_1\left(|A|^2 \cdot |B|^2\right).$$

Proof Since A and B are normal, it follows from Lemma 2.2 that $\|AB\| = \||A| \cdot |B|\|$. So we get

$$\|AB\|^2 = \big\|\big(|A| \cdot |B|\big)\big\|^2 = s_1^2\big(|A| \cdot |B|\big) = \lambda_1\big(\big(|A| \cdot |B|\big)^* \cdot \big(|A| \cdot |B|\big)\big),$$

as desired. □

Lemma 2.4 *If $A, B \in M_n$ are normal matrices, then for integers $m \geq 2$*

$$s_1^m(AB) = \|AB\|^m \leq \lambda_1\big(|A|^m \cdot |B|^m\big).$$

Proof By Lemma 2.3, we have $\|AB\|^2 = \lambda_1(|A|^2 \cdot |B|^2)$, and

$$\|AB\|^m = \big(\|AB\|^2\big)^{m/2} = \lambda_1^{m/2}\big(|A|^2|B|^2\big).$$

Applying Lemma 2.1 to the right side above, we have the following:

$$\lambda_1^{m/2}\big(|A|^2 \cdot |B|^2\big) \leq \lambda_1\big(|A|^m \cdot |B|^m\big).$$

Thus we get $\|AB\|^m \leq \lambda_1(|A|^m \cdot |B|^m)$ for integers $m \geq 2$, as desired. □

Here we note that $|A^m| = |A|^m$ holds for any normal matrix.

The following lemma needs the notion of the Grassmann power $\Lambda^k A$ (or antisymmetric tensor product), which can be found in [8, p.18].

Lemma 2.5 *If $A \in M_n$, $1 \leq k \leq n$, then for any natural number m, the following holds*:

$$\prod_{j=1}^{k} s_j\big(A^m\big) \leq \prod_{j=1}^{k} s_j^m(A), \quad \textit{and} \quad \prod_{j=1}^{n} s_j\big(A^m\big) = \prod_{j=1}^{n} s_j^m(A).$$

i.e.,

$$s\big(A^m\big) \prec_{\log} s^m(A).$$

Proof For $1 \leq k \leq n$, consider the kth antisymmetric tensor product $\Lambda^k A$ of $A \in M_n$. It is known [8, p.18] that $\Lambda^k(A^m) = (\Lambda^k A)^m$ and

$$s_1\big(\Lambda^k\big(A^m\big)\big) = s_1\big(\big(\Lambda^k A\big)^m\big) = \big\|\big(\Lambda^k A\big)^m\big\| \leq \big\|\Lambda^k A\big\|^m = \big(s_1\big(\Lambda^k A\big)\big)^m.$$

Thus

$$\prod_{j=1}^{k} s_j\big(A^m\big) \leq \prod_{j=1}^{k} s_j^m(A), \quad k = 1, \ldots, n.$$

In particular,

$$\prod_{j=1}^{n} s_j\big(A^m\big) = \prod_{j=1}^{n} s_j^m(A) = \big|\det(A)\big|^m,$$

which, equivalently, says that $s(A^m) \prec_{\log} s^m(A)$. This completes the proof. □

3 Main results

In this section, we shall present the main results of this paper.

Theorem 3.1 *If $A, B \in M_n$ are normal matrices, then*

$$s\left(e^{A+B}\right) \prec_{\log} \lambda\left(\left|e^A\right| \cdot \left|e^B\right|\right).$$

Proof Let $A, B \in M_n$ be normal matrices. It is clear that $\Lambda^k e^{A/m}$, $\Lambda^k e^{B/m}$ are normal for $1 \le k \le n$. By replacing A, B by $\Lambda^k e^{A/m}$, $\Lambda^k e^{B/m}$ in Lemma 2.4, respectively, we can obtain the following for integers $m \ge 2$:

$$\begin{aligned} s_1^m\left(\Lambda^k\left(e^{A/m}e^{B/m}\right)\right) &= s_1^m\left(\Lambda^k e^{A/m}\Lambda^k e^{B/m}\right) \\ &\le \lambda_1\left(\left|\Lambda^k e^A\right|\left|\Lambda^k e^B\right|\right) = \lambda_1\left(\Lambda^k\left(\left|e^A\right|\left|e^B\right|\right)\right). \end{aligned}$$

Here we note that $|\Lambda^k A| = \Lambda^k|A|$ because $|\Lambda^k A|^2 = \Lambda^k|A|^2$. So we obtain

$$\prod_{j=1}^k s_j^m\left(e^{A/m}e^{B/m}\right) \le \prod_{j=1}^k \lambda_j\left(\left|e^A\right| \cdot \left|e^B\right|\right).$$

From Lemma 2.5, we have

$$\prod_{j=1}^k s_j\left[\left(e^{A/m}e^{B/m}\right)^m\right] \le \prod_{j=1}^k s_j^m\left(e^{A/m}e^{B/m}\right).$$

Thus,

$$\prod_{j=1}^k s_j\left[\left(e^{A/m}e^{B/m}\right)^m\right] \le \prod_{j=1}^k \lambda_j\left(\left|e^A\right|\left|e^B\right|\right).$$

The Lie product formula [8, p.254] says that for any matrices A, B

$$\lim_{m\to\infty}\left(e^{A/m}e^{B/m}\right)^m = e^{A+B}.$$

Thus taking $m \to \infty$ in the inequality above yields

$$\prod_{j=1}^k s_j\left(e^{A+B}\right) \le \prod_{j=1}^k \lambda_j\left(\left|e^A\right|\left|e^B\right|\right).$$

Finally we note that

$$\prod_{j=1}^n s_j\left(e^{A+B}\right) = \left|\det\left(e^{A+B}\right)\right| = \left|\det\left(e^A e^B\right)\right| = \prod_{j=1}^n \lambda_j\left(\left|e^A\right|\left|e^B\right|\right).$$

Thus we get

$$s\left(e^{A+B}\right) \prec_{\log} \lambda\left(\left|e^A\right| \cdot \left|e^B\right|\right).$$

This completes the proof. □

From Theorem 3.1, we know that

$$s\left(e^{A+B}\right) \prec_{\log} \lambda\left(\left|e^A\right| \cdot \left|e^B\right|\right).$$

On the other hand, the following equation holds:

$$\lambda\left(\left|e^A\right| \cdot \left|e^B\right|\right) = \lambda\left(\left|e^A\right|^{1/2} \cdot \left|e^B\right| \cdot \left|e^A\right|^{1/2}\right) = s\left(\left|e^A\right|^{1/2} \cdot \left|e^B\right| \cdot \left|e^A\right|^{1/2}\right).$$

The above two inequalities yield the following:

$$s\left(e^{A+B}\right) \prec_{\log} s\left(\left|e^A\right|^{1/2} \cdot \left|e^B\right| \cdot \left|e^A\right|^{1/2}\right).$$

Thus, we can get the following corollary by using the Fan Dominance Principle [10, p.56].

Corollary 3.2 *If $A, B \in M_n$ are normal matrices, then*

$$\left\|\left|e^{A+B}\right|\right\| \leq \left\|\left|\left|e^A\right|^{1/2} \cdot \left|e^B\right| \cdot \left|e^A\right|^{1/2}\right|\right\|,$$

for all unitarily invariant norms $\||\cdot|\|$.

From Theorem 3.1, we can also have the following result.

Theorem 3.3 *If $A, B \in M_n$ are normal matrices, then*

$$\left|\lambda\left(e^{A+B}\right)\right| \prec_{\log} \lambda\left(\left|e^A\right| \cdot \left|e^B\right|\right).$$

Proof By Weyl's majorant theorem we have $|\lambda(A)| \prec_{\log} s(A)$. Hence Theorem 3.1 implies the desired inequality in Theorem 3.3. □

Note that Theorem 3.3 strengthens the Golden-Thompson inequality:

$$\left|\operatorname{tr}\left(e^{A+B}\right)\right| \leq \operatorname{tr}\left(e^A e^B\right)$$

for Hermitian matrices A, B.

Theorem 3.4 *If A, B are normal matrices, then*

$$\left|\operatorname{tr}\left(e^{A+B}\right)\right| \leq \operatorname{tr}\left(\left|e^A\right| \cdot \left|e^B\right|\right).$$

Proof Because $x \prec_{\log} y$ implies $x \prec_w y$, it follows from Theorem 3.3 that

$$\left|\lambda\left(e^{A+B}\right)\right| \prec_w \lambda\left(\left|e^A\right| \cdot \left|e^B\right|\right).$$

Taking the traces above, we have

$$\left|\operatorname{tr}\left(e^{A+B}\right)\right| \leq \operatorname{tr}\left(\left|e^A\right| \cdot \left|e^B\right|\right).$$

So we get the desired inequality. This completes the proof. □

Of course, Theorem 3.4 is an extension of Golden-Thompson inequality:

$$\mathrm{tr}\left(e^{A+B}\right) \le \mathrm{tr}\left(e^{A}e^{B}\right)$$

for Hermitian matrices A, B.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HL carried out the theorems and corresponding proofs, DZ checked the proofs carefully, and provided numerical examples and valuable suggestions. All authors read and approved the final manuscript.

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